A NATURAL GROMOV-WITTEN VIRTUAL FUNDAMENTAL CLASS

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Abstract. We describe a program for proving that the Gromov-Witten moduli spaces of compact symplectic manifolds carry a unique virtual fundamental class that satisfies certain naturality conditions. The virtual fundamental class is constructed using only Ruan-Tian perturbations by introducing stabilizing divisors, using Čech homology, and systematically applying the naturality conditions. In low dimensions or low genus, no gluing theorems are needed.

1. Introduction

The central object of Gromov-Witten theory is the virtual fundamental class (VFC) of the moduli space of stable maps. In the algebraic geometry context, the virtual fundamental class was constructed soon after the subject began. It is difficult to extend the construction to the symplectic category, in large part because the methods of geometric analysis are ill-suited to apply to maps from families of degenerating curves. The issue was originally addressed in the late 1990s ([FO], [LiT], [LiuT], [R], [S]). As the subject has developed, there has been an increasing need for an approach that is conceptual and also easy to use. In recent years major efforts have been made to define a virtual fundamental class using Kuranishi structures [FOOO], [J], [MW], and polyfolds [HWZ]. Recently, J. Pardon has developed a very nice approach using implicit atlases [Par].

The program described in this article is an alternative approach, building on ideas of Cieliebak-Mohnke [CM] and guided by the functorial aspects of the VFC. Parts of this program are currently conjectural. The strategy is to first observe that existing results give a virtual fundamental class for a set of especially nice almost complex structures (which we call super-fine and regular), and then successively enlarge the space of parameters on which the VFC is defined. This strategy works well in high dimensions ($\dim X \geq 12$) or low genus ($g \geq 1$) where, as observed by Tehrani and Zinger in [TZ], dimension counts obviate any need for gluing theorems. (We do not address the lower dimensional case here.) To fully exploit functoriality, we consider a variety of related Gromov-Witten theories.

Here is the general setting. In symplectic Gromov-Witten theory, each moduli space is part of a family parameterized by the space $\mathcal{J}(X)$ of almost complex structures on X that are tamed by a fixed symplectic structure ω, or more generally, the space of all tamed pairs (J,ω). One must also specify the type of domain curve and type of target. The choices involved can be phrased as follows.

A "GW moduli problem" for a symplectic manifold X, or more generally a symplectic pair (X,V), consists of

(a) A choice of a category of families of complex curves,
(b) A choice of a (possibly empty) normal crossing divisor $V \subset X$,
(c) A choice of a set $\mathcal{J}$ of that parameterizes a set of pseudo-holomorphic maps.

From this data one constructs a GW family $\overline{\mathcal{M}}(X)$ of moduli spaces over $\mathcal{J}$ whose fiber $\overline{\mathcal{M}}^J(X)$ over $J \in \mathcal{J}$ is the (compact) set of all isomorphism classes of J-holomorphic maps from curves of the specified type. We will assume that (a) is a category of curves with n marked points and a

Deligne-Mumford space $\overline{\mathcal{M}}_n$; then there is an stabilization-evaluation map for the family:

$$\begin{array}{ccc} \overline{\mathcal{M}}(X) & \xrightarrow{se} & \overline{\mathcal{M}}_n \times X^n \\ \pi \downarrow & & \\ \mathcal{J} & & \end{array} \tag{1.1}$$

Meta-Theorem: Every fiber of π carries a virtual fundamental class $[\overline{\mathcal{M}}^J(X)]^{vir} \in \check{H}_*(\overline{\mathcal{M}}^J(X))$ in rational Čech homology.

The existence of a virtual fundamental class immediately gives the Gromov-Witten invariants by pushing forward by the se map.

The purpose of our program is to make this meta-theorem precise, to outline a proof, and to describe some functorial and uniqueness properties of the virtual fundamental class.

The following examples emphasize the fact that there is not one, but many GW theories, depending on the choices (a)–(c).

Example 1.1. It is standard to consider the compact moduli spaces $\overline{\mathcal{M}}^J_{A,g,n}(X)$ of stable J-holomorphic maps from a genus g n-marked curve to X that represent $A \in H_2(X)$. These give a diagram (1.1) where

$$se : \overline{\mathcal{M}}_{A,g,n}(X) \to \overline{\mathcal{M}}_{g,n} \times X^n.$$

The GW invariants are

$$GW_{A,g,n}(X) = se_*[\overline{\mathcal{M}}^J_{A,g,n}(X)]^{vir} \in H_*(\overline{\mathcal{M}}_{g,n} \times X^n).$$

Each $\overline{\mathcal{M}}_{A,g,n}(X)$ is an open and closed subset of $\overline{\mathcal{M}}(X) = \underset{A,g,n}{\sqcup} \overline{\mathcal{M}}_{A,g,n}(X)$.

Example 1.2. One can also consider maps with decorated domains. One example, described in Section 2, is obtained by fixing a finite group G and considering curves with twisted G-covers. Again, there is a space of stable maps and a stabilization-evaluation map

$$se : \overline{\mathcal{M}}^G_{A,g,n}(X) \to \overline{\mathcal{M}}^G_{g,n} \times X^n.$$

Example 1.3. One can also decorate the target by choosing a normal crossing divisor $V \subset X$ and making two modifications: a) restrict the parameter space $\mathcal{J}$ to the subspace $\mathcal{J}(X,V)$ of V-compatible almost complex structures, and b) mark all the points in the inverse image of V, and decorate them with a vector s that records contact multiplicities to the various branches of V. Using the relative stable map compactification, one obtains moduli spaces $\overline{\mathcal{M}}(X,V)$ of relatively stable maps and a refined se map

$$se : \overline{\mathcal{M}}_{A,g,n,s}(X,V) \to \overline{\mathcal{M}}_{g,n+\ell(s)} \times X^n \times \mathbb{P}_s(NV)$$

as described in Section 7. There is also a G-twisted version

$$se : \overline{\mathcal{M}}^G_{A,g,n,s}(X,V) \to \overline{\mathcal{M}}^G_{g,n+\ell(s)} \times X^n \times \mathbb{P}_s(NV).$$

To include all these examples, we consider families of moduli spaces

$$\begin{array}{c} \overline{\mathcal{M}}^{G,P}_{A,g,n,s}(X,V) \\ \pi \downarrow \\ P \subset \mathcal{J} \end{array} \tag{1.2}$$

parameterized by some subset P of almost complex structures or Ruan-Tian perturbations (see Section 3.1), and where $\overline{\mathcal{M}}^G(X,V)$ denotes the moduli space of G-twisted maps into (X,V), for a finite group G and a normal crossing divisor V in X. Each such family of moduli spaces has a natural stabilization-evaluation map se and an expected dimension d depending on (A, g, n, s). These families are related by functorial maps of several types:

(i) An inclusion $\iota : Q \hookrightarrow R$ of compact subsets of $\mathcal{J}$ yields a diagram

$$\begin{tikzcd} \overline{\mathcal{M}}^{G,Q}_{A,g,n,s}(X,V) \arrow[r, "\iota"] \arrow[d] & \overline{\mathcal{M}}^{G,R}_{A,g,n,s}(X,V) \arrow[d] \\ Q \arrow[r, "\iota"] & R \end{tikzcd} \tag{1.3}$$

(ii) Forgetting the twisted G-cover induces a map

$$\varphi_G : \overline{\mathcal{M}}^{G}_{A,g,n,s}(X,V) \to \overline{\mathcal{M}}_{A,g,n,s}(X,V). \tag{1.4}$$

over the subspace $\mathcal{J}(X,V)$ of parameters that do not depend on the G-structure.

(iii) Forgetting a smooth divisor V with the corresponding contact points induces a map

$$\varphi_V : \overline{\mathcal{M}}_{A,g,n,s}(X,V) \to \overline{\mathcal{M}}_{A,g,n}(X) \tag{1.5}$$

over the space of parameters $\mathcal{J}(X,V)$ that do not depend on the $\ell(s)$ contact points. Similarly, forgetting a component V of a normal crossing divisor $V \cup W$ induces a map

$$\varphi_W : \overline{\mathcal{M}}_{A,g,n,s}(X, V\cup W) \to \overline{\mathcal{M}}_{A,g,n,s_W}(X,W) \tag{1.6}$$

over $\mathcal{J}(X, V \cup W)$, where s_W is the subset of s that records the intersection multiplicities with W.

A parameter $J \in P$ is called *regular* if the linearization of the J-holomorphic map equation on each stratum is a surjective operator, for every J-holomorphic map $f \in \overline{\mathcal{M}}^{J}_{A,g,n,s}(X,V)$. Two classes of regular J are especially easy to work with: a regular J is *domain-stable* if the domain of every f in the moduli space is a stable curve, and is *domain-fine* if each domain is a stable curve with no non-trivial automorphisms.

Ideally, one could hope to find a parameter space P in which the fiber $\overline{\mathcal{M}}^{J}$ over each regular J is a manifold of the expected dimension d. For such J, one expects that the restriction of (1.4) to the fiber to be a map

$$\varphi_G : \overline{\mathcal{M}}^{G,J}_{A,g,n,s}(X,V) \to \overline{\mathcal{M}}^{J}_{A,g,n,s}(X,V)$$

of degree equal to the degree $\deg_G$ of the finite branch cover $\varphi_G : \overline{\mathcal{M}}^{G}_{g,n+\ell(s)} \to \overline{\mathcal{M}}_{g,n+\ell(s)}$.

Focusing on the case where s is $\mathbf{1} = (1, \ldots, 1)$, one might guess that for a smooth divisor D constructed by Donaldson's Theorem, (1.5) is a map

$$\varphi_D : \overline{\mathcal{M}}^{J}_{A,g,n,\mathbf{1}}(X,D) \to \overline{\mathcal{M}}^{J}_{A,g,n,\mathbf{1}}(X)$$

of degree $\ell(s)! = (A \cdot D)!$. In Section 11, we show that this property holds for a non-empty class of "sufficiently positive divisors" D (cf. Definition 8.1) provided that either $\dim_{\mathbb{R}} X \geq 10$ or $g \leq 1$ (outside this range, there may be correction terms coming from maps with components in D.)

Similarly, one might conjecture that (1.6) restricts to be a map

$$\varphi_D : \overline{\mathcal{M}}^{J}_{A,g,n,s,\mathbf{1}}(X, V \cup D) \to \overline{\mathcal{M}}^{J}_{A,g,n,s}(X,V),$$

also of degree $(A \cdot D)!$. In Section 11 we show that this property holds if the branch D belongs to a more restricted, again non-empty, class of sufficiently positive divisors, and either $g \leq 1$ or every stratum of $V \cup D$ is at least 8 dimensional.

Each subset Q of the parameter space P in (1.2) determines a moduli space $\overline{\mathcal{M}}^{Q}(X) = \pi^{-1}(Q)$ over Q. As shown in Section 4, the family (1.2) has a metric space topology making π a proper map to P with its C^0 topology. One can then pass to Čech homology with rational coefficients, which defined for all metric spaces and is equal to Steenrod homology with rational coefficients for all compact metric spaces. For facts about these homology theories we refer the reader to [Ma1], [Ma2] and [Mil]. In

Čech homology, a compactification $\overline{M} = M$ of an oriented topological manifold M has a fundamental class $[\overline{M}]$ provided that the set $S = \overline{M} \setminus M$ of "boundary strata" has codimension 2, regardless of how these strata fit together (see [DK, page 357], and [**?**] for details).

Definition 1.4. A *virtual fundamental class* for a moduli space $\overline{\mathcal{M}}^P = \overline{\mathcal{M}}^P_{A,g,n,s}(X,V)$ over a path-connected manifold $P \subset \mathcal{J}$ associates to each compact subset $Q \subseteq P$ an element

$$[\overline{\mathcal{M}}^Q]^{vir} \in \check{H}_*(\overline{\mathcal{M}}^Q;\mathbb{Q})$$

such that a regularity axiom and three naturality axioms hold:

A1. If $J \in P$ is regular then $[\overline{\mathcal{M}}^J]^{vir}$ is the fundamental class $[\overline{\mathcal{M}}^J]$.

A2. A proper inclusion $\iota : Q \to R$ induces $\iota_*[\overline{\mathcal{M}}^Q]^{vir} \;=\; [\overline{\mathcal{M}}^R]^{vir}$.

A3. Under (1.4), $(\varphi_G)_*[\overline{\mathcal{M}}^{G,Q}]^{vir} \;= \deg_G \cdot [\overline{\mathcal{M}}^Q]^{vir}$.

A4. Suppose that $A \neq 0$, that D is a smooth sufficiently positive divisor, that Q is a subset of $\mathcal{J}(X,D) \subset \mathcal{J}(X)$, and that $\dim_{\mathbb{R}} X \geq 10$ or $g \leq 1$. Then under (1.5) with $s = \mathbf{1} = (1,\ldots,1)$,

$$\frac{1}{\ell(s)!}\,(\varphi_D)_*[\overline{\mathcal{M}}^Q_{A,g,n,\mathbf{1}}(X,D)]^{vir} \;=\; [\overline{\mathcal{M}}^Q_{A,g,n}(X)]^{vir}.$$

More generally, suppose that $A \neq 0$, $V \cup D$ is a normal crossing extension of V with D sufficiently positive, $Q \subset \mathcal{J}(X, V \cup D)$, and that either $g \leq 1$ or all strata of $V \cup D$ are at least 8 dimensional. Then under (1.6) and $s_D = \mathbf{1} = (1,\ldots,1)$,

$$\frac{1}{\ell(s_D)!}\,(\varphi_D)_*[\overline{\mathcal{M}}^Q_{A,g,n,s,\mathbf{1}}(X,V\cup D)]^{vir} \;=\; [\overline{\mathcal{M}}^Q_{A,g,n,s}(X,V)]^{vir}.$$

With this terminology, our main result can be stated simply:

Theorem 1.5. *Assume X is a closed symplectic manifold and V a normal crossing divisor in X. Whenever $A \neq 0$ and*

$$\textit{either } g \leq 1, \textit{ or } X \textit{ and every stratum of } V \textit{ are at least 12 dimensional} \tag{1.7}$$

then there is a unique virtual fundamental class associated to $\overline{\mathcal{M}}_{A,g,n,s}(X,V) \to \mathcal{J}(X,V)$. In particular,

(a) For each $J \in \mathcal{J}(X)$, there is a class

$$[\overline{\mathcal{M}}^J_{A,g,n,s}(X)]^{vir} \;\in\; \check{H}_*(\overline{\mathcal{M}}^J_{A,g,n,s}(X);\mathbb{Q}).$$

(b) For each $J \in \mathcal{J}(X,V)$, there is a class

$$[\overline{\mathcal{M}}^J_{A,g,n,s}(X,V)]^{vir} \;\in\; \check{H}_*(\overline{\mathcal{M}}^J_{A,g,n,s}(X,V);\mathbb{Q}).$$

For the cases not covered by Theorem 1.5 — those where the dimension of the target is low and the genus of the domain is high — Axiom A4 must be modified by the addition of correction terms. This complication will be addressed elsewhere. The case $A = 0$ is much easier and can be treated with standard techniques.

The image of such a virtual fundamental class under an *se* map is called a *Gromov-Witten class.*

Corollary 1.6. *The Gromov-Witten class*

$$GW_{A,g,n,s}(X,V) = se_*[\overline{\mathcal{M}}^J_{A,g,n,s}(X,V)]^{vir} \in H_*(\overline{\mathcal{M}}_{g,n} \times X^n \times \mathbb{P}_s(NV))$$

depends only on the deformation class of $(J,\omega) \in \mathcal{J}(X,V)$.

Proof. Two tame structures $J_0 = (J, \omega)$ and $J_1 = (J', \omega')$ are deformation equivalent if they lie in a path $P = \{J_t\}$ in $\mathcal{J}(X)$. Let $\iota_0 : \{J_0\} \to P$ and $\iota_1 : \{J_1\} \to P$ be the inclusions. Then by Axiom A2

$$se_* \iota_{0*}[\overline{\mathcal{M}}^{J_0}(X,V)]^{vir} = se_* \iota_{1*}[\overline{\mathcal{M}}^{J_1}(X,V)]^{vir} = se_*[\overline{\mathcal{M}}^{P}(X,V)]^{vir}.$$

□

Theorem 1.5 is proved by repeatedly applying three moves: (i) introducing G-structures and stabilizing divisors, and passing to the associated moduli spaces, (ii) enlarging the space of almost complex structures and (iii) applying the continuity property of Čech homology. Stabilizing divisors are analogs of the very ample line bundles used to construct the virtual fundamental class in algebraic geometry. We follow Cieliebak and Mohnke's approach [CM] of using divisors from Donaldson's Theorem, and also include ideas from Siebert [S] and McDuff-Wehrheim [MW].

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2. Families of curves

Gromov-Witten moduli spaces $\overline{\mathcal{M}}(X)$ are families of pseudo-holomorphic maps. To understand them, one must first understand the case where X is a single point. In this case, $\overline{\mathcal{M}}(X)$ is a family of curves with an appropriate topology. Thus it is not enough to consider curves in isolation; one must consistently work with families of curves and maps between such families. This viewpoint, reviewed in this section, is standard in algebraic geometry, but its power has not been fully used in the geometric analysis literature on Gromov-Witten theory.

For our purposes, a *family of genus g n-marked curves over S* is a proper surjective map of complex analytic spaces with sections $\sigma_1, \dots, \sigma_n$

$$\begin{tikzcd} \mathcal{C} \arrow[d, "\pi"'] \\ S \arrow[u, bend right, "\sigma_i"'] \end{tikzcd} \tag{2.1}$$

whose fibers $C_s = \pi^{-1}(s)$ are closed, connected curves of arithmetic genus g and that has the following structure. There is an nodal set $\mathcal{N} \subset \mathcal{C}$ such that (i) π is a locally trivial fibration in the complement of each neighborhood of $\mathcal{N}$ and (ii) for each point $c \in \mathcal{N}$ there are coordinates $(x, y, v) \in \mathbb{C}^2 \times V$ on a neighborhood of $c = (0, 0, 0)$ and $(z, v) \in \mathbb{C} \times V$ on a neighborhood of $\pi(c)$ in which π is given by $\pi(x, y, v) = (xy, v)$. Finally, the images of the sections σ_i are disjoint and disjoint from $\mathcal{N}$.

Maps between families are given by the obvious commutative square; this gives a notion of isomorphisms and automorphisms of families. The pullback of a family (2.1) along a map $\varphi : T \to S$ is a family $\varphi^*\mathcal{C}$ over T. One can then envision a *universal curve*: a family $\overline{\mathcal{U}} \to \overline{\mathcal{M}}$ over some space $\overline{\mathcal{M}}$ such that any family (2.1) is isomorphic to the pullback along a unique map $\varphi : S \to \overline{\mathcal{M}}$.

$$\begin{tikzcd} \mathcal{C} \arrow[r] \arrow[d] & \overline{\mathcal{U}} \arrow[d] \\ S \arrow[r, "\varphi"] & \overline{\mathcal{M}} \end{tikzcd} \tag{2.2}$$

In this generality, universal curves do not exist. The key impediment is the existence of families that contain a fiber C_s with non-trivial automorphisms. If the map φ to $\overline{\mathcal{M}}$ is to be unique, then the pullback must be $C_s/\mathrm{Aut}(C_s)$ instead of C_s. Worse, if $\mathrm{Aut}(C_s)$ contains a subgroup isomorphic to $\mathbb{C}^*$ then $\overline{\mathcal{M}}$ cannot exist as a Hausdorff space. Thus the usual approach is to:

(i) Restrict to *stable families*: a curve C is *stable* if $\mathrm{Aut}(C)$ is finite and a family is stable if its fibers are all stable.

(ii) Require only the weaker universal property that there is a unique map φ as in (2.2) so that the pullback family is a finite quotient of $\mathcal{C} \to S$.

Property (ii) makes $\overline{\mathcal{M}}$ a "coarse moduli space" and necessitates working with orbifolds. (Alternative approaches using Artin stacks are not well-suited for use with geometric analysis.)

Example 2.1. Deligne-Mumford spaces. Consider genus g curves C with n distinct marked points $x_1, \dots, x_n$, none of which is a node. The x_i, together with the nodes, are called the special points of C. Smooth curves of this sort are stable if $2g - 2 + n > 0$; in general C is stable if each irreducible component with genus 0 has at least 3 special points, and each with genus 1 has at least one special point. For $2g - 2 + n > 0$ families of stable curves modulo automorphisms are classified by maps into the Deligne-Mumford space $\overline{\mathcal{M}}_{g,n}$ (see [L2] for a nice survey of this). Deligne, Mumford and Knudsen proved (see [L2]):

- $\overline{\mathcal{M}}_{g,n}$ is a projective, complex analytic orbifold;
- the locus $\partial\overline{\mathcal{M}}_{g,n} = \overline{\mathcal{M}}_{g,n} \setminus \mathcal{M}_{g,n}$ parametrizing singular curves is a normal crossing divisor in the orbifold sense.

There is also a universal curve $p : \overline{\mathcal{U}}_{g,n} \to \overline{\mathcal{M}}_{g,n}$ which is isomorphic to $\overline{\mathcal{U}}_{g,n} = \overline{\mathcal{M}}_{g,n+1}$ where p is the map that forgets the extra marked point x_0 adn collapses unstable components. Each stable genus g curve C with n marked points has a unique stable model $\mathrm{st}(C) = [C] \in \overline{\mathcal{M}}_{g,n}$ and a unique map $\iota : C \to \overline{\mathcal{U}}_{g,n}$ that fits in the commutative diagram

$$\begin{tikzcd} C \arrow[r, "\iota"] \arrow[d] \arrow[dr, "\mathrm{st}"'] & \overline{\mathcal{U}}_{g,n} \arrow[d, "p"] \\ pt. \arrow[r] & \overline{\mathcal{M}}_{g,n} \end{tikzcd} \tag{2.3}$$

(ι is defined by $\iota(z) = \mathrm{st}(C, z) \in \overline{\mathcal{M}}_{g,n+1}$ for non-special points z and extended by unique continuation). However, the fiber over a stable curve $[C] \in \overline{\mathcal{M}}_{g,n}$ is isomorphic to $C/\mathrm{Aut}\, C$ and the map ι is not injective if $\mathrm{Aut}\, C \neq 1$; this means that $\overline{\mathcal{M}}_{g,n}$ is only a "coarse moduli space".

Remark 2.2. Diagram (2.3) also applies to prestable curves in the stable range $2g - 2 + n > 0$ and we further extend it into the unstable range by *formally* defining $\overline{\mathcal{M}}_{g,n}$ to be the topological space $\overline{\mathcal{M}}_{g,3-2g}$ obtained by adding the minimum number of marked points needed to stabilize. Thus we define $\overline{\mathcal{M}}_{0,0}$, $\overline{\mathcal{M}}_{0,1}$ and $\overline{\mathcal{M}}_{0,2}$ all to be $\overline{\mathcal{M}}_{0,3}$, which is a single point, and define $\overline{\mathcal{M}}_{1,0}$ to be $\overline{\mathcal{M}}_{1,1}$.

In the unstable range, these spaces must be used cautiously: for them, the fiber of the universal curve (2.3) is a point and the map ι is only defined modulo $\mathrm{Aut}\, C$, which is a Lie group of positive dimension.

One can also consider families of curves with additional structure. By defining a decorated curve to be a curve C together with a map $\rho : T \to C$ we can include extra marked points, a principal G-bundle, a divisor or log structure, etc. Morphisms and automorphisms of decorated curves are defined by the obvious commutative square. Again, a decorated curve is called stable if its group of automorphisms is finite.

In the examples we will use, decorations have the form $\rho : \tilde{C} \to C$ where $\tilde{C}$ is another curve. For each, there is a moduli space $\overline{\mathcal{M}}^{dec}_{g,n}$ that classifies isomorphism classes of stable pairs (C, ρ) where C is a genus g nodal curve with n marked points. These decorated moduli spaces come with a universal curve $p : \overline{\mathcal{U}}^{dec}_{g,n} \to \overline{\mathcal{M}}^{dec}_{g,n}$ and forgetful maps with the following properties:

(a) $\overline{\mathcal{M}}^{dec}_{g,n}$ and $\overline{\mathcal{U}}^{dec}_{g,n}$ are complex projective orbifolds.

(b) For each stable decorated curve (C,ρ) there is a classifying map ι and a stabilization map st such that $st = p\circ\iota$ whose image is a fiber isomorphic to $(C,\rho)/\mathrm{Aut}(C,\rho)$:

$$\begin{tikzcd}
(C,\rho) \arrow[r, "\iota"] \arrow[dr, "\mathrm{st}"'] & \overline{\mathcal{U}}^{dec}_{g,n} \arrow[d, "p"] \\
& \overline{\mathcal{M}}^{dec}_{g,n}
\end{tikzcd} \tag{2.4}$$

(c) the maps (2.4) are natural with respect to "forget decorations" maps $\widetilde{t}$ and t and "forget marked point" maps p:

$$\begin{tikzcd}
\overline{\mathcal{U}}^{dec}_{g,n} \arrow[r, "\widetilde{t}"] \arrow[d, "p"'] & \overline{\mathcal{U}}_{g,n} \arrow[d, "p"] \\
\overline{\mathcal{M}}^{dec}_{g,n} \arrow[r, "t"] & \overline{\mathcal{M}}_{g,n}
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
\overline{\mathcal{M}}^{dec}_{g,n+1} \arrow[r, "\widetilde{t}"] \arrow[d, "p"'] & \overline{\mathcal{M}}_{g,n+1} \arrow[d, "p"] \\
\overline{\mathcal{M}}^{dec}_{g,n} \arrow[r, "t"] & \overline{\mathcal{M}}_{g,n}
\end{tikzcd} \tag{2.5}$$

Example 2.3. Given a curve C with n marked points $\{x_i\}$, the operation of "adding ℓ additional marked points" can be viewed as the choice of a degree 1 holomorphic map $\rho:\tilde{C}\to C$ where $\tilde{C}$ has $n+\ell$ marked points $\{\tilde{x}_i\}$ with $\rho(\tilde{x}_i)=x_i$ for $i=1,\dots,n$. Then $\overline{\mathcal{M}}^{dec}_{g,n}$ is $\overline{\mathcal{M}}_{g,n+\ell}$.

Example 2.4. Similarly, ρ could be a configuration of ℓ unordered, non-special points on C (i.e. an effective divisor on C); then $\overline{\mathcal{M}}^{dec}_{g,n}$ is the ℓ-fold configuration space $\mathcal{C}^{\ell}_{g,n}\to\overline{\mathcal{M}}_{g,n}$. We could also consider the relative Hilbert scheme $\mathcal{H}ilb^{\ell}\to\overline{\mathcal{M}}_{g,n}$. These two spaces are homeomorphic, but have a different smooth (orbifold) structure; the second is a smooth resolution of the first. (e.g. in the case $C=\mathbb{P}^1$, $\mathcal{H}ilb^{\ell}(\mathbb{P}^1)=\mathbb{P}^d\to\mathrm{Sym}^d\mathbb{P}^1$).

In these examples, (C,ρ) is stable whenever $\tilde{C}$ is, even if C is not stable. Thus decorations can be used to stabilize curves, as was done in Remark 2.2.

Given that automorphisms cause problems, and that decorated curves have fewer automorphisms, one should ask: is there a choice of decorations that completely eliminates automorphisms? More specifically, is there a category, whose objects are families of decorated curves, in which no object has non-trivial automorphisms? The resulting moduli space would then be a *fine* moduli space: every family would be the pullback of a classifying map, unique up to isomorphism, that is an isomorphism on each fiber.

The existence of fine moduli spaces of curves was unknown until the 2003 work of Abramovich, Corti and Vistoli [ACV]. They constructed moduli spaces $\overline{\mathcal{M}}^{G}_{g,n}$ of stable twisted G-covers for finite groups G. These are G-covers $\rho:\widetilde{C}\to C$ (possibly ramified, but only over special points of C) where $\widetilde{C}$ has marked points with stacky structure recording the ramification order of the cover, and that are also "balanced" (satisfy a matching condition at the nodes, cf. [ACV, §2.2 and §4.3]). They also showed that there exists certain finite groups G such that $\overline{\mathcal{M}}^{G}_{g,n}$ is a fine moduli space. Their results are our final example.

Example 2.5. In genus zero $\overline{\mathcal{M}}_{0,n}$ is already a fine moduli space whenever $n\geq 3$. For any finite group G the moduli spaces $\overline{\mathcal{M}}^{G}_{g,n}$ (denoted $\mathcal{B}^{tei}_{g,n}(G)$ in [ACV, Defn 7.6.2]) of stable twisted G covers have the properties (a), (b) and (c) above plus the following extra properties (cf. [ACV]):

(d) There is a natural G-action on $\overline{\mathcal{M}}^{G}_{g,n}$, and the forgetful map $\varphi_G:\overline{\mathcal{M}}^{G}_{g,n}\to\overline{\mathcal{M}}_{g,n}$ is a Galois cover.
(e) The universal curve $\overline{\mathcal{U}}^{G}_{g,n}$ is an open and closed subset of $\overline{\mathcal{M}}^{G}_{g,n+1}$.

(f) For each g there exits a finite group G such that $\overline{\mathcal{M}}_{g,n}^G$ and $\overline{\mathcal{U}}_{g,n}^G$ are smooth complex projective manifolds, $\overline{\mathcal{M}}_{g,n}^G$ is a fine moduli space and $\varphi_G : \overline{\mathcal{M}}_{g,n}^G \to \overline{\mathcal{M}}_{g,n}$ is a finite surjective morphism (cf. [ACV, Thm 7.6.4]).

Note that (f) implies the remarkable fact that *the moduli space* $\overline{\mathcal{M}}_{g,n}$ *is a global quotient orbifold* — the quotient of a compact Kähler manifold by a finite group (this was first proved by Looijenga [L1]). For later use, also note that the forgetful map φ_G induces a map in rational homology

$$\varphi_{G*} : \mathrm{H}_*(\overline{\mathcal{M}}_{g,n}^G; \mathbb{Q}) \longrightarrow \mathrm{H}_*(\overline{\mathcal{M}}_{g,n}; \mathbb{Q})$$

that relates the rational fundamental classes by the formula

$$\varphi_{G*})[\overline{\mathcal{M}}_{g,n}^G] = \deg \varphi_G \cdot [\overline{\mathcal{M}}_{g,n}]. \tag{2.6}$$

3. Moduli spaces of holomorphic maps

Let X be a closed symplectic manifold and let $\mathcal{J}(X)$ denote the space of tame structures (J, ω) on X, consisting of an almost complex structure J and a symplectic form ω, with the C^0 topology. For each pair (J, ω) the moduli space

$$\overline{\mathcal{M}}^J(X)$$

consists of equivalence classes of stable maps $f : C \to X$ from a nodal marked domain C into X that are solutions to the J-holomorphic map equation

$$\overline{\partial}_J f = 0.$$

Such a map is *stable* if the automorphism group $\mathrm{Aut}(f, C)$ is finite, and two such maps are equivalent if they are related by pre-composition by an isomorphism of the domain. In particular, if C is a stable curve then any map $f : C \to X$ is stable. In fact, the stability condition on $f : C \to X$ is topological: there are no unstable domain components that represent 0 in homology. The moduli space has components $\overline{\mathcal{M}}_{A,g,n}(X)$ indexed by the homology class $A = f_*[C] \in H_2(X, \mathbb{Z})$, the arithmetic genus g and number of marked points n of C, and is stratified by the topological type of the domain as explained in the Appendix.

Each map has an energy

$$E(f) = \frac{1}{2} \int_C |df|^2 d\mathrm{vol}_C$$

calculated using the Riemannian metric $g(u, v) = \frac{1}{2}[\omega(u, Jv) + \omega(v, Ju)]$. For J-holomorphic maps the energy is the cohomological pairing

$$E(f) = \omega(f_*[C]) = \omega(A). \tag{3.1}$$

When compactness is important, we will restrict attention to the portion of the moduli space *below energy level* E, written and defined as

$$\overline{\mathcal{M}}^{J,E}(X) \;=\; \bigcup_{A,g,n} \Big\{ f \in \overline{\mathcal{M}}_{A,g,n}^J(X) \,\Big|\, E(f) \leq E \text{ and } 3g - 3 + n \leq E \Big\}.$$

for (J, ω) in $\mathcal{J}(X)$. Occasionally we may need to use a separate upper bound E_1 for $E(f)$ than the bound E_2 for $3g - 3 + n$, in which case $E = (E_1, E_2)$.

As (J,ω) varies in $\mathcal{J}(X)$, the moduli spaces $\overline{\mathcal{M}}^J(X)$ fit together in an universal family $\overline{\mathcal{M}}(X)$ over $\mathcal{J}(X)$. This comes with several natural maps: the projection pr whose fiber at a fixed structure J is the moduli space $\overline{\mathcal{M}}^J(X)$ and, for each (A,g,n), a stabilization-evaluation map se

$$\begin{tikzcd} \overline{\mathcal{M}}_{A,g,n}(X) \arrow[r, "se"] \arrow[d, "\mathrm{pr}"] & \overline{\mathcal{M}}_{g,n} \times X^n \\ \mathcal{J}(X) & \end{tikzcd} \tag{3.2}$$

whose first component takes the equivalence class of $f : C \to X$ to the stabilization $st(C)$ in the Deligne-Mumford space $\overline{\mathcal{M}}_{g,n}$ and whose second component evaluates f at the n marked points (for $2g-2+n \leq 0$ we define $st(C)$ as in Remark 2.2).

To incorporate maps with decorated domains, let $\overline{\mathcal{M}}^{dec}_{A,g,n}(X)$ be the space of equivalence classes $[f, C, \rho, J]$ where $f : C \to X$ is J-holomorphic map in $\overline{\mathcal{M}}_{A,g,n}(X)$ and ρ is a decoration on C. The equivalence relation is given by diagrams

$$\begin{tikzcd} \tilde{C}' \arrow[r, "\cong"'] \arrow[d, "\rho'"] & \tilde{C} \arrow[d, "\rho"] & \\ C' \arrow[r, "\cong"] & C \arrow[r, "f"] & X \end{tikzcd} \tag{3.3}$$

As before, $[f, C, \rho, J]$ is stable if $\mathrm{Aut}(f, C, \rho)$ is finite or, equivalently, if the restriction of (f, C, ρ) to every irreducible component of C either has finite automorphism group or represents a nontrivial homology class in X.

Decorated moduli space come with a forgetful map $\varphi : \overline{\mathcal{M}}^{dec}_{A,g,n}(X) \to \overline{\mathcal{M}}_{A,g,n}(X)$ that forgets the decorations and which is natural with respect to the diagram (3.4), so there are maps

$$\begin{tikzcd} \overline{\mathcal{M}}^{dec}_{A,g,n}(X) \arrow[r, "se"] \arrow[d, "\varphi"] & \overline{\mathcal{M}}^{dec}_{g,n} \times X^n \arrow[d] \\ \overline{\mathcal{M}}_{A,g,n}(X) \arrow[r, "se"] & \overline{\mathcal{M}}_{g,n} \times X^n \end{tikzcd} \tag{3.4}$$

of moduli spaces over $\mathcal{J}(X)$.

3.1. **Graphs maps and Ruan-Tian perturbations.**

Restrict now to maps decorated by a twisted G-cover of their domains as in Example 2.5 (G could be the trivial group), and let $\overline{\mathcal{M}}^G(X) \to \mathcal{J}(X)$ denote their moduli space that fits in the diagram (3.4).

A map $f : C \to X$ whose domain is stable with Aut $(C) = 1$ has a graph

$$F : C \to \overline{\mathcal{U}}_{g,n} \times X$$

defined by $F(z) = (\iota(z), f(z))$ where ι is the map in (2.3) from C to a fiber of the universal curve. More generally, if $f : C \to X$ is a map whose domain is a stable twisted G-cover $\rho : \widetilde{C} \to C$ (possibly with $G = 1$) then the "graph" of f is defined using Diagram (2.4) by

$$F = \iota_\rho \times (f \circ \rho) : \widetilde{C} \to \overline{\mathcal{U}}^G_{g,n} \times X. \tag{3.5}$$

This also extends to the case when $\widetilde{C} \to C$ is not necessarily stable as in Remark 2.2, in which case F factors through the stable model $\mathrm{st}(\widetilde{C} \to C)$. Observe that if $\mathrm{Aut}(C, \rho) = 1$ then ι_ρ, and therefore F, is an embedding.

Next recall that the universal curve $\overline{\mathcal{U}}^G_{g,n}$ is projective; denote it by $\overline{\mathcal{U}}$ and fix an embedding $\overline{\mathcal{U}} \hookrightarrow \mathbb{P}^M$. For each fixed J, let $\mathcal{V}(X)$ be the space sections of the bundle $\mathrm{Hom}^{0,1}(\pi_1^* T\mathbb{P}^M, \pi_2^* TX)$ over

$\mathbb{P}^M \times X$ (i.e such that $J \circ \nu + \nu \circ j = 0$). Each $\nu \in \mathcal{V}(X)$ defines a deformation J_ν of the product almost complex structure on $\overline{\mathcal{U}} \times X$ by writing

$$J_\nu = j \times J \oplus (-\nu \circ j) = \begin{pmatrix} j & 0 \\ -\nu \circ j & J \end{pmatrix} \tag{3.6}$$

Definition 3.1. *Let $\mathcal{JV}(X)$ denote the space of smooth almost complex structures J_ν in the form (3.6); its elements can be written as pairs (J, ν) with $J \in \mathcal{J}(X)$ and ν as above.*

The equation $\overline{\partial}_{J_\nu} F = 0$ on the graph $F : C \to \overline{\mathcal{U}} \times X$ or equivalently

$$\overline{\partial}_J f(z) = \nu(z, f(z)) \tag{3.7}$$

on the map $f : C \to X$ will be called a *Ruan-Tian perturbation* of the J-holomorphic map equation. In the case of G-decorated maps, the corresponding equation on the pair (f, ρ) is

$$\overline{\partial}_J \tilde{f}(z) = \nu(\iota_\rho(z),\ \tilde{f}(z)) \qquad \text{where } \tilde{f} = f \circ \rho. \tag{3.8}$$

Note that the map $(J, \nu) \mapsto J$ gives a fibration with a section $J \mapsto (J, 0)$:

$$\begin{tikzcd} \mathcal{JV}(X) \arrow[r] & \mathcal{J}(X) \arrow[l, bend right] \end{tikzcd} \tag{3.9}$$

There is a corresponding extension of the universal moduli space $\overline{\mathcal{M}}^G(X)$ over $\mathcal{JV}(X)$ with a diagram (3.4) where $\mathcal{J}(X)$ is replaced by $\mathcal{JV}(X)$. Note that $\mathcal{JV}(X)$, and therefore the perturbed pseudo-holomorphic map equation depends on the type of decorations used.

The graph construction allows us to extend the space of allowable deformations (perturbations) of the pseudo-holomorphic map equation in a *very specific way to domain dependent ones*, defined by pullback from $\overline{\mathcal{U}} \times X \hookrightarrow \mathbb{P}^M \times X$ via the graph construction. As we will later see, the graph construction, combined with a fixed embedding $\overline{\mathcal{U}} \hookrightarrow \mathbb{P}^M$ has several other nice analytical consequences.

4. The topology of $\overline{\mathcal{M}}_{g,n}(X)$

The topology of the universal moduli spaces $\overline{\mathcal{M}}_{g,n}(X)$ is constrained by the requirement that the maps used in Gromov-Witten theory be continuous. These constraints lead directly to a metric space topology on moduli space with all the desired properties — see Theorem 4.3. Moreover, this topology is the same as the Gromov topology commonly used in the literature (see the Appendix). Everything in this section applies to the moduli spaces $\overline{\mathcal{M}}^G_{g,n}(X)$, but for simplicity we will often omit the G from the notation.

To topologize $\mathcal{J}(X)$, fix a Riemannian metric on X and use the Sobolev $W^{\ell,p}$ norms with $\ell p \geq \dim X$.

Lemma 4.1. *The space $\mathcal{J}^\ell$ of tame $W^{\ell,p}$ pairs (J, ω) on X and the space $\mathcal{JV}^\ell$ of triples (J, ν, ω) with ν as in Definition 3.1 are smooth separable Banach manifolds.*

Proof. Let V be the Banach space all $W^{\ell,p}$ 2-forms that satisfy $d\omega = 0$ weakly. The equation $J^2 = -Id.$ defines a fiber bundle F that is a submanifold of $End(TX)$. Let $\mathcal{F}^\ell$ be the completion of the space of smooth sections of F in the $W^{\ell,p}$ norm. Because of the Sobolev embedding $W^{\ell,p} \subset C^0$ for $\ell p \geq \dim X$, $\mathcal{F}^\ell$ (or the completion in any norm stronger than C^0) is a smooth Banach manifold [P]. Then $\mathcal{J}^\ell$ is an open set (determined by the tame condition and the non-degeneracy of ω) of the manifold $\mathcal{F}^\ell \times V$ and hence is a manifold. □

To topologize the (universal) moduli space $\overline{\mathcal{M}}_{A,g,n}(X) \to \mathcal{JV}^\ell$ recall that $\overline{\mathcal{M}}_{A,g,n}(X)$ is the set of isomorphism classes of triples $(f, C, \widetilde{J})$ where C is a (not necessarily stable) genus g curve with n marked points, $\widetilde{J}$ denotes a triple $(J, \nu, \omega) \in \mathcal{JV}^\ell$, and $f : C \to X$ is a (J, ν)-holomorphic map. This set comes with several natural maps: (i) a projection π to the space of parameters $\mathcal{J} = \mathcal{JV}^\ell$,

(ii) stabilization-evaluation maps maps se, (iii) the energy function $E(f)$, (iv) forgetful maps φ_k and φ_G that forget the last k marked points or the decorations G and contract those components that become unstable. All of these maps are invariant under reparametrization and therefore descend to the moduli spaces, giving commutative diagrams:

$$\begin{tikzcd} \overline{\mathcal{M}}^G_{A,g,n}(X) \arrow[r, "se"] \arrow[d, "\pi"] & \overline{\mathcal{M}}^G_{A,g,n} \times X^n \\ \mathcal{J} & \end{tikzcd} \qquad \begin{tikzcd} \overline{\mathcal{M}}^G_{A,g,n+k}(X) \arrow[r, "se"] \arrow[d, "\varphi_k"] & \overline{\mathcal{M}}^G_{g,n+k} \times X^{n+k} \arrow[d, "\varphi_k"] \\ \overline{\mathcal{M}}^G_{A,g,n}(X) \arrow[r, "se"] & \overline{\mathcal{M}}^G_{g,n} \times X^n \end{tikzcd} \qquad \begin{tikzcd} \overline{\mathcal{M}}^G_{A,g,n}(X) \arrow[d, "\varphi_G"] \\ \overline{\mathcal{M}}_{A,g,n}(X) \end{tikzcd} \tag{4.1}$$

These maps respect the action of the symmetric group S_n permuting the marked points, and when the domains are twisted G-curves also respect the G action.

The maps in (4.1) induce another important collection of maps. First, each $f \in \overline{\mathcal{M}}^G_{A,g,n}(X)$ is continuous, so its "graph" (cf. (3.5))

$$\Gamma_f = Image(F) = se_0(\varphi_1^{-1}(f)) \subset \overline{\mathcal{M}}^G_{g,n+1} \times X \tag{4.2}$$

is a compact subset of $\overline{\mathcal{M}}^G_{g,n+1} \times X$. Thus $f \mapsto \Gamma_f$ defines a map

$$\Gamma : \overline{\mathcal{M}}^G_{A,g,n}(X) \to Subsets_c(\overline{\mathcal{M}}^G_{g,n+1} \times X) \tag{4.3}$$

where $Subsets_c(Z)$ denotes the set of compact subsets of Z. Similarly, for each $k \geq 2$ there is a "multi-point graph map"

$$\Gamma^k : \overline{\mathcal{M}}^G_{A,g,n}(X) \to Subsets_c(\overline{\mathcal{M}}^G_{g,n+k} \times X^k)$$

defined by $\Gamma^k_f = se(\varphi_k^{-1}(f))$ using (4.1).

Definition 4.2. *A topology on the collection of all moduli spaces*

$$\overline{\mathcal{M}}^G_{g,n}(X)$$

is called natural *if (i) the energy function and all the maps π, se, φ_G, φ_k and Γ^k above are continuous, and (ii) the actions of G and S_n are continuous.*

The following theorem gives the key topological properties of Gromov-Witten moduli spaces, extending Theorem 5.6.6 of [MS]. Recall that a map is *perfect* if it is continuous, surjective, closed and all fibers are compact.

Theorem 4.3. *There is a natural topology on the universal moduli spaces $\overline{\mathcal{M}}^G_{g,n}(X) \to \mathcal{J}\mathcal{V}^\ell(X)$ that is metrizable and for which*

(a) Each map $\pi : \overline{\mathcal{M}}^{G,E}_{g,n}(X) \to \mathcal{J}\mathcal{V}^\ell(X)$ is proper.
(b) The maps φ_k in (4.1) are proper and perfect.

Proof. The proof is given in the appendix. □

Theorem 4.3 shows that the universal moduli space is Hausdorff. Statement (a) is a version of the Gromov Compactness Theorem; it implies that the fiber $\overline{\mathcal{M}}^J_{A,g,n}(X)$ over each $J \in \mathcal{J}(X)$ is compact. The Appendix also contains a proof of the fact (Corollary A.4) that the topology in Theorem 4.3 is, in fact, the usual Gromov topology defined in the literature.

5. The VFC for domain-fine moduli spaces

This section shows how standard results imply the existence of virtual fundamental classes for one very nice class of moduli spaces: the "domain-fine" moduli spaces defined below. These are especially easy to work with because the stabilization map takes their domain isomorphically to a fiber of the universal Deligne-Mumford curve, and because gluing theorems apply without complications.

The following two terms will be used repeatedly in this and later sections.

Definition 5.1. *A stable map $f : C \to X$*

- *is* domain-stable *(ds) if C is a stable curve, and*
- *is* domain-fine *(df) if in addition,* $\mathrm{Aut}\, C = 1$.

A moduli space $\overline{\mathcal{M}}(X)$ is called domain-stable (resp. domain-fine) over $U \subset \mathcal{J}^\ell$ if, for every $J \in U$, each map in $\overline{\mathcal{M}}^J(X)$ is domain-stable (resp. domain-fine).

Both properties are preserved under adding decorations: if $\overline{\mathcal{M}}_{A,g,n}(X; J)$ is domain-fine or domain-stable, then so is $\overline{\mathcal{M}}^G_{A,g,n+k}(X; J)$ for any $k \geq 0$ and any finite group G.

Examples.

1: For the trivial class $A = 0$, the moduli space $\overline{\mathcal{M}}_{0,g,n}(X)$ is domain-stable over all of $\mathcal{JV}^\ell$.

2: For genus $g = 0$, all domain-stable maps are domain-fine.

3: If X is a curve of genus $g \geq 2$, all stable maps $f : C \to X$ are domain-stable (those of positive degree have genus $C \geq 2$, and degree 0 stable maps have stable domains).

Lemma 5.2. *For fixed topological data (A, g, n), the sets*

$$\mathcal{J}^\ell_{ds} = \left\{ J \,|\, \overline{\mathcal{M}}^J_{A,g,n}(X) \text{ is domain-stable} \right\} \qquad\qquad \mathcal{J}^\ell_{df} = \left\{ J \,|\, \overline{\mathcal{M}}^J_{A,g,n}(X) \text{ is domain-fine} \right\}$$

are open in the C^0 topology on $\mathcal{J}^\ell$. The spaces $\mathcal{JV}^\ell_{ds}$ and $\mathcal{JV}^\ell_{df}$, defined similarly, are also open.

Proof. Under Gromov convergence, the order of the automorphism group is upper semi-continuous and limits of unstable domain components are unstable. Thus each *dr* (resp. *df*) map f has a neighborhood with the same property. For $J \in \mathcal{J}^\ell_{ds}$ (or $\mathcal{J}^\ell_{df}$) these open sets cover the moduli space $\overline{\mathcal{M}}^J_{A,g,n}(X)$, and hence by compactness cover the moduli spaces $\pi^{-1}(U)$ for an open neighborhood U of J. The same argument applies to $\mathcal{JV}^\ell_{ds}$ and $\mathcal{JV}^\ell_{df}$. □

The next theorem describes the structure of moduli spaces of domain-fine perturbed J-holomorphic maps for fixed (A, g, n). These facts are well-known; we sketch the proof and refer the reader to [RT1] and [RT2] for details. As in those papers, a parameter $J \in \mathcal{J}^\ell$ is called *regular* for (A, g, n) if the linearization of the J-holomorphic map equation on each stratum with fixed topological type is a surjective operator, for every J-holomorphic map $f \in \overline{\mathcal{M}}^J_{A,g,n}(X)$.

Theorem 5.3. *If $J \in \mathcal{JV}^\ell$ is both domain-fine and regular for (A, g, n) then $\mathcal{M}^J_{A,g,n}(X)$ is an oriented manifold of dimension*

$$\iota = 2c_1(X)A + (\dim_{\mathbb{R}} X - 6)(1 - g) + 2n, \tag{5.1}$$

with a compactification $\overline{\mathcal{M}}^J = \mathcal{M}^J \cup B^J$ whose "boundary" B^J is a finite union of strata, each a manifold of dimension $\leq \iota - 2$.

Furthermore, over a regular path γ in $\mathcal{JV}^\ell_{df}$ the moduli space $\mathcal{M}^\gamma_{A,g,n}(X)$ is an oriented cobordism of dimension $\iota + 1$ with a compactification $\overline{\mathcal{M}}^\gamma = \mathcal{M}^\gamma \cup B^\gamma$ with codimension 2 boundary B^γ. Finally, the set of regular J is open and dense in $\mathcal{JV}^\ell_{df}$, and the set of regular paths is open and dense in the space of paths in $\mathcal{JV}^\ell_{df}$.

Proof. For domain-fine maps, one can use the variation in ν to show that the linearization of the equation (3.7) in (J,ν) is onto (essentially because the graph F of f is an embedding, thus somewhere injective). Standard results then imply that there is a dense set of regular values in $\mathcal{JV}_{df}(X)$ over which each stratum of $\overline{\mathcal{M}}^J_{A,g,n}(X)$ is smooth with dimension equal to the index of the linearization; the top stratum $\mathcal{M}^J_{A,g,n}(X)$, consisting of maps with smooth domains, has dimension (5.1), while the stratum consisting of maps whose domains have k nodes has dimension $2k$ less than (5.1). The top stratum is oriented by the determinant line bundle as in [RT1] or [MS]. The compactness statements follow from Theorem 4.3a. $\square$

The properties of the moduli space listed in Theorem 5.3 imply that, for regular domain-fine J, the compactified moduli space carries a fundamental cycle in rational Čech homology (as stated, the theorem does not provide enough information to obtain a fundamental class in singular homology). The key point is that, in Čech homology, every oriented topological manifold $\mathcal{M}^J$ carries a fundamental class $[\mathcal{M}^J] \in \check{H}_\iota(\mathcal{M}^J, \mathbb{Q})$ (compactness is not needed), and when B^J has codimension 2 as in Theorem 5.3, the long exact sequence for the pair $(\overline{\mathcal{M}}^J, B^J)$ gives an isomorphism

$$\check{H}_\iota(\mathcal{M}^J, \mathbb{Q}) \cong \check{H}_\iota(\overline{\mathcal{M}}^J, \mathbb{Q}).$$

Under this isomorphism, $[\mathcal{M}^J]$ corresponds to a class in $\check{H}_\iota(\overline{\mathcal{M}}^J, \mathbb{Q})$ that we denote $[\overline{\mathcal{M}}^J]$. The existence of this class was noted by Donaldson and Kronheimer [DK, page 357], and a detailed presentation will be given in [**?**].

These facts about Čech homology can be used to define a virtual fundamental class for the moduli spaces $\overline{\mathcal{M}}^J_{A,g,n}(X)$ that are the fibers of

$$\pi : \overline{\mathcal{M}}_{A,g,n}(X) \longrightarrow \mathcal{JV}^\ell_{df} \tag{5.2}$$

over the domain-fine $\mathcal{JV}^\ell_{df}$ as follows. Using Theorem 4.3, the moduli space (5.2) is a space with two metric topologies: one constructed as in the proof of Lemma 4.3 with $\mathcal{J}$ replaced by $\mathcal{JV}^\ell_{df}$, and a "rough topology" obtained by the same construction but using the C^0 distance on $\mathcal{JV}^\ell_{df}$ in formula (A.1). Fix $J_0 \in \mathcal{JV}^\ell_{df}$. By Lemma 5.2 there is a C^0 ball $U \subset \mathcal{JV}^\ell$ containing J_0 that lies in $\mathcal{JV}^\ell_{df}$. Theorem 5.3 then shows that:

- For a dense set of regular (J,ν) in U the moduli space over (J,ν) has a fundamental class
$$[\overline{\mathcal{M}}^{J,\nu}_{A,g,n}(X)] \in \check{H}_\iota(\overline{\mathcal{M}}^{J,\nu}_{A,g,n}(X), \mathbb{Z}). \tag{5.3}$$
- For a dense set of regular paths γ in U with endpoints (J_0,ν_0) and (J_1,ν_1), the images of the maps induced by the inclusions into the moduli space $\overline{\mathcal{M}}^\gamma$ over γ are equal:
$$[\overline{\mathcal{M}}^{J_0,\nu_0}_{A,g,n}(X)] = [\overline{\mathcal{M}}^{J_1,\nu_1}_{A,g,n}(X)] \in \check{H}_\iota(\overline{\mathcal{M}}^\gamma_{A,g,n}(X), \mathbb{Z}). \tag{5.4}$$

Čech homology has a second property that is important for our purposes: it satisfies the continuity axiom:

$$\varprojlim_i \check{H}_*(Y_i;\mathbb{Q}) = \check{H}_*(Y;\mathbb{Q}) \tag{5.5}$$

for any inverse system $\{\cdots \to Y_3 \to Y_2 \to Y_1\}$ of compact metric spaces with limit Y (cf. [Mil]). For computations, it is useful to note that for locally compact Hausdorff spaces, rational Čech homology coincides with the Steenrod homology theory described in [Mil] and in Section 4 of [Ma2]). Taking $\{Y_i\}$ to be a sequence of regular moduli spaces leds to our first theorem about virtual fundamental classes.

Theorem 5.4. *Fix (A,g,n) and $\mathcal{JV}_{df}$ as in Lemma 5.2. The fundamental class (5.3) extends uniquely to a Čech homology class*

$$[\overline{\mathcal{M}}^J_{A,g,n}(X)]^{vir} \in \check{H}_*(\overline{\mathcal{M}}^J_{A,g,n}(X), \mathbb{Q}) \tag{5.6}$$

defined for every domain fine $J \in \mathcal{JV}_{df}$ with the property that for any smooth path γ in $\mathcal{JV}_{df}$ from J_0 to J_1 the images under the maps induced by the inclusion

$$[\overline{\mathcal{M}}_{A,g,n}^{J_0}(X)]^{vir} = [\overline{\mathcal{M}}_{A,g,n}^{J_0}(X)]^{vir} \in \check{H}_*(\overline{\mathcal{M}}_{A,g,n}^{\gamma}(X), \mathbb{Q}) \tag{5.7}$$

are equal. In particular, the GW class

$$GW_{A,g,n}(X) \overset{\text{def}}{=} (\text{st} \times \text{ev})_*[\overline{\mathcal{M}}_{A,g,n}^{J}(X)]^{virt} \in H_*(\overline{\mathcal{M}}_{A,g,n} \times X^n, \mathbb{Q}) \tag{5.8}$$

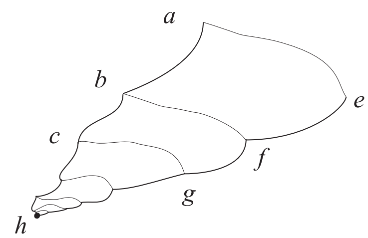
is independent of J on each path-component of $\mathcal{J}_{df}$.

Proof. For simplicity we write J to mean a pair (J, ν). For each $J \in \mathcal{JV}_{df}$, consider the balls $B_k \subset \mathcal{JV}^\ell$ consisting of all J' whose distance from J in the $W^{\ell,p}$ norm is less than $1/k$. By Lemma 5.2, B_k lies in $\mathcal{JV}_{df}$ for large k. Furthermore, each B_k is path connected, contains a dense set of regular values J for which (5.3) holds, and any two regular values are connected by a regular path for which (5.4) holds.

Choose a sequence $J_k \in B_k$ of regular points converging in C^0 to J and regular paths $\gamma_k \subset B_k$ from J_k to J_{k+1}. For each m

$$K_m \;=\; \{J\} \;\cup\; \bigcup_{k \geq m} \gamma_k$$

is compact, and $\overline{\mathcal{M}}_m = \pi^{-1}(K_m) = \overline{\mathcal{M}}_{A,g,n}^{K_m}(X)$ is a sequence of nested compact metric spaces (cf. Theorem 4.3) whose intersection is the compact space $\overline{\mathcal{M}}_{A,g,n}^{J}(X)$.



For each regular value $J_k \in K_m$ the images under the inclusions $\overline{\mathcal{M}}^{J_k}(X) \hookrightarrow \overline{\mathcal{M}}_m$ determine a rational Čech homology class

$$VFC_m \overset{\text{def}}{=} [\overline{\mathcal{M}}_{A,g,n}^{J_k}(X)] \in \check{H}_\iota(\overline{\mathcal{M}}_m, \mathbb{Q}). \tag{5.9}$$

As in (5.3) and (5.4), this class is independent of k, and these homology classes are consistently related by the inclusions $\overline{\mathcal{M}}_{m_1} \hookrightarrow \overline{\mathcal{M}}_{m_2}$ for $m_1 \geq m_2$. The continuity axiom of rational Čech homology then shows that there is a well-defined limit

$$[\overline{\mathcal{M}}^J(X)]^{vir} = \varprojlim_m VFC_m \in \varprojlim_m \check{H}_*(\overline{\mathcal{M}}_m, \mathbb{Q}) = \check{H}_*(\overline{\mathcal{M}}_{A,g,n}^{J}(X), \mathbb{Q})$$

at each point $J \in \mathcal{JV}_{df}$. If K_m' is another such broken path, we can find regular paths between J_k and J_k' inside B_k as shown in the figure. Then (5.4) shows that the two classes VFC_m are compatible in the homology of the moduli space over $K_m \cup K_m' \cup \Gamma_m$.

Relation (5.7) follows the same way by joining broken paths with different limit points in the same path component of $\mathcal{J}_{df}$. Finally, observe that the rational classes (5.8) are locally constant in J, so are constant on path connected components of $\mathcal{J}_{df}$. $\square$

6. The VFC for domain-stable moduli spaces

The construction of Section 5 defines a virtual fundamental class over those elements of $\mathcal{JV}^\ell$ with domain-fine moduli spaces. We next extend the construction to the larger class of domain-stable moduli spaces by decorating domains with twisted G-covers.

As described after (3.2), an element of $\overline{\mathcal{M}}^G(X)$ is an isomorphism class of data (f, C, ρ, J) consisting of a map $f : C \to X$ and a G-twisted curve $\rho : \tilde{C} \to C$. The G-structures define a lifted graph map (3.5) into $\overline{\mathcal{U}}^G \times X \subset \mathbb{P}^M \times X$ and an enlarged space of perturbations $\mathcal{JV}^G(X)$; these perturbations depend on both f and ρ. Over $\mathcal{J}_{ds}(X)$ (but not over $\mathcal{JV}^G_{ds}(X)$!) the map $(f,\rho)\mapsto f$ that forgets the twisted G-cover defines a map t that appeared in (4.1) and that fits into the diagram

$$\begin{tikzcd}
\overline{\mathcal{M}}^G(X) \arrow[r] \arrow[d, "t"'] & \overline{\mathcal{M}}^G(X) \arrow[dd] \\
\overline{\mathcal{M}}(X) \arrow[d] & \\
\mathcal{J}(X) \arrow[r] & \mathcal{JV}^G(X)
\end{tikzcd} \tag{6.1}$$

Equivalently, t is the quotient map for the action of G on $\overline{\mathcal{M}}^G(X)$ by $g[f,\rho]=[f,\rho g^{-1}]$. There is also a G-action on $\mathcal{JV}^G(X)$ induced from the action on universal curve $\overline{\mathcal{U}}^G$, but the righthand vertical map in (6.1) is not G-invariant. In this context, the results of Section 5 hold on the set $\mathcal{JV}^G_{df}(X)$ of $J\in\mathcal{JV}^G(X)$ for which the moduli space $\overline{\mathcal{M}}^{G,J}_{A,g,n}(X)$ is domain-fine:

Lemma 6.1. *Fix (A,g,n) and a finite group G. Then Theorems 5.3 and 5.4 hold for the moduli spaces $\overline{\mathcal{M}}^G_{A,g,n}(X)$ over $\mathcal{JV}^G_{df}(X)$. Hence there is a well-defined virtual fundamental class*

$$[\mathcal{M}^{G,J}_{A,g,n}(X)]^{vir} \tag{6.2}$$

Proof. As described at the end of Section 3, the pair (f,ρ) is (J,ν)-holomorphic if and only if its graph F is J_ν-holomorphic. Because of the domain-fine assumption, $\mathrm{Aut}(C,\rho)=1$ and hence F is embedding into the manifold $\mathbb{P}^M\times X$. McDuff and Salamon show (Chapter 3 of [MS]) that generic perturbations of J_ν on $\overline{\mathcal{U}}^G\times X$ give regularity for somewhere injective maps. From their proof, one sees that it is enough to use perturbations of the form (3.6). As in Theorem 5.3, regular domain-fine moduli spaces carry a virtual fundamental class. The proof of Theorem 5.4 then applies without change.

Note that, in proving regularity, the relevant linearization to the perturbed equation is obtained by varying $f\circ\rho$ through *G-invariant* maps. This variation is determined by the variation in f because the set of equivalence classes of twisted G-covers $\rho:\tilde{C}\to C$ over a fixed C is finite. (The linearization of the J-holomorphic map equation for the map $\tilde{f}=f\circ\rho$ is a different operator with a different index). $\square$

We next verify that the classes (6.2) are consistently defined when one replaces G by a larger group. Given an extension $0\to G\to K\to H\to 0$ of H by G, there is an action of G on the space of K-twisted maps whose quotient induces a map

$$t:\overline{\mathcal{M}}^K_{A,g,n}(X)\to\overline{\mathcal{M}}^H_{A,g,n}(X) \tag{6.3}$$

defined over $\mathcal{J}(X)$ and more generally over $\mathcal{JV}^H(X)$ where $\mathcal{JV}^H(X)\hookrightarrow\mathcal{JV}^K(X)$ is the map $\nu\to t^*\nu$ induced by the map $t:\overline{\mathcal{U}}^K\to\overline{\mathcal{U}}^H$ at the level of universal curves.

Lemma 6.2. *Assume $J\in\mathcal{J}(X)$ is domain-fine for both $\overline{\mathcal{M}}^H_{A,g,n}(X)$ and $\overline{\mathcal{M}}^K_{A,g,n}(X)$. Then the virtual fundamental classes of (6.3) are defined and related by*

$$\frac{1}{|H|}[\overline{\mathcal{M}}^{H,J}_{A,g,n}(X)]^{vir} \;=\; \frac{1}{|K|}\;t_*[\overline{\mathcal{M}}^{K,J}_{A,g,n}(X)]^{vir}\in\check{H}_*(\overline{\mathcal{M}}^{H,J}_{A,g,n}(X),\mathbb{Q}) \tag{6.4}$$

Proof. Since the domain-fine is an open condition, there exists a sufficiently small C^0 neighborhood U of J in $\mathcal{JV}^H(X)$ over which both moduli spaces remain domain-fine. By Lemma 6.1 there is an open dense set of regular $J'=(J,\nu)$ in $U\subset\mathcal{JV}^H_{df}(X)$ and for these the moduli space $\overline{\mathcal{M}}^{H,J'}_{A,g,n}(X)$ is a compact oriented topological manifold. For the pullback J' to $\mathcal{JV}^K(X)$ the moduli space $\overline{\mathcal{M}}^K(X)$ is by assumption also domain-fine, so the G action on it is free and therefore the linearized operator for the space of K-twisted maps (described in the proof of Lemma 6.1) is also regular.

Thus $\overline{\mathcal{M}}^{K,J'}_{A,g,n}(X)$ is also a compact oriented topological manifold and the quotient by the G-action gives the map

$$t : \overline{\mathcal{M}}^{K,J'}_{A,g,n}(X) \to \overline{\mathcal{M}}^{H,J'}_{A,g,n}(X)$$

which has virtual degree $|G| = |K|/|H|$. Thus (6.4) holds for an open dense set of $J' \in \mathcal{JV}^H_{df}(X) \cap \mathcal{JV}^K_{df}(X)$, and hence for all $J \in \mathcal{J}^H_{df}(X) \cap \mathcal{J}^K_{df}(X)$ by the continuity construction of Theorem 5.4. □

Next consider a moduli space $\overline{\mathcal{M}}^J_{A,g,n}(X)$ which is only domain stable. Then $\overline{\mathcal{M}}^{H,J}_{A,g,n}(X)$ is domain stable for any finite group H. Let G be any finite group with property (f) of Example 2.5, and consider the moduli space $\overline{\mathcal{M}}^{H\times G}(X)$ of twisted $H \times G$-structures, which is then domain-fine, and therefore Lemma 6.1 applies to it. Referring to diagram (6.3), define

$$[\overline{\mathcal{M}}^{H,J}_{A,g,n}(X)]^{vir} \;=\; \frac{1}{|G|}\, t_*[\overline{\mathcal{M}}^{H\times G,J}_{A,g,n}(X)]^{vir} \in \check{H}_*(\overline{\mathcal{M}}^{H,J}_{A,g,n}(X),\mathbb{Q}). \tag{6.5}$$

Theorem 6.3. *The moduli spaces $\overline{\mathcal{M}}^H_{A,g,n}(X)$ over the space of domain-stable $\mathcal{J}_{ds}(X)$ admit a virtual fundamental class (6.5) that*

(i) is independent of the G used in its definition,
(ii) is consistent under the quotient map as in (6.4),
(iii) for $H = 1$ it extends the one over the domain-fine $\mathcal{J}_{df}(X)$ defined by Theorem 5.4.

Proof. Define the virtual fundamental class by (6.5). If G_1, G_2 are two finite groups that satisfy condition (f) of Example 2.5, then so does $G_1 \times G_2$. Applying Lemma 6.2 to the diagram $\overline{\mathcal{M}}^{H\times G_1\times G_2}(X) \to \overline{\mathcal{M}}^{H\times G_i}(X) \to \overline{\mathcal{M}}^H(X)$ shows that the virtual fundamental class (6.5) induced by G_1 and G_2 are both equal to the one induced by $G_1 \times G_2$ so (i) holds. Part (ii) follows in a similar fashion from Lemma 6.2. Part (iii) follows by taking $H = 1$ in Lemma 6.2. □

A little more work extends the virtual fundamental class of Theorem 6.3 over the space $\mathcal{JV}^G(X)$ in Diagram 6.1. For each $(J,\nu) \in \mathcal{JV}^G(X)$, G acts on the moduli space $\overline{\mathcal{M}}^{G,J,[\nu]}(X)$ over the orbit $[\nu]$. For a dense set of generic (J,ν), the action of G on (J,ν) will be free and the moduli space $\overline{\mathcal{M}}^{G,J,[\nu]}(X)/G$ will be regular. Applying the limiting argument of Theorem 5.4 then extends the VFC over $\mathcal{JV}^G_{ds}/G$ and defines

$$[\overline{\mathcal{M}}^J_{A,g,n}(X)]^{vir} \;=\; \frac{1}{|G|}\lim_{\nu\to 0}[\overline{\mathcal{M}}^{G,J,[\nu]}_{A,g,n}(X)/G] \in \check{H}_*(\overline{\mathcal{M}}^J_{A,g,n}(X),\mathbb{Q}).$$

for all $J \in \mathcal{J}_{ds}(X)$.

7. Relative moduli spaces

The set of moduli spaces $\overline{\mathcal{M}}^G_{A,g,n}(X)$ is contained in the larger set of moduli spaces relative a divisor $V \subset X$. For $G = 1$, this was done in the [IP1] for smooth divisors, and was recently generalized in [I2] to normal crossing divisors. This section reviews those aspects of the theory that are needed later.

A smooth divisor V is an embedded codimension 2 submanifold of X that is J-holomorphic for some tame pair (J,ω). More generally, a normal crossing divisor V in X is the union of closed immersed codimension 2 submanifolds that are J-holomorphic for some tame pair (J,ω), and are in general position; see Definition 1.3 in [I2] for precise details. For each pair (X,V) there is a stratification of X whose depth $k \geq 0$ strata V^k correspond to points in X where at least k different local branches of V meet. Moreover, for each integer $m \geq 0$ there is an associated deformation space $\mathcal{Z}_m \to D^m$ over the polydisk whose total space $\mathcal{Z}_m$ is a subset of a finite dimensional manifold. For smooth divisors V, the generic fiber of $\mathcal{Z}_m$ is diffeomorphic to X and the central fiber is $X_m = X \cup \mathbb{P}_V \cup \cdots \cup \mathbb{P}_V$, obtained by attaching m copies of the compactified normal bundle $\mathbb{P}_V = \mathbb{P}(\mathbb{C} \oplus NV)$, identifying the zero section of one to the infinity section of the next. For general normal crossing divisors V there is a similar deformation space whose central fiber X_m is a level m "building" constructed from the strata of V. In all cases, there is also a map $\pi_V : X_m \to X$ that collapses all copies of $\mathbb{P}_V$ to V.

The relative moduli spaces are constructed in much the same way as the usual moduli spaces $\overline{\mathcal{M}}_{A,g,n}(X)$ with three main differences (see [IP1] and [I2] for details):

(a) One restricts (J, ω) to be in the subspace $\mathcal{J}^\ell(X, V) \subset \mathcal{J}^\ell(X)$ of triples that satisfy a condition ("V-adapted") on the 1-jet of J along V.

In particular, this implies that V is J-holomorphic, and thus some J-holomorphic maps into X can have components mapped into V.

(b) For each map $f \in \overline{\mathcal{M}}_{A,g,n}(X)$ with no components or nodes mapped to V, one marks all the points in $f^{-1}(V)$ and records their intersection multiplicities s with the various branches of V to obtain an open set $\mathcal{M}_{A,g,n,s}(X, V)$ of maps with extra $\ell(s)$ marked points and contact data s.

(c) For each s, there is a compactification $\overline{\mathcal{M}}_{A,g,n,s}(X, V)$ as described in [I2].

As a set, $\overline{\mathcal{M}}_{A,g,n,s}(X, V)$ consists of equivalence classes of certain type of maps $f : C \to X_m$ where maps are equivalent if they are related by an isomorphism of their domains and the rescaling of their target X_m induced by the $\mathbb{C}^*$ action on each positive level $\mathbb{P}_V$. While all these f are stable as maps into X_m, their projections $\widehat{f} = \pi_X \circ f$ to X may have unstable components that are mapped to a point of V. Each such component, called a *trivial component*, has its image in a fiber of $\mathbb{P}_V$ and its domain is an unstable rational curve with precisely two marked points. When V is smooth, the maps f in the compactification have no nontrivial components in the total divisor of X_m, thus all nontrivial components of f have well-defined contact data to the total divisor and f satisfies a matching condition along the singular divisor of X_m (cf. [IP1]). When V has normal crossings, trivial components of f in the singular divisor cannot be avoided, but then the nontrivial components satisfy an enhanced matching condition (cf. [I2]). This matching can be expressed in terms of the inverse image of a certain diagonal Δ under an evaluation map Ev which fits in the diagram

$$\pi \times se \times \mathrm{Ev} : \overline{\mathcal{M}}_{A,g,n,s}(X) \to \mathcal{J}(X, V) \times \left(\overline{\mathcal{M}}_{g,n+\ell(s)} \times X^n\right) \times N_s V \tag{7.1}$$

and which is a lift of the usual evaluation map. (As described in [I2], Ev keeps track of the multiplicities s as well as of the leading coefficients of f at the extra $\ell(s)$ contact points to V.) Such a map f is called *relatively stable* if its automorphism group is finite, or equivalently if f has at least one nontrivial domain component in each positive level.

The relative moduli spaces come with several functorially-defined maps. There is a diagram

$$\begin{tikzcd} \overline{\mathcal{M}}_{A,g,n,s}(X,V) \arrow[r, "\varphi_V"] \arrow[d, "\pi"] & \overline{\mathcal{M}}_{A,g,n+\ell(s)}(X) \arrow[d, "\pi"] \\ \mathcal{J}(X,V) \arrow[r, hook] & \mathcal{J}(X) \end{tikzcd} \tag{7.2}$$

where φ_V is the map that takes f to the map $\widehat{f}$ obtained from $\pi_X \circ f$ after collapsing all the trivial domain components of its domain:

$$\begin{tikzcd} C \arrow[r, "f"] \arrow[d, "\mathrm{ct}"] & X_m \arrow[d, "\pi_X"] \\ \widehat{C} \arrow[r, "\widehat{f}"] & X \end{tikzcd} \tag{7.3}$$

Here $\widehat{C} = \mathrm{ct}(C)$ is called the *contracted domain* of f. It is different from the stable model $\mathrm{st}(C)$, which is obtained by contracting all unstable domain components, not just the trivial ones.

The normal crossing divisors in X form a directed system, partially ordered by inclusion; the smallest element is the empty divisor. There is a corresponding directed system of moduli spaces. Whenever $V \subset V \cup V'$ there is a diagram generalizing (7.2)

$$\begin{tikzcd} \overline{\mathcal{M}}_{A,g,n,s\cup s'}(X, V\cup V') \arrow[r, "\varphi_{V'}"] \arrow[d, "\pi"'] & \overline{\mathcal{M}}_{A,g,n+\ell',s}(X,V) \arrow[d, "\pi"] \\ \mathcal{J}(X, V\cup V') \arrow[r, hook] & \mathcal{J}(X,V) \end{tikzcd} \tag{7.4}$$

that forgets some components and that is compatible with the map (7.1).

In the special case of Diagram 7.2 when $(X,V) = (\mathbb{P}_V, V_0)$, the projection $\pi : \mathbb{P}_V \to V$ induces a map

$$\begin{tikzcd} \overline{\mathcal{M}}_{A,g,n,s}(\mathbb{P}_V, V) \arrow[r, "\varphi_V"] \arrow[d, "\pi"'] & \overline{\mathcal{M}}_{\pi_* A,g,n+\ell(s)}(V) \arrow[d, "\pi"] \\ \mathcal{J}(\mathbb{P}_V, V) \arrow[r] & \mathcal{J}(V) \end{tikzcd} \tag{7.5}$$

except when $\pi_* A = 0$, $g = 0$ and $n + \ell(s) < 3$. In this context, we enlarge the definition of "trivial component" of f to include unstable genus zero curves whose image under $\widehat{f} : \widehat{C} \to V$ is a constant.

Remark 7.1. As discussed in Remark 2.2, it is convenient to extend the definition of the moduli space $\overline{\mathcal{M}}_{A,g,n}(X)$ into the unstable range $A = 0$, $2g - 2 + n \le 0$ by defining $\overline{\mathcal{M}}_{0,g,n}(X)$ to be the topological space $X \times \overline{\mathcal{M}}_{g,n}$. With this definition, the map (7.5) extends to the unstable range.

Most of the discussion in the previous sections extends to the relative case. The next several paragraphs describe the minor modifications needed.

A. The topology of moduli space. The refined Gromov convergence described in [I2] similarly induces a metrizable topology on the relative moduli space. This topology is the same as the one obtained by regarding the relative moduli space as a subset of $\overline{\mathcal{M}}_{A,g,n+\ell(s)}(\mathcal{Z})$ (with the images of all maps landing in the fibers of $\mathcal{Z}$). A sequence $f_n : C_n \to X$ of relatively stable maps in $\mathcal{M}_s(X,V)$ converge if there exists an m and a sequence of rescaling parameters $\lambda_n \in (\mathbb{C}^*)^m$ of the target such that the rescaled maps $R_{\lambda_n} f_n$, regarded as maps from C_n into the fiber of $\mathcal{Z}$ over λ_n converge to a relatively map $f_0 : C_0 \to X_m$ into the central fiber of $\mathcal{Z}$ over $\lambda = 0$. This now involves a choice of local trivialization/identification of both the domains and targets of the maps away from their singular locus.

With the topology of Lemma 4.1, the bottom map in (7.2) is a continuous injection. We first give the relative moduli space $\overline{\mathcal{M}}_{A,g,n,s}(X,V)$ the topology induced by pullback by the map φ_V in (7.2) from the Gromov topology on the absolute moduli space $\overline{\mathcal{M}}(X)$. The graph maps into $\overline{\mathcal{U}} \times \mathcal{Z}$ similarly induce the metric topology on $\overline{\mathcal{M}}(X,V)$.

B. Domain-stable and Domain-fine. Definition 5.1 extends to relatively stable maps as follows.

Definition 7.2. *A map $f \in \overline{\mathcal{M}}^J_{A,g,n,s}(X,V)$ is* domain-stable *(resp.* domain-fine*) if $\widehat{f}$ is domain-stable (resp. domain-fine).*

With this definition a moduli space $\overline{\mathcal{M}}_{A,g,n,s}(X,V)$ is domain-fine if and only if its image in $\overline{\mathcal{M}}_{A,g,n+\ell(s)}(X)$ under the map (7.2) is domain-fine.

C. Twisted G-structures. In the decorated version of the relative theory, the moduli space $\overline{\mathcal{M}}^G_s(X,V)$ consists of equivalence classes $[f,\rho]$ where $\rho : \widetilde{C} \to C$ is a twisted G-cover and $f : C \to X_m$ is a smooth map in a level m building as before. The equivalence relation is now up to isomorphisms of the twisted G-covers $\rho : \widetilde{C} \to C$ as well as the $(\mathbb{C}^*)^m$ rescaling action on the target X_m. The moduli space $\overline{\mathcal{M}}_s^G(X,V)$ comes with a natural G-action whose quotient $\overline{\mathcal{M}}_s^G(X,V)/G = \overline{\mathcal{M}}_s(X,V)$ is the original undecorated relative moduli space.

D. Ruan-Tian perturbations. In Diagram 7.2, the image of the bottom map may not contain any regular J. Thus we will enlarge $\mathcal{J}^\ell(X,V)$ to space $\mathcal{JV}^\ell(X,V)$ of perturbations using the construction of Section 3.1. Note, however, that this new space $\mathcal{JV}^\ell(X,V)$ does *not* embed in the space $\mathcal{JV}^\ell$ of Section 3.1.

Regarding relatively stable maps as maps into the fibers of $\mathcal{Z}$, there is a graph map $F : \tilde{C} \to \overline{\mathcal{U}} \times \mathcal{Z}$ as in (3.5) with $\overline{\mathcal{U}} = \overline{\mathcal{U}}^G_{g,n+\ell(s)}$. Again, we fix an embedding $\overline{\mathcal{U}} \to \mathbb{P}^M$. Let $\mathcal{JV}^\ell(X,V)$ be the space of tame pairs (J_ν, ω) on $\overline{\mathcal{U}} \times \mathcal{Z}$ where J_ν, generalizing (3.6), preserves the fibers of both the universal curve $\overline{\mathcal{U}}$ and of $\mathcal{Z}$, and also satisfies the V-compatibility condition (see Definition 3.2 of [IP1] and its extension [I2]). Note that the graph map factors through $\widehat{C}$, so trivial components remain trivial under these perturbations.

Diagram (6.1) similarly extends to the relative case.

E. Twisted decorated relative moduli spaces. To describe a smooth model of the relative stable map compactification $\overline{\mathcal{M}}_s(X,V)$ for domain-fine maps, we also need to include a choice of roots of the leading coefficients of f at all the contact points with the total divisor (see [I1] and [I2]). Adding a choice of these roots defines another resolution $\overline{\mathcal{M}}^s(X,V) \to \overline{\mathcal{M}}_s(X,V)$ of the relative moduli space and more generally $\overline{\mathcal{M}}^{G,s}(X,V) \to \overline{\mathcal{M}}^G_s(X,V)$ of the decorated relative moduli spaces for any finite group G. These fit as the top row in the diagram:

$$\begin{tikzcd} \overline{\mathcal{M}}^{G,s}(X,V) \arrow[r, "t"] \arrow[d, "\pi"'] & \overline{\mathcal{M}}^{s}(X,V) \arrow[d, "\pi"'] \\ \overline{\mathcal{M}}^{G}_s(X,V) \arrow[r, "t"] & \overline{\mathcal{M}}_{s}(X,V) \end{tikzcd} \tag{7.6}$$

The moduli spaces in the top row come a natural action of $\boldsymbol{\mu}_s$, the product of the cyclotomic groups of roots of unity at each one of the contact points to V, whose quotient gives rise to the moduli spaces in the bottom row, while the moduli spaces in the left column come with a natural G action whose quotient are the spaces in the right column. Of course, there is also a symmetric group action S_ℓ reordering the contact points to V.

Example 7.3. For a fixed nodal marked curve (C,x), the relative moduli space $\overline{\mathcal{M}}_s(C,x)$ is the space of admissible covers constructed by Mumford-Harris, and while its resolution $\overline{\mathcal{M}}^s(C,x)$ is the moduli space of twisted (balanced) covers constructed by Abramovich-Vistoli. Moreover, when $\mathrm{Aut}(C,x) = 1$, the space of twisted G-covers of (C,x) considered in [ACV] is a particular example of the relative moduli space $\overline{\mathcal{M}}^s(C,x)$ of twisted covers $\rho : \widetilde{C} \to C$.

When $G = 1$ and V is smooth, the boundary strata of $\overline{\mathcal{M}}(X,V)$ consisting of maps into a level 1 building $X_1 = X \cup \mathbb{P}_V$ can then be described in terms of its resolution

$$\overline{\mathcal{M}}^s(X,V) \times_{\mathrm{Ev}} \overline{\mathcal{M}}^s(\mathbb{P}_V, V_\infty \cup V_0) \longrightarrow \overline{\mathcal{M}}(X,V) \tag{7.7}$$

obtained by (a) ordering the marked points corresponding to the nodes along the singular locus $V = V_\infty$, and (b) choosing a root of the leading coefficients of f at these nodes. The local model of the relative moduli space near f is

$$\lambda = a_1 b_1 \mu_1^{s_1} = \cdots = a_\ell b_\ell \mu_\ell^{s_\ell} \tag{7.8}$$

where a_i, b_i are the leading coefficients of f at the node x_i. The zero locus of equation (7.8) is not smooth at the origin when $\ell \geq 2$. One obtains a smooth resolution of (7.8) by a base change after replacing the variables (a_i, b_i) with (α_i, β_i) where $\alpha_i^{s_i} = a_i$ and $\beta_i^{s_i} = b_i$. The antidiagonal $\boldsymbol{\mu}_\ell$ action preserves the $\alpha_i\beta_i$ and gives rise to the balancing condition. There is also a symmetric group action S_ℓ that reorders the ℓ nodes.

With this set-up, the dimension counts and transversality results proved in [IP1] and [I2] give the following analog of Theorem 5.3.

Lemma 7.4. *For a Baire set of $J \in \mathcal{JV}^G_{df}(X,V)$, the relative moduli space $\mathcal{M}^J = \mathcal{M}^{G,s,J}_{A,g,n}(X,V)$ is an oriented manifold of dimension*

$$\iota = 2c_1(X)A - 2V\cdot A + (\dim_{\mathbb{R}} X - 6)(1-g) + 2(n+\ell(s)) \tag{7.9}$$

with a compactification $\overline{\mathcal{M}}^J = \mathcal{M}^J \cup B^J$ whose "boundary" B^J is a finite union of strata, each a manifold of dimension $\leq \iota - 2$. Furthermore, over a regular path γ in $\mathcal{JV}^\ell_{df}$, the moduli space $\mathcal{M}^\gamma$ an oriented cobordism of dimension $\iota+1$ with a compactification $\overline{\mathcal{M}}^\gamma = \mathcal{M}^\gamma \cup B^\gamma$ with codimension 2 boundary B^γ. Finally, the set of regular J is open and dense in $\mathcal{JV}^\ell_{df}(X,V)$, while the set of regular paths is open and dense in the set of paths in $\mathcal{JV}^\ell_{df}(X,V)$.

As in Section 5, Lemma 7.4 is all that is needed to obtain a virtual fundamental class in Čech homology; no knowledge of how the strata fit together is needed.

Theorem 7.5. *The collection of moduli spaces in diagram (7.6) admit compatible VFCs over $\mathcal{J}_{ds}(X,V)$, and for any path between J_0 and J_1 in $\mathcal{J}_{ds}(X,V)$ the VFC over J_0 and J_1 have the same image under the inclusion maps as in (5.6). In particular*

$$[\overline{\mathcal{M}}^J_{A,g,n,s}(X,V)]^{vir} = \frac{1}{|\boldsymbol{\mu}_s||G|}\pi_*[\overline{\mathcal{M}}^{G,s,J}_{A,g,n}(X,V)]^{vir} \in \check{H}_*(\overline{\mathcal{M}}^J_{A,g,n,s}(X,V);\mathbb{Q})$$

extends the one of that appears in (6.5) when $V=\emptyset$.

Remark 7.6. The symmetric group S_ℓ acts on the extra $\ell=\ell(s)$ contact points to V inducing an action on the moduli space $\overline{\mathcal{M}}_s(X,V)$ over $\mathcal{J}(X)$. In the arguments that follow, it will be convenient to also consider the quotient by this S_ℓ action, in which case the resulting relative moduli space

$$\overline{\mathcal{M}}^J_{[s]}(X,V) = \overline{\mathcal{M}}^J_s(X,V)\Big/ S_{\ell(s)} \tag{7.10}$$

will be called the *symmetrized relative moduli space* and the subscript s will change to $[s]$. It has a corresponding VFC

$$[\overline{\mathcal{M}}^J_{A,g,n,[s]}(X,V)]^{vir} = \frac{1}{\ell(s)!}\pi_*[\overline{\mathcal{M}}^J_{A,g,n,[s]}(X,V)]^{vir} \in \check{H}_*(\overline{\mathcal{M}}^J_{A,g,n,[s]}(X,V);\mathbb{Q})$$

defined as the pushforward by the quotient map $\pi : \overline{\mathcal{M}}_s(X,V) \to \overline{\mathcal{M}}_{[s]}(X,V)$ over $\mathcal{J}_{ds}(X)$. With this definition, the symmetrized relative GW invariant corresponds to an unordered sequence of ℓ multiplicities $[s]$ and is equal to

$$GW_{[s]} = \frac{1}{\ell!}GW_s \in H_*(X^n \times (\overline{\mathcal{M}}_{g,n+\ell}\times V^\ell)/S_\ell;\mathbb{Q})$$

The upshot of this discussion is that the symmetrized relative moduli space $\overline{\mathcal{M}}_{[s]}(X,V)$ also has smooth representatives constructed by breaking the symmetry and turning on a generic (non-equivariant) perturbation. To fully break the symmetry, one needs to order the marked points and the components of the domain, add a suitable group G for genus $g\geq 1$, and finally also add the roots of the leading coefficients.

8. Stabilizing divisors

This section introduces the notion of a stabilizing divisor for a pair (X, V) where V is a normal crossing divisor in a closed symplectic manifold X. Stabilizing divisors will be crucial in the next two sections. Here we present some of their important properties and show that Donaldson's Theorem implies that stabilizing divisors exist in abundance on any symplectic manifold.

To start, we fix X, a normal crossing divisor V and a subset B of $\mathcal{J}(X, V)$. To enumerate the strata of moduli spaces, let

$$\mathcal{C}_{E,B}$$

be the collection of all triples (A, g, k) with $A \in H_2(X)$ such that:

(i) $A \neq 0$.
(ii) $E(A, g) = \max\{\omega(A), 3g - 3\} \leq E$ ("energy less than E").
(iii) For some $J \in B$, $\overline{\mathcal{M}}^J_{A,g}(V^k)$ is not empty, i.e. A is represented by a J-holomorphic map from a genus g curve into the stratum V^k of (X, V) with depth $k \geq 0$ (including the top stratum $V^0 = X$).

Definition 8.1. *A smooth codimension 2 submanifold $D \subset X$ is an E-*stabilizing divisor *for (X, V, ω) on B if*

(i) D is transverse to V and there exists $(\omega, J') \in \mathcal{J}(X, V)$ in B such that D is J'- holomorphic.
(ii) D is sufficiently positive in the sense that

$$D \cdot A \geq c_1(\widetilde{V}^k)A + \dim_{\mathbb{C}} V^k + 2g + 1 \qquad \textit{for all } (A, g, k) \in \mathcal{C}_{E,B} \tag{8.1}$$

where $\widetilde{V}^k$ is the smooth resolution of the closed stratum V^k of V (cf. [I2]*).*

If D is a stabilizing divisor for (X, V) we let

$$\mathcal{J}_D(X, V)$$

be the set of all $(J, \omega) \in \mathcal{J}(X, V)$ such that D is J-holomorphic.

Proposition 8.2. *Suppose that D is an E-stabilizing divisor for (X, V) at the point $B = (J_0, \omega_0)$ in $\mathcal{J}(X, V)$. Then there is a C^0 ball U around B in $\mathcal{J}_D(X, V)$ such that for an open dense and path-connected set $\widehat{U}$ of $J \in U$:*

(a) the only J-holomorphic genus g maps into D with $E(A, g) \leq E$ are constant;
(b) the relative moduli spaces $\overline{\mathcal{M}}^J_{A,g,n,s}(X, V \cup D)$ with $E(A, g) \leq E$ are domain-stable;

Statements (a) and (b) remain true in generic 1-parameter families $\{J_t\}$ in U.

Proof. This follows by a standard dimension count argument; note that (a) and (b) are C^0-open conditions on J. For simplicity, we first prove this for the case $V = \emptyset$, in which case (8.1) becomes

$$D \cdot A \geq c_1(X)A + \dim_{\mathbb{C}} X + 2g + 1 \qquad \text{for all } (A, g) \in \mathcal{C}_{E,J_0} \tag{8.2}$$

First notice that (8.2) is a topological condition on what kind of strata appear in the moduli space $\overline{\mathcal{M}}^{J_0}(X)$ below energy level E. By Lemma A.5, there is a C^0-ball U around J_0 with $\mathcal{C}_{E,U} = \mathcal{C}_{E,J_0}$, so the inequality (8.2) holds for all non-trivial J-holomorphic maps with $J \in U$.

Assume next that $\dim D > 0$ (the result is trivially true otherwise). Then below energy level E, the expected dimension of the moduli space $\mathcal{M}_{A,g,0}(D)$ is

$$\begin{aligned}
\dim_{\mathbb{C}} \mathcal{M}_{A,g,0}(D) &= c_1(D)A + (\dim_{\mathbb{C}} D - 3)(1 - g) \\
&= c_1(X)A - DA + 2g - 3 + \dim_{\mathbb{C}} D - (\dim_{\mathbb{C}} D - 1)g \\
&\leq -\dim_{\mathbb{C}} X + \dim_{\mathbb{C}} D - 3 - (\dim_{\mathbb{C}} D - 1)g \;<\; -1
\end{aligned}$$

where for the first inequality we used (8.2) for any $J \in U$. The standard transversality argument at a somewhere injective map [MS] implies that for generic J in U there are no simple smooth J-holomorphic maps into D, and thus no multiple covers either, and this is true in a generic 1-dimensional family of J's. Therefore the only J-holomorphic curves in D below energy level E are constants, and these have stable domains. Furthermore, in the relative moduli space $\mathcal{M}(\mathbb{P}_D, D_\infty \cup D_0)$ we see only multiple covers of the fiber (relative both 0 and ∞) and these all have stable domain except for the trivial covers. Similarly, in $\mathcal{M}(\mathbb{P}_D, D_\infty)$ we see only multiple covers of the fiber relative ∞ only, and these have stable domain unless $g = 0$, $\ell(s) \geq 1$ and $n + \ell(s) \leq 2$, i.e. are trivial components.

Next, by (8.2) the expected dimension d of the relative moduli space $\mathcal{M}_{A,g,n,s}(X, D)$ is

$$\begin{aligned} d &= c_1(X)A + (\dim_{\mathbb{C}} X - 3)(1-g) + n + \ell(s) - D \cdot A \\ &= c_1(X)A - D \cdot A + 2g - 2 + n + \ell(s) + (\dim_{\mathbb{C}} X - 1)(1-g) \\ &\leq -\dim_{\mathbb{C}} X + \ell(s) + n - 3 + (\dim_{\mathbb{C}} X - 1)(1-g) \end{aligned}$$

If the domain is unstable, that is $g \leq 1$ and $\ell(s) + n < 3 - 2g$, this gives

$$\dim_{\mathbb{C}} \mathcal{M}_{A,g,n,s}(X, D) < -1.$$

Thus for generic $J \in U$ (even in a generic path) there are no simple, smooth maps with unstable domain in $\mathcal{M}_s(X, D)$, and therefore there are no multiple covers other than the constants (below energy level E). Recalling the first part of the argument, we conclude that for generic J all maps in $\overline{\mathcal{M}}_s(X, D)$ have stable domains, and the same remains true in generic paths of J's. This proves (b).

Next consider the general case when V is a normal crossing divisor in X. Then the proof follows the same outline as above, except that now V induces a stratification of X: the depth k piece V^k is where at least k branches of V meet (here $k \geq 0$ includes $V^0 = X$); V^k is smooth away from higher depth stratum and comes with a smooth resolution $\widetilde{V}^k \to V^k$. Because we are now restricted to $J \in \mathcal{J}(X, V \cup D)$, to get transversality for the first part of the argument we need to work separately with simple maps into each stratum $V^k \cap D$ which are not contained in any higher depth stratum. Any such J-holomorphic map lifts to a map in the resolution $\widetilde{V}^k$, with image not contained in the higher depth stratum of $\widetilde{V}^k$. Furthermore, if A denotes its homology class, then $c_1(\widetilde{V}^k \cap D)A = c_1(\widetilde{V}^k)A - D \cdot A$. The dimension count then shows that the only maps into D below energy level E are constants. For the second part of the argument, for each smooth, simple map $f \in \overline{\mathcal{M}}(X, V \cup D)$ consider its projection $\widehat{f}$ to $\overline{\mathcal{M}}(X, V)$ under the map that collapses all the levels over D, but not those over V. There are two possibilities for this projection. If $\widehat{f}$ is a map into D, then by the first part of the argument it is constant; thus f is a map into a fiber of $(\mathbb{P}_D, D_0 \cup D_\infty)$; these all have stable domains except for the trivial maps. Otherwise $\widehat{f}$ is a map into one of the strata V^k that does not lie entirely in the higher depth stratum of $V \cup D$; then $\widehat{f}$ has a lift to a map in a relative moduli space $\overline{\mathcal{M}}_s(\widetilde{V}^k, \widetilde{V^k \cap D})$, where $\widetilde{V^k \cap D}$ is the resolution of the lift of the divisor $V^{k+1} \cap D$ to $\widetilde{V}^k$. The dimension of this moduli space is $c_1(\widetilde{V}^k)A - D \cdot A + (\dim_{\mathbb{C}} V^k - 3)(1-g) + \ell(s) + n$ which is similarly negative by (8.1) if the domain is unstable.

Finally, note that (a) and (b) in the statement of the Proposition are both open conditions, thus the subset $\widehat{U}$ of U on which they hold is open and dense set. Moreover, since U is path connected and properties (a) and (b) hold for generic path in U then $\widehat{U}$ is also path connected. $\square$

The arguments in the next section require a controlled existence statement for stabilizing divisors. The following lemma shows that stabilizing divisors exist for a dense set of tame pairs (J, ω) in the spaces $\mathcal{J}(X)$ and $\mathcal{J}(X, V)$. In the statement, (ω_0, J_0) is a compatible pair, but the general elements of the ball B_ε are *tame, but not necessarily compatible pairs.*

Lemma 8.3. *Fix an energy level E and a compatible structure (J_0, ω_0) on X and let B_ε denote C^0-ball around (J_0, ω_0) as above. Then there exists an $\varepsilon_0 > 0$ such that, for any $0 < \varepsilon < \varepsilon_0$,*

(a) There exists an E-stabilizing divisor D for X on $B_\varepsilon \subset \mathcal{J}(X)$.
(b) If V is a normal crossing divisor for (J_0, ω_0), there exists an E-stabilizing divisor D for (X, V) on $B_\varepsilon \cap \mathcal{J}(X, V)$.

(c) *Given* $(J_1, \omega_0), (J_2, \omega_0) \in B_\varepsilon$ *and normal crossing divisors* V_1, V_2 *in* (X, J_i, ω_0)*, there is a path* D_t *of* E*-stabilizing divisors for* X *on* B_ε *whose endpoints* D_i *are* E*-stabilizing divisors for* (X, V_i) *on* $B_\varepsilon \cap \mathcal{J}(X, V_i)$.

Proof. All three statements follow from Donaldson's Theorems [D1] [D2] and its extensions by Auroux [A1]; related statements were also proved in [CM]. We include here an outline of the proof for the specific form stated above.

First, by Lemma A.5 there exits an $\varepsilon_0 > 0$ such that the C^0 closed ball B_0 centered at (J_0, ω_0) and of radius ε_0 satisfies $\mathcal{C}_{B,E} = \mathcal{C}_{J_0,E}$. Furthermore, there is a lower bound $\omega_0(A) \geq 2\alpha_0 > 0$ for all $(A, g, k) \in \mathcal{C}_{J_0,E}$ Thus after shrinking ε_0 we may assume that

$$\omega(A) \geq \alpha_0 > 0 \tag{8.3}$$

for all $(\omega, J) \in B_0$ and (A, g, k) in the finite set $\mathcal{C}_{B,E}$.

Now fix a C^0 ball $B = B_\varepsilon$ about the compatible pair (ω_0, J_0) with $\varepsilon < \varepsilon_0$, sufficiently small so that all $J \in B$ are tamed by all the ω's in B (being a tamed pair is an open condition). Choose a compatible pair (J', ω') in a ball $B' = B_{\varepsilon/4}$ with ω' representing a rational cohomology class. Donaldson's Theorem implies that there exists constants $m_\varepsilon, C_\varepsilon$ such that for any $m \geq m_\varepsilon$, there exists a smooth divisor D_m representing $m\omega'$ and which is $C_\varepsilon m^{-1/2}$-J'-holomorphic. Hence for sufficiently large m:

- there exists an almost complex structure J_m on (X, D_m) such that $|J_m - J'|_{C^0} < \varepsilon/4$ (see Appendix) and therefore $J_m \in B \cap \mathcal{J}(X, D_m)$.
- D_m satisfies (8.2) on B because $D_m \cdot A = m\omega'(A) \geq m\alpha_0$ by (8.3).

Therefore D_m is an E-stabilizing divisor for X on B, as statement (a) asserts.

Next, given a normal crossing divisor V in (X, ω_0, J_0), similarly we can find a compatible pair (J', ω') in $B' = B_{\varepsilon/2}$ such that ω' is rational and J' is adapted to V (so $(\omega', J') \in \mathcal{J}(X, V)$). For any $\eta > 0$ small similarly by the results of Donaldson and Auroux, there exists constants C_ε, m_ε (which depend also on η) such that for each $m \geq m_\varepsilon$ there is a smooth divisor D_m with the following properties: (i) D_m is Poincaré dual to $m\omega'$ (ii) D_m is $C_\varepsilon m^{-1/2}$-J' holomorphic and (iii) D_m is η-transverse to every stratum of V. As above, D_m satisfies (8.2) on B for all large m. Then we can construct a J_m within $\varepsilon/2$ in C^0 from J_0 (cf Appendix) such that J_m is $D_m \cup V$ adapted. Therefore D_m is an E-stabilizing divisor for (X, V) on $B_\varepsilon \cap \mathcal{J}(X, V)$. Note that the construction in the Appendix gives rise to a J_m which is tamed by ω_0 for ε small but perhaps not compatible.

The proof of (c) is similar. For $i = 1, 2$ we can similarly find V_i-adapted, compatible pairs (ω', J'_i) with the same rational ω', and a path (ω', J'_t) of compatible pairs in $B_{\varepsilon/4}$ between them. Applying Donaldson's Theorem to the path (ω', J'_t) shows that, for each $\eta > 0$ small enough, and each large m, there is a family $D_{t,m}$ of smooth divisors with the following properties: (i) they are Poincare dual to $m\omega'$ (ii) $D_{t,m}$ are $C_\varepsilon m^{-1/2}$-J'_t-holomorphic and (iii) at the endpoints $D_{i,m}$ are η-transverse to V_i for $i = 1, 2$. Again, (8.3) also implies that (iv) $D_{t,m}$ satisfy (8.2) on B for all $0 \leq t \leq 1$. Then the construction in the Appendix produces a perturbed path J''_t of almost complex structures on X still within $\varepsilon/2$ of J_0 (thus tamed by ω_0) such that J''_t is adapted to $D_{t,m}$ for all t and moreover at the endpoints J''_i is adapted to $V_i \cup D_{i,m}$. Therefore $D_{t,m}$ is a path of E-stabilizing divisors with the properties stated in (c). $\square$

Remark 8.4. By first deforming ω to a rational form and then using Auroux's generalization [A2] of Donaldson's theorem, we can similarly find stabilizing divisors representing the Poincare dual of $\tau + k\omega$ for any $\tau \in H^2(X, \mathbb{Z})$ and $k > k_\tau$ sufficiently large. The case $\tau = c_1(TX)$ is useful in certain applications.

9. The stabilization procedure

When D is an E-stabilizing divisor for a normal crossing divisor (X, V) it is useful to define a space of "weakly compatible" almost complex structures: we let $\mathcal{J}_D^E(X, V)$ denote the subspace of $\mathcal{J}(X, V)$ consisting of all J such that

$$\begin{aligned}&\text{(a) } D \text{ is } J\text{-complex, and}\\ &\text{(b) all } J\text{-holomorphic maps } f \text{ into } D \text{ with } E(f) \leq E \text{ and } g \leq E \text{ are constant.}\end{aligned} \tag{9.1}$$

Here (b) weakens part of the definition of $V \cup D$-compatibility, the 1-jet condition on J along D in Definition 3.2 of [IP1]. The 1-jet condition was used only to prove that the linearized normal operator D_f^N is complex linear (Lemma 3.3 of [IP1]). But conditions (a) and (b) above also imply complex linearity because D_f^N is complex linear for constant maps. Thus all results in [IP1] hold for weakly compatible J.

For the remainder of this section, fix a normal crossing divisor V in X and a compatible pair (ω_0, J_0) in $\mathcal{J}(X, V)$. Then fix topological data (A, g) and an energy level E with $E(A, g) < E$. We will work under the hypothesis of Theorem 1.5, namely

Assumption 9.1. *Let V be a normal crossing divisor on X. Assume that $A \neq 0$ and that either $g \leq 1$, or X and every strata of the normal crossing divisor V (if non-empty) has dimension at least 12.*

Finally, fix a C^0 ball $B = B_\varepsilon \subset \mathcal{J}(X, V)$ centered at J_0 and small enough that Lemma 8.3 applies.

Let D be an E-stabilizing divisor for (X, V) on B, which exists by Lemma 8.3b. The proof of Theorem 7.5 combines with the discussion above to give the following:

Lemma 9.2. *Assume D be an E-stabilizing divisor for (X, V) on B, and $E(A, g) \leq E$. Then for each $J \in \mathcal{J}_D(X, V) \cap B$ there exists a virtual fundamental cycle*

$$[\overline{\mathcal{M}}^J_{A,g,n,s}(X, V \cup D)]^{vir} \in \check{H}_*(\overline{\mathcal{M}}^J_{A,g,n,s}(X, V \cup D); \mathbb{Q}) \tag{9.2}$$

that is independent of J in the sense that, for any path γ between $J_0, J_1 \in \mathcal{J}_D(X, V) \cap B$, the image under the inclusions are equal:

$$[\overline{\mathcal{M}}^{J_0}_{A,g,n,s}(X, V \cup D)]^{vir} = [\overline{\mathcal{M}}^{J_1}_{A,g,n,s}(X, V \cup D)]^{vir} \in \check{H}_*(\overline{\mathcal{M}}^{\gamma}_{A,g,n,s}(X, V \cup D); \mathbb{Q}) \tag{9.3}$$

Proof. Applying Proposition 8.2 at each point of the non-empty path-connected set $\mathcal{J}_D(X, V) \cap B$ and taking the union shows that there is an open, dense path-connected subset $\hat{U}$ of $\mathcal{J}_D(X, V) \cap B$ so that

(i) $\hat{U} \subset \mathcal{J}(X, V; D)$ and
(ii) $\overline{\mathcal{M}}_{A,g,n,s}(X, V \cup D)$ is domain-stable for all n, s and all $J \in \hat{U}$.

Theorem 7.5 then gives a well-defined VFC on $\widehat{U}$. This then extends by continuity to the entire $\mathcal{J}_D(X, V) \cap B$ with the fact that $\widehat{U}$ is open, dense and path connected subset of $\mathcal{J}_D(X, V) \cap B$, and that any path in $\mathcal{J}_D(X, V) \cap B$ can be approximated by a path in $\widehat{U}$ (the set of paths in $\widehat{U}$ are dense in the set of all paths in $\mathcal{J}_D(X, V) \cap B$). □

Now consider the component of the moduli space in (9.2) with $s' = s \cup [1]$, that is, the compactification of the space of stable maps that intersect V with multiplicity vector s and intersect D at $A \cdot D$ unordered points each with multiplicity 1. As in Diagrams (4.1) and (7.2) there is a projection

$$\begin{array}{c}\overline{\mathcal{M}}_{A,g,n,s'}(X, V \cup D)\\ \Big\downarrow{\scriptstyle \varphi_D}\\ \overline{\mathcal{M}}_{A,g,n,s}(X, V)\end{array} \tag{9.4}$$

where φ_D is the composition of the map in diagram (7.4) that forgets the branch D with the map that also forgets the marking of the contact points to D.

Definition 9.3. *Assume (X,V) satisfies Assumption 9.1. If D is any E-stabilizing divisor for (X,V) on B, for any $J \in \mathcal{J}_D(X,V)\cap B$ the image of the homology class (9.2) under φ_D defines a class*

$$VFC_D^J \overset{def}{=} \varphi_{D*}[\overline{\mathcal{M}}^J_{A,g,n,s\cup[1]}(X,V\cup D)]^{vir} \in \check{H}_*(\overline{\mathcal{M}}^J_{A,g,n,s}(X,V);\mathbb{Q})$$

Here [1] means that we first divide by the symmetric group action reordering the contact points to D as described in Remark 7.6.

As defined VFC_D^J depends on the choice of D (as well as V and A,g,s which are assumed fixed in the discussion below). The remainder of this section is devoted to showing that VFC_D^J is independent of D in an appropriate sense.

The next step is to prove that these virtual fundamental classes are consistent for different choices of D. We start with a proposition whose proof is based on a separate, independent result proved in Section 11.

Proposition 9.4. *If D and D' are E-stabilizing divisors for V on B then for any $J \in B\cap\mathcal{J}_{D\cup D'}(X,V)$ (if one exists)*

$$VFC_D^J = VFC_{D'}^J \in \check{H}_*(\overline{\mathcal{M}}^J_{A,g,n,s}(X,V),\mathbb{Q}). \tag{9.5}$$

Proof. By a slight modification of Proposition 8.2, there is an open dense and path connected subset $\widehat{U}$ of $B\cap\mathcal{J}_{D\cup D'}(X,V)$ on which

(a) the moduli spaces $\overline{\mathcal{M}}^J_{A,g,n,s\cup[1]}(X,V\cup D)$, $\overline{\mathcal{M}}^J_{A,g,n,s\cup[1]}(X,V\cup D')$ and $\overline{\mathcal{M}}^J_{A,g,n,s\cup[1]}(X,V\cup D\cup D')$ are all domain-stable and
(b) there are no J-holomorphic maps into D or D' other than constants with energy less or equal to $E(A,g)$.

Proposition 11.2 below shows that these conditions, together with Assumption 9.1, imply that

$$VFC_D^J = VFC_{D\cup D'}^J = VFC_{D'}^J$$

for all $J\in\widehat{U}$. This equality extends by continuity to all $J\in B\cap\mathcal{J}_{D\cup D'}(X,V)$. $\square$

Theorem 9.5. *Assume D and D' are E-stabilizing divisors for (X,V) on B. Then for any $J \in B\cap\mathcal{J}_D(X,V)$ and $J'\in B\cap\mathcal{J}_{D'}(X,V)$ there exists a path γ in $B\cap\mathcal{J}(X,V)$ from J to J' such that*

$$VFC_D^J = VFC_{D'}^{J'} \in \check{H}_*(\overline{\mathcal{M}}^\gamma_{A,g,n,s}(X,V),\mathbb{Q})$$

after inclusion.

Proof. For simplicity, set $\mathcal{J}=\mathcal{J}(X,V)$. Use Lemma 8.3 (c) to find a path D_t of E-stabilizing divisors on B for (X,V) and whose endpoints are E-stabilizing divisors for $(X,V\cup D)$ and respectively $(X,V\cup D')$ on B thus $B\cap\mathcal{J}_{D\cup D_0}\neq\emptyset$ and $B\cap\mathcal{J}_{D'\cup D_1}\neq\emptyset$. Since $B\cap(\cup_t\mathcal{J}_{D_t})$ is path connected, we can therefore find a path $\gamma_0=\{J_t\}$ such that $J_t\in B\cap\mathcal{J}_{D_t}$ and with endpoints $J_0\in B\cap\mathcal{J}_{D\cup D_0}$ and $J_1\in B\cap\mathcal{J}_{D\cup D_1}$. The $VFC_{D_t}^{J_t}$ is well defined and becomes constant in t after inclusion in $\check{H}_*(\overline{\mathcal{M}}^{\gamma_0}(X),\mathbb{Q})$

Moreover, since the endpoints $J_0\in B\cap\mathcal{J}_{D\cup D_0}$ and $J_1\in B\cap\mathcal{J}_{D'\cup D_1}$ then Proposition 9.4 applies to give

$$VFC_D^{J_0} = VFC_{D\cup D_0}^{J_0} = VFC_{D_0}^{J_0} \qquad VFC_{D'}^{J_1} = VFC_{D'\cup D_1}^{J_1} = VFC_{D_1}^{J_1}$$

Next, $B\cap\mathcal{J}_D$ is path connected, so we can find a path γ_1 in $B\cap\mathcal{J}_D$ from J to J_0 on which the VFC_D is well defined and moreover

$$VFC_D^{J'} = VFC_D^{J_0}$$

after inclusion in $\check{H}_*(\overline{\mathcal{M}}^{\gamma_1}(X),\mathbb{Q})$. Similarly, we can find a path γ_2 in $B\cap\mathcal{J}_{D'}$ from J_1 to J' such that after inclusion in $\check{H}_*(\overline{\mathcal{M}}^{\gamma_2}(X),\mathbb{Q})$

$$VFC_D^{J_1} = VFC_D^{J'}$$

Now let γ denote the concatenation of the path $\gamma_1\#\gamma_0\#\gamma_2$, which is a path in $B\cap\mathcal{J}(X,V)$ from J to J'. After inclusion into $\check{H}(\overline{\mathcal{M}}^{\gamma}(X),\mathbb{Q})$, combining the last three displayed equation above then gives the desired equality

$$VFC_D^J = VFC_D^{J_0} = VFC_{D_0}^{J_0} = VFC_{D_1}^{J_1} = VFC_{D'}^{J_1} = VFC_{D'}^{J'}$$

in $\check{H}(\overline{\mathcal{M}}^{\gamma}(X),\mathbb{Q})$. $\square$

10. Defining the VFC via stabilizing divisors

We can now, at last, complete the proof of Theorem 1.5 by defining a virtual fundamental class for any pair (X,V). The context remains the same: X is a closed symplectic manifold and $V\subset X$ is a (possibly empty) normal crossing divisor for some compatible structure (ω,J,g) on X. We will also fix an energy level E and work with data (A,g) below energy E. Finally, we assume that (X,V) satisfies Assumption 9.1.

Proposition 10.1. *Let V and (X,ω,J) be as above. Then as long as $E(A,g)\le E$:*

(a) For any shrinking sequence B_i of C^0 balls in $\mathcal{J}(X,V)$ centered at (ω,J) there exists stabilizing divisors D_i for (X,V) on B_i and a sequence $\varepsilon_i\to 0$ such that

(i) for generic $(\omega,J_i,\nu_i)\in\mathcal{JV}_{D_i}(X,V)$ with $(\omega,J_i)\in B_i$ and $|\nu_i|<\varepsilon_i$, the symmetrized relative moduli spaces

$$\overline{\mathcal{M}}^{J_i,\nu_i}_{A,g,n,s_V\cup[1]}(X,V\cup D_i)$$

defined as in (7.10)*, are domain-stable and carry fundamental classes in Čech homology.*

(ii) After forgetting D_i, their actual fundamental cycles pass to the limit to define a rational Čech homology element

$$\lim_{i\to\infty}\varphi_{D_i*}[\overline{\mathcal{M}}^{J_i,\nu_i}_{A,g,n,s_V\cup[1]}(X,V\cup D_i)]\in\check{H}_*(\overline{\mathcal{M}}^{J}_{A,g,n,s_V}(X,V);\ \mathbb{Q}).\tag{10.1}$$

(b) The class (10.1) is independent of the choices made in (a) and is invariant under deformations of the compatible pair (ω,J) inside $\mathcal{J}(X,V)$.

Proof. For this argument we are working in the neighborhood of a fixed moduli space $\overline{\mathcal{M}}^J(X,V)=\overline{\mathcal{M}}^J_{A,g,n,s}(X,V)$; in particular A,g,n,s will be fixed, where $E(A,g)\le E$.

Consider the sequence of C^0-balls $B_n=B(J,\frac{1}{n})$. For each n, by Lemma 8.3 (b) there exits an E-stabilizing divisor D_n for (X,V) on B_n thus $\mathcal{J}_{D_n}(X,V)\cap B_n\neq\emptyset$. Pick $J_n\in\mathcal{J}_{D_n}(X,V)\cap B_n$ and consider the class $VFC_{D_n}^{J_n}$ defined in Definition 9.3.

Now use Theorem 9.5 to construct paths γ_n in B_n from J_n to J_{n+1}, and let K_m be the union of J_0 with the paths γ_n for all $n\ge m$, as in the proof of Theorem 5.4. Then under the inclusion $K_n\hookrightarrow K_m$:

$$VFC_{D_n}^{J_n}=VFC_{D_m}^{J_m}\in\check{H}_*(M_m,\mathbb{Q})$$

for all $n\ge m$, where M_m is the moduli space $\overline{\mathcal{M}}(X,V)$ over K_m. This means that the VFC passes to the limit to give a class in $\check{H}_*(\overline{\mathcal{M}}^J(X,V),\mathbb{Q})$. Independence of the sequence D_n,J_n is proven by the ladder argument: if (D_n,J_n) and (D_n',J_n') are two such sequences, Theorem 9.5 provides paths δ_n in B_m that together with the original paths form the ladder.

This shows that (10.1) is well defined, independent of choices made. This class is invariant under deformations of (ω,J) in the sense that for any smooth path γ of compatible pairs $(\omega_t,J_t)\in\mathcal{J}(X,V)$ the two VFC have equal inclusions into the Čech homology of the moduli space over the path:

$$VFC_{J_0}=VFC_{J_1}\in\check{H}_*(\overline{\mathcal{M}}^B_{A,g,n,s}(X,V);\mathbb{Q})$$

This follows from a similar continuity argument using the fact that γ is compact, by taking B_n a finite cover of it with balls of radius $1/n$ centered at points in γ. $\square$

With (X,V) and $E\geq E(A,g)$ as above, consider a shrinking sequence B_i of C^0 balls around (ω,J) in $\mathcal{J}(X,V)$, and pick E-stabilizing divisors D_i for (X,V) on B_i.

Definition 10.2. The virtual fundamental class *of* $\overline{\mathcal{M}}^J(X,V)$ *is the rational Čech homology class*

$$[\overline{\mathcal{M}}^J_{A,g,n,s}(X,V)]^{vir} \overset{def}{=} \lim_{i\to\infty}\varphi_{D_i*}[\overline{\mathcal{M}}_{A,g,n,s\cup[1]}(X,V\cup D_i)]^{vir}\in \check{H}_*(\overline{\mathcal{M}}^J(X,V);\mathbb{Q}) \tag{10.2}$$

and the corresponding relative GW invariant *is*

$$GW_{A,g,n,s}(X,V)=se_*[\overline{\mathcal{M}}^J_{A,g,n,s}(X,V)]^{vir}\in H_*(\overline{\mathcal{M}}_{g,n+\ell(s)}\times N_sV) \tag{10.3}$$

By Proposition 10.2b, the virtual fundamental class is well defined, independent of the choices made in its construction. Thus we have completed the proof of Theorem 1.5 of the introduction. In particular, in terms of GW invariants, we have the following:

Corollary 10.3. *The relative invariant (10.3) is well-defined and invariant under deformations. When X is semi-positive and V is empty, it agrees with the invariant $GW(X)$ defined in* [RT2]*, and when V is smooth it agrees with the relative invariant of $GW(X,V)$ defined in* [IP1].

11. Independence of the stabilizing divisor

It remains to show that the VFC of Definition 9.3 is independent of the stabilizing divisor D, and agrees with the VFC defined by Theorem 7.5 for domain-stable moduli spaces. These two facts follow from Proposition 11.3 and Proposition 11.1, respectively.

Fix a homology class $A\neq 0$ and a genus g and set $E=E(A,g)$. Also fix a normal crossing divisor V (possibly empty), and an E-stabilizing divisor D for V. Recall from Proposition 8.2 that there is a non-empty set of J in $\mathcal{J}_D(X,V)$ for which there are no non-constant J-holomorphic maps into D with energy less than or equal to E, and the compactified moduli space $\overline{\mathcal{M}}^J_{A,g,s\cup[\mathbf{1}]}(X,V\cup D)$ is domain-stable. For each such J, under Assumption 9.1, the following proposition applies, showing that the virtual fundamental classes of the stabilized and unstabilized moduli spaces agree whenever the unstabilized moduli space is already domain-stable.

Proposition 11.1. *Fix $A\neq 0$ and a normal crossing divisor V. Assume that D is a smooth divisor transverse to V, and that $J\in\mathcal{J}_D(X,V)$ is such that*

(a) the moduli space $\overline{\mathcal{M}}^J_{A,g,n,s_V}(X,V)$ is domain-stable,
(b) there are no non-constant J-holomorphic maps into D with energy less or equal to $E(A,g)$, and
(c) either $g\leq 1$ or every stratum of $V\cup D$ is at least 8 dimensional.

Then for each $J\in\mathcal{J}_D(X,V)$ is in the set described in Proposition 8.2

$$[\overline{\mathcal{M}}^J_{A,g,n,s_V}(X,V)]^{vir}=\varphi_{D*}[\overline{\mathcal{M}}^J_{A,g,n,s_V\cup[\mathbf{1}]}(X,V\cup D)]^{vir}\in\check{H}_*(\overline{\mathcal{M}}^J_{A,g,n,s_V}(X,V);\mathbb{Q}) \tag{11.1}$$

where φ_{D*} is the map in diagram (9.4).

Proof. First consider the case where $V=\emptyset$. Because D is J-holomorphic, the moduli space $\overline{\mathcal{M}}^J_{A,g,n}(X)$ of stable maps has a refined stratification. As described in the appendix, the strata are indexed by decorated graphs, whose vertices, which correspond to irreducible component C_i of C, are labeled by a genus, number of marked points, and the homology class $A_i=[f(C_i)]$; these are refined by also including labels indicating which C_i are mapped into D, and, on the components not mapped to D, labels recording their contact multiplicities to D.

The *principal stratum* $\mathcal{N}_2$ is the open subset of $\mathcal{M}^J_{A,g,n}(X)$ consisting of maps $f : C \to X$ with smooth domain that are transverse to D and such that $f^{-1}(D)$ is disjoint from the special points of C. By marking all points in $f^{-1}(D)$, one obtains a subset $\tilde{\mathcal{N}}$ inside the relative moduli space $\mathcal{M}^J_{A,g,n,(\mathbf{1})}(X,D)$ with a forgetful map $\tilde{\mathcal{N}} \to \mathcal{N}_2$. The symmetric group S_ℓ acts by permuting the added ℓ marked points, and $\mathcal{N}_2$ is homeomorphic to the subset $\mathcal{N}_1 = \tilde{\mathcal{N}}/S_\ell$, which is the principal stratum of the quotient space $\overline{\mathcal{M}}^J_{A,g,n,[\mathbf{1}]}(X,D)$ defined by (7.10).

The same stratification scheme applies to the moduli spaces obtained by replacing J by (J,ν), where ν is any perturbation whose normal component to D vanishes along D, and that is invariant under the S_ℓ action. These conditions are satisfied if ν is pulled back from a Ruan-Tian perturbation on $\overline{\mathcal{U}}^G_{g,n} \times X$ whose normal component vanishes on $\overline{\mathcal{U}}^G_{g,n} \times D$. For every such pair (J,ν), the map that forgets the divisor D and the extra ℓ points (and collapses any unstable domain components that are mapped to points) is a continuous map

$$\varphi_D : \overline{\mathcal{M}}^{J,\nu}_{A,g,n,[\mathbf{1}]}(X,D) \to \overline{\mathcal{M}}^{J,\nu}_{A,g,n}(X) \tag{11.2}$$

that restricts to a homeomorphism from $\mathcal{N}_1$ to $\mathcal{N}_2$. The spaces in (11.2) can be regarded as compactifications of $\mathcal{N}_1$ and $\mathcal{N}_2$ respectively. So it suffices to show that there exists a sequence of parameters ν converging to zero such that:

(i) All maps in the principal strata $\mathcal{N}_1$ and $\mathcal{N}_2$ of the spaces in 11.2 are regular, and hence $\mathcal{N}_1$ and $\mathcal{N}_2$ are oriented manifolds of dimension ι.

(ii) All the other strata of each compactification are manifolds of dimension at most $\iota - 2$.

(iii) The restriction of φ_D to the principal stratum $\mathcal{N}_1$ preserves orientation.

It then follows that the Čech homology map induced by (11.2) takes the virtual fundamental class of the right-hand space to the virtual fundamental class of the left-hand space (see [**?**] for details).

But facts (i)-(ii) are precisely what Tehrani and Zinger proved in [TZ, §3.1]. Their key observation is that one can use condition (c) and a dimension count to show that, for certain (J,ν), the compactified moduli space contains no maps that have genus $g \le 2$ components mapped into D. The dimension counts in [TZ, §3.1] imply that under assumptions (b) and (c), the actual dimension of each stratum except the principal stratum is still at most $\iota - 2$.

The proof involves considering the restrictions of maps $f : C \to X$ to those irreducible components C_i mapped into D. Note that, for each (J,ν) as above, the regularity of $f_i : C_i \to D$ as a map into D is not the same as the regularity of f_i as a map into X. The difference is captured by the normal component of the linearization, whose index, under assumption (b), equals $2(1-g)$, which is non-negative for $g = 0,1$. From this fact and dimension counts, Tehrani and Zinger establish (ii) above. We refer the reader to the proof of Theorem 1 in [TZ, §3.1] for the required transversality and orientation arguments, including extensions without assumption (a).

The argument above extends to the case when V is a normal crossing divisor, now using the stratification of X induced by V as in the proof of Proposition 8.2. In fact, under assumptions (a)-(c), one does not even need to consider the sections of the normal bundle to D that arise from renormalization in order to construct a compactification of $\mathcal{N}_1 = \mathcal{M}_{A,g,n,s_V,[\mathbf{1}]}(X, V \cup D)$ with codimension 2 boundary strata. On the other hand, as above, $\mathcal{N}_1$ is identified with the principal stratum $\mathcal{N}_2$ of $\mathcal{M}_{A,g,n,s_V}(X,V)$. Again, dimension counts extending those in [TZ, §3.1] show that, under assumptions (a)-(c), all strata in the boundary $\overline{\mathcal{M}}_{A,g,n,s_V}(X,V) \setminus \mathcal{N}_2$ have codimension at least 2. □

Remark 11.2. Without assumption (c), the two virtual fundamental cycles cannot be related using dimension counts alone. In general, an analysis of (11.2) shows that

$$[\overline{\mathcal{M}}_{A,g,n,s}(X,V)]^{vir} \quad = \quad \varphi_{D*}[\overline{\mathcal{M}}_{A,g,n,s,[1]}(X, V \cup D)]^{vir} + \text{ correction terms,}$$

where the correction terms come from the contribution of strata of $\overline{\mathcal{M}}(X)$ consisting of maps with components in D whose dimension is equal to, or higher than, the dimension of the virtual fundamental class. There are several ways to calculate these contributions, but all involve gluing, either using the trivial decomposition of X as the union of (X, D) and the projectivized normal bundle $\mathbb{P}_D$ (with D identified with the infinity section of $\mathbb{P}_D$), or alternatively using an obstruction bundle gluing argument as in [HT].

Proposition 11.3. *Assume V is a normal crossing divisor in (X, ω, J), while D and D' are two smooth J-holomorphic divisors such that V, D, D' are in general position. Fix $A \neq 0$ and assume that*

(a) the moduli space $\overline{\mathcal{M}}^J_{A,g,n,s\cup[\mathbf{1}]}(X, V \cup D)$ is domain stable, and

(b) there are no non-constant J-holomorphic maps into D or D' with energy less or equal to $E(A, g)$.

(c) either $g \leq 1$ or every stratum of $V \cup D \cup D'$ is at least 8 dimensional.

Then $\overline{\mathcal{M}}^J_{A,g,n,s\cup[\mathbf{1}]\cup[\mathbf{1}]}(X, V \cup D \cup D')$ is also domain stable, and under the forgetful map

$$\varphi_{D'} : \overline{\mathcal{M}}^J_{A,g,n,s\cup[\mathbf{1}]\cup[\mathbf{1}]}(X, V \cup D \cup D') \to \mathcal{M}^J_{A,g,n,s\cup[\mathbf{1}]}(X, V \cup D)$$

defined by (9.4) *with V replaced by $V \cup D$ and D by D', we have the equality*

$$\varphi_{D'*}[\overline{\mathcal{M}}^J_{A,g,n,s,\cup[\mathbf{1}]\cup[\mathbf{1}]}(X, V \cup D \cup D')]^{vir} = [\overline{\mathcal{M}}^J_{A,g,n,s\cup[\mathbf{1}]}(X, V \cup D)]^{vir} \tag{11.3}$$

as elements of $\check{H}_*(\overline{\mathcal{M}}^J_{A,g,n,s_V\cup[\mathbf{1}]}(X, V \cup D); \mathbb{Q})$.

Proof. The assumptions immediately imply the first assertion because any constant components in D' or $D \cap D'$ must have stable domains. Therefore each of the moduli spaces that appear in (11.3) has a well-defined virtual fundamental class, obtained by using Ruan-Tian perturbations. However, the Ruan-Tian perturbations needed to achieve transversality are *a priori* different for these two moduli spaces. To relate the two virtual fundamental classes, we must restrict to a common class of perturbations for which the forgetful map

$$\varphi_{D'} : \overline{\mathcal{M}}^{J,\nu}_{A,g,n,s\cup[1]}(X, V \cup D \cup D') \to \overline{\mathcal{M}}^{J,\nu}_{A,g,n,s}(X, V \cup D)$$

is well-defined and continuous. This is the case, for example, when we restrict to the subset of $(J, \nu) \in \mathcal{JV}_D(X, V)$ for which the normal component to D' of the perturbation ν vanishes. It is straightforward to verify that this subset is a non-empty Banach submanifold of $\mathcal{J}_D(X, V)$.

It then suffices to find a sequence of such perturbations ν converging to zero, and open subsets

$$\mathcal{N}_1 \subseteq \mathcal{M}^{J,\nu}_{A,g,n,s\cup[1]}(X, V \cup D \cup D') \quad \text{and } \mathcal{N}_2 \subseteq \mathcal{M}^{J,\nu}_{A,g,n,s}(X, V \cup D)$$

with the following properties:

(i) each $\mathcal{N}_1$ and $\mathcal{N}_2$ consist only of regular points of their corresponding moduli space, and therefore are oriented manifolds of dimension ι.

(ii) φ restricts to an orientation preserving homeomorphism between $\mathcal{N}_1$ and $\mathcal{N}_2$

(iii) the complement of $\mathcal{N}_1$ in $\overline{\mathcal{M}}^{J,\nu}_{A,g,n,s\cup[1]}(X, V \cup D \cup D')$ is a set of dimension $\iota - 2$.

(iv) the complement of $\mathcal{N}_2$ in $\overline{\mathcal{M}}^{J,\nu}_{A,g,n,s\cup[1]}(X, V \cup D)$ is a set of dimension $\iota - 2$.

As above, the sets $\mathcal{N}_1$ and $\mathcal{N}_2$ can be chosen to be the principal strata of the refined stratifications of the two moduli spaces, corresponding to maps $f : C \to X$ with smooth domain, transverse to D and D', not entirely contained in $V \cup D \cup D'$ and missing the singular locus of $V \cup D \cup D'$. The only difference between $\mathcal{N}_1$ and $\mathcal{N}_2$ is that for $\mathcal{N}_1$ all the points in $f^{-1}(V \cup D \cup D')$ are already marked (and decorated with the multiplicity of contact), while for $\mathcal{N}_2$ only the points in $f^{-1}(V \cup D)$ are marked and decorated.

The considerations of [TZ, §3.1] now easily extend. Again, the key observation is that condition (c) implies that all irreducible components that are mapped to D, D' or $D \cap D'$ have genus $g \leq 1$. For these, the index of the normal operator is nonnegative by (b), and therefore the actual dimension of each stratum of either compactification, except the principal stratum, is again at most $\iota - 2$. $\square$

Appendix

This appendix establishes several results used in previous sections that do not appear in the literature. These are basic facts about the topology of the moduli space of J-holomorphic maps and the existence of V-compatible almost complex structures.

A.1. Proof of Theorem 4.3. Theorem 4.3 states several properties about the topology of the moduli space. The proofs are presented below, organized into four steps. Along the way, two general facts about metric spaces and maps $f : X \to Y$ are used repeatedly:

(i) If (Y, d) is a metric space, then $f : X \to Y$ induces a pseudo-metric f^*d on X; this defines a topology on X for which f is continuous.

(ii) A metric on X induces a metric $d_{\mathcal{H}}$ – the Hausdorff distance – on the set $Z = Subsets_c(X)$ of its non-empty compact subsets that is compact whenever (X, d) is compact.

Step 1. Using Facts (i) and (ii), define an initial topology on $\overline{\mathcal{M}}_{A,g,n}(X)$ by pulling back the metrics on $\mathcal{J}(X)$, $\overline{\mathcal{M}}_{g,n}$ and $\overline{\mathcal{M}}_{g,n+1} \times X$ by $\pi \times \mathrm{st} \times \Gamma$. More precisely, define the pseudo-metric d_0 by setting

$$d_0\big((C, f, J), (C', f', J')\big) \;=\; d_{\mathcal{J}}(J, J') \;+\; d_{\overline{\mathcal{M}}_{g,n}}\big(\mathrm{st}(C), \mathrm{st}(C')\big) \;+\; d_{\mathcal{H}}(\Gamma_f, \Gamma_{f'}) \tag{A.1}$$

with Γ_f as in (4.2).

Lemma A.1. *Assume f, h are stable (perturbed) pseudo-holomorphic maps with same topological data (A, g, n). If $\mathrm{Aut}\ C_f = 1$, then $d_0(f, h) = 0$ if and only if f and h differ by a reparametrization.*

Proof. One direction is clear because d_0 is reparametrization invariant. Conversely, assume $d_0(f, h) = 0$. Then $\mathrm{st}(f) = \mathrm{st}(h) \in \overline{\mathcal{M}}_{g,n}$, $\pi(f) = \pi(h)$ and $\Gamma_f = \Gamma_h$. By assumption, the graph Γ_f is an embedded nodal curve in $\overline{\mathcal{M}}_{g,n+1} \times X$. Because C_f is already stable with $\mathrm{Aut}\ C_f = 1$ and $\mathrm{st}(h) = \mathrm{st}(f)$ so we can assume, after precomposing f by an isomorphism, that C_f is obtained from C_h by collapsing the unstable rational components C_h^+ of C_h; moreover, $C_f = C_f/\mathrm{Aut}C_f$ is canonically isomorphic to the fiber of the universal curve over $[C_f] \in \overline{\mathcal{M}}_{g,n}$. Since $\Gamma_f = \Gamma_h \subset \overline{\mathcal{U}}_{g,n} \times X$ then the restrictions of f and h to the stable part C_f of their domain must now be equal. The energies then satisfy $E(h) = E(f) + E(h^+)$ where h^+ is the restriction of h to C_h^+. On the other hand, $E(f) = E(h)$ because f and h represent the same homology class. Since h is a stable map, $C_h^+ = \emptyset$ since otherwise $E(h^+) > 0$. We conclude that f and h have the same domain and $f = h$. □

Step 2. Define a sequence of pseudo-metrics on $\overline{\mathcal{M}}_{A,g,n}(X)$ by setting $d_k(f, f') = d_{\mathcal{H}}\big(\Gamma^k_f, \Gamma^k_{f'}\big)$ and then let

$$d(f, f') \;=\; d_0(f, f') \;+\; \sum_{k=2}^{\infty} 2^{-k}\, \frac{d_k(f, f')}{1 + d_k(f, f')}$$

Lemma A.2. *d is a metric on $\overline{\mathcal{M}}_{A,g,n}(X)$.*

Proof. Assume by contradiction that $f, h \in \overline{\mathcal{M}}_{A,g,n}(X)$ such that $d_k(f, h) = 0$ for all k but f is not a reparametrization of h. We can then add a finite collection of marked points x to the domain of f to ensure that its domain is stable with trivial automorphism group. Then $d_0(\Gamma_{(f,x)}, \Gamma^k_h) \le d_k(f, h) = 0$, and so $d_0((f, x), (h, y)) = 0$ for some collection y of marked points. But then by the previous lemma (f, x) and (h, y) must differ by a reparametrization, contradiction. □

Step 3. After symmetrizing the metric by the actions of the finite groups G and S_n, we may assume that d is invariant under the action of $G \times S_n$. It follows that all maps π, se, φ_k and Γ^k above are continuous.

Step 4. The proof is completed by using the following version of the Gromov Compactness Theorem.

Theorem A.3. *Every sequence $\{f_n : C_n \to X\}$ of J_n-holomorphic maps with fixed arithmetic genus and number of marked points, uniformly bounded energy, and with $J_n \to J$ in $\mathcal{J}^\ell$ has a subsequence that converges in C^0 up to reparameterization to a J-holomorphic map $f : C \to X$.*

Because C^0 convergence implies the convergence of the graphs $\Gamma^k_{f_n}$, this immediately shows that $\pi : \overline{\mathcal{M}}^{G,E}_{g,n}(X) \to \mathcal{J}^\ell$ is proper. The maps φ_k in (4.1) are also proper because their fibers are closed subsets of C^k. Furthermore, φ_k is perfect: it is continuous, surjective and, by an observation of Palais [P], a proper continuous map to a metric space is closed. Thus all parts of Theorem 4.3 hold. □

Rather surprisingly, convergence in the metric of Theorem 4.3 implies convergence in the Gromov topology with higher regularity. For a precise statement, a brief diversion is necessary to clarify what convergence with higher regularity means in the present context.

Suppose $f_m \in \overline{\mathcal{M}}^E(X)$ satisfies $d(f_m, f_0) \to 0$. Then, as in the proof of Lemma A.2 we can add marked points to the domain of f_0 to obtain an $n+k$-marked stable curve C_0 with Aut $C_0 = 1$. The assumption that $d^k(f_m, f_0) \to 0$ then means that one can choose k marked points to make the domain of each f_m an $n+k$ curve C_m. By the semicontinuity properties used in the proof of Lemma 5.2, the C_m are stable with Aut $C_m = 1$ for large m. We can then regard the maps f_m and f_0 as maps defined on fibers of the universal curve over an open neighborhood U of $[C_0]$ in $\overline{\mathcal{M}}_{g,n+k}$.

Now for each compact set K in $C_0 \setminus \{nodes\}$ we can find a fiber-preserving biholomorphic map

$$\tau : K \times U \to \overline{\mathcal{U}}_K$$

where $\overline{\mathcal{U}}_K$ is the portion of the universal curve over U and outside a fixed open neighborhood of the nodal set $\mathcal{N} \subset \overline{\mathcal{U}}$. Then $g_m = f_m \circ \tau : K \to X$ is a sequence of maps with the same domain.

Elliptic regularity results now apply. In particular, if $\{f_m\}$ is a sequence of J_m-holomorphic maps from K to X with $f_m \to f_0$ in C^0 and $J_m \to \mathcal{J}_0$ in $W^{\ell,p}$ for $\ell \geq 1$ and $\ell p > 2$, then $f_m \to f_0$ in $W^{\ell,p}$ [IS]. This applies for all $\ell \geq 1$ and all compact sets.

Letting $\mathcal{J}^\infty(X)$ denote the space $\bigcap_\ell \mathcal{J}^\ell(X)$ of smooth tame structures on X, we conclude that if $J_m \to J_0$ in C^∞ then $f_m \to f_0$ in C^0, in Hausdorff distance and in C^∞ on every compact set $K \subset C_0 \setminus \{nodes\}$, which is the standard definition of Gromov convergence used in the literature. Thus we have:

Corollary A.4. *The topology on $\overline{\mathcal{M}}^E(X)$ over $\mathcal{J}^\infty(X)$ given by Theorem 4.3 is equivalent to the usual Gromov topology.*

It is straightforward to extend these results to decorated and relative versions of the moduli spaces.

A.2. The stratification of the universal moduli space. The universal moduli space $\overline{\mathcal{M}}(X)$ is stratified by the topological type t of its elements (f, C). As in [MS], each type can be represented by the dual graph whose vertices, which correspond to irreducible component C_i of C, are labeled by a genus, number of marked points, and the homology class $A_i = [f(C_i)]$. For relative moduli spaces, one also keeps track of the contact information to V. The types are partially ordered: $t \prec t'$ if t' is obtained from t by smoothing a node.

Each graph type t defines a open stratum $\mathcal{M}_t$ of the universal moduli space, and we set $\overline{\mathcal{M}}_t = \bigcup_{\tau \preceq t} \mathcal{M}_\tau$. Fix E and let $\mathcal{T}_{U,E}$ denote the collection of all topological types that are represented by maps f in the universal moduli space $\overline{\mathcal{M}}^{U,E}(X) \to U$ over $U \subset \mathcal{J}$ with $\omega(A) \leq E$ and with $3g - 3 + n \leq E$. Gromov compactness implies $\mathcal{T}_{U,E}$ is finite for any U with compact closure in C^0. It also implies the following fact.

Lemma A.5. *Each $J \in \mathcal{J}(X)$ has a C^0 neighborhood U in which*

$$\mathcal{T}_{U,E} = \mathcal{T}_{J,E}$$

In particular, every triple (A, g, n) represented by a map in $\overline{\mathcal{M}}^{U,E}(X)$ is already represented by one in $\overline{\mathcal{M}}^{J,E}(X)$.

Proof. The inclusion $\mathcal{T}_{J,E} \subset \mathcal{T}_{U,E}$ is obvious. For the other inclusion, assume the contrary: that there is a C^0-convergent sequence $J_k \to J_0$ and maps $f_k : C_k \to X$ in $\overline{\mathcal{M}}_t^{J_k,E}(X)$ for some t with $\overline{\mathcal{M}}_t^{J,E}(X)$ empty. The inequality $3g_k - 3 + n_k \leq E$ gives a uniform bound on the topological type of $\mathrm{st}(C_k)$. We also have the energy bound $E(f_k) = \omega(A) \leq E$ and a uniform lower bound $\alpha > 0$ on the energy of each non-constant J_k-holomorphic sphere. Since f_k is a stable map, this uniformly bounds the number of unstable rational components of C_k. Hence there are finitely many possible domain types for C_k (unstable genus 1 components can occur only if C_k is smooth, genus 1 with no marked points).

By Gromov compactness a subsequence of the f_k converges in C^0, in homology, and in energy to a J-holomorphic stable limit $f_0 : C_0 \to X$ that is in $\overline{\mathcal{M}}_t^{J,E}$; contradiction. $\square$

A.3. Adapting J to a normal crossing divisor. In Section 8 we introduced a stabilizing divisor D, replaced the normal crossing divisor V with $V \cup D$, and replaced J by a nearby J' compatible with $V \cup D$. This subsection explains how to find such a J'. It begins at the level of linear algebra.

Fix a vector space W of dimension $2n$ with a hermitian structure (ω, J, g). The Grassmann manifold $Gr_{2\ell}$ of codimension 2ℓ subspaces of W is compact and has a canonical Riemannian metric on $Gr_{2\ell}$ induced by the metric g on W. Furthermore, the subset $Gr_{2\ell}^J$ of the J-invariant subspaces is a submanifold.

Definition A.6. *We say that $V \in Gr_{2\ell}$ is* ε-holomorphic *if it lies in the ε-tubular neighborhood of $Gr_{2\ell}^J$.*

Similarly, an ordered configuration $V = \{V_i\}$ of k codimension 2 linear subspaces of W is a point in the product $G_k(W) = Gr_2 \times \cdots \times Gr_2$ and V is ε-holomorphic if it lies in the ε-neighborhood of the submanifold

$$G_k^J(W) \;=\; Gr_2^J \times \cdots \times Gr_2^J \;\subset\; G_k(W) \tag{A.2}$$

of J-invariant configurations. It is also useful to consider the singular variety $S \subset G_k(W)$ of configurations not in general position, and say that V is *α-general* if $\mathrm{dist}(V, S) \geq \alpha$. Observe that, because $G_k^J(W)$ is a submanifold, there are constants $c_1, \varepsilon_0 > 0$ so that if V is an ε-holomorphic configuration with $\varepsilon < \varepsilon_0$ then there is a J-invariant configuration V' within distance $c_1\varepsilon$ of V, and in fact V is also ω-symplectic. Moreover, if V is α-general and ε is sufficiently small (depending on α) then the complex configuration V' is $\alpha/2$-general.

It is useful to think of configurations as assembled from 2-dimensional subspaces in the following way.

Definition A.7. *Let $V = \{V_i\}$ be a configuration of k codimension 2 subspaces of W in general position and with common intersection $V_{all} = \cap_i V_i$. An* adapted splitting *of V is V_{all} together with a set of k 2-dimensional complementary subspaces N_i such that*

$$W = V_{all} \oplus \oplus_i N_i \tag{A.3}$$

and for any $I \sqcup I' \subset \{1, \ldots, k\}$,

$$\bigcap_{i \in I} V_i \;=\; \Big(\bigcap_{i \in I \sqcup I'} V_i\Big) \oplus \bigoplus_{j \in I'} N_j. \tag{A.4}$$

Adapted splittings always exist. For example, one can choose an invertible linear transformation $L : W \to \mathbb{C}^n$ such that $L(V)$ is the standard configuration V^0 of the first k complex coordinate hyperplanes $\{x^i = 0\}$ in $\mathbb{C}^n$ and take N_i to be the inverse image of the complex line in the i direction (these complex lines form the standard splitting N^0 of V^0). An adapted splitting (A.3) is called J-invariant if V_{all} and each N_i are J-invariant; it follows that each V_i and all intersections (A.4) are J-invariant. Thus there is a dual perspective: given a general point $N = (V_{all}, N_1, \ldots, N_k)$ in the Grassmann

$$\widehat{G}_k(W) \;=\; Gr_{2k} \times Gr_{2n-2} \times \cdots \times Gr_{2n-2}$$

we obtain a configuration $V = \{V_i\}$ where

$$V_i \;=\; V_{all} \cup \bigoplus_{j\neq i} N_j. \tag{A.5}$$

Again $\widehat{G}_k(W)$ contains a submanifold $\widehat{G}^J_k(W)$ of J-invariant subspaces and a variety $\widehat{S}$ of splittings not in general position, and we say $N \in \widehat{G}_k(W)$ is ε-holomorphic if it lies in the ε neighborhood of $\widehat{G}^J_k(W)$ and is α-general if it lies outside the α-neighborhood of $\widehat{S}$.

Lemma A.8. *Fix a hermitian vector space (W, ω, J, g) and $\alpha > 0$. Then there exist constants c_α, C_α and $\varepsilon_0 > 0$ with the following property: for every α-general and ε-holomorphic configuration $V \subset W$ with $\varepsilon < \varepsilon_0$, there exists a V-adapted splitting $N_V = \{N_i\}$ that is c_α-general and $C_\alpha\varepsilon$-holomorphic.*

Proof. Whenever V is α-general we can find a linear transformation L that takes V into the standard configuration V^0 with norms $|L|$ and $|L^{-1}|$ bounded by a constant C'_α depending on α but independent of V. Pulling back the standard splitting N^0 gives an adapted splitting $L^{-1}(N^0)$ for V that is c_α-general for a constant c_α independent of V. If V is J-invariant, we can find a complex linear transformation L that pulls back N^0 to a J-invariant c_α-general splitting N_V. Finally, if V is only ε J-holomorphic, for ε sufficiently small (depending on α) homotoping V to the nearby J-invariant V' gives a path L_t of linear transformations of length $c_1\varepsilon$ and so a homotopy from the splitting N_V to a J-invariant one $N_{V'}$ of length $C_\alpha\varepsilon$. $\square$

Now consider a compact manifold X^{2n} with an almost Kähler structure (ω, J, g) and a topological normal crossing divisor $V = \{V_i\}$, that is, assume that V satisfies Definition 1.3 of [I2] without the requirement that V be J-holomorphic. In particular, the branches of V are in general position and each intersection $V_I = V_{i_1} \cap \cdots \cap V_{i_k}$ is a submanifold. Also fix an adapted splitting N_V (this is called the "normal bundle to V" in [I2]). The local model of X near a point $p \in V_I$ is then described via a local diffeomorphism sending the branches V_i into the i^{th} coordinate planes in $\mathbb{C}^n$ as in [I2], but without any compatibility conditions with (ω, J).

Along each depth k stratum V_I where k branches of V meet, the restriction of the configuration N_V to V_I defines a smooth section of the Grassmann bundle $\widehat{G}_k(TX)$ over V_I that lies in the subbundle $\widehat{G}^J_k(TX)$ at each point $p \in V_I$ where the configuration N_V and therefore V is J-holomorphic. In fact, using the local models, we can extend the splitting (A.3) over a neighborhood of V_I (by extending smoothly and then projecting), making the extension agree with the existing splitting on those strata $V_{I'}$ whose closure contains V_I. This gives sections σ_I defined on a neighborhood of V_I for each I, and these are compatible: if $I' \subset I$ then V_I lies in the closure of $V_{I'}$, and on the intersection of their tubular neighborhoods there is a forgetful map (that forgets the branches V_i for i in $I \setminus I'$)

$$\widehat{G}_{|I|}(TX) \hookrightarrow \widehat{G}_{|I'|}(TX) \tag{A.6}$$

which takes σ_I to $\sigma_{I'}$.

Proposition A.9. *Assume (X, J, g, ω) and $\alpha > 0$ is fixed. Then there exist constants $C_\alpha, \varepsilon_0 > 0$ with the following property: for any $\varepsilon < \varepsilon_0$ and any topological normal crossing divisor (V, N_V) in X that is α-general and ε-holomorphic there is an almost complex structure J_V on X with $|J - J_V| \le C_\alpha\varepsilon$ such that*

(a) (V, N_V) is J_V-holomorphic, and
(b) J_V is V-compatible in the sense of [IP1] *and* [I2].

Proof. Both (X, V) and the Grassmanian manifolds are compact so we can find uniform bounds for all pointwise estimates below.

For any V which is ε-holomorphic and α-general, we can deform each section σ_I to a section φ_I of the complex configurations space $\widehat{G}^J_{|I|}(TX)$ and, by further deformations, make the $\{\varphi_I\}$ compatible under the inclusions $\widehat{G}^J_{|I'|}(TX) \hookrightarrow \widehat{G}^J_{|I|}(TX)$ corresponding to (A.6). Because for ε small these deformations take place in a small tubular neighborhood of $\widehat{G}^J_{|I|}(TX)$ in $\widehat{G}_{|I|}(TX)$ (away from the singular locus S

of non transverse configurations), the deformation is unique up to homotopy and $\text{dist}(\sigma_I, \varphi_I) \le C_\alpha \varepsilon$ for some uniform constant C_α (independent of ε and V), using the canonical metric on $\widehat{G}_{|I|}(TX)$ induced by the metric g on X.

At each point p in a neighborhood of V, the section φ corresponds to a J-complex configurations N_φ in T_pX that is close to N_V in the sense that the g-orthogonal projection

$$\pi_\varphi : N_V \to N_\varphi \tag{A.7}$$

satisfies $\|\pi_\varphi - \text{Id.}\| \le C\varepsilon$. Here π_φ is defined by taking TV_I and each N_i, for $i \in I$ onto the corresponding subspaces of N_φ, therefore is well defined on $T_pX = T_pV_I \oplus_{i\in I} N_i$. In particular, π_φ is an isomorphism for small ε. Define J_V by

$$J_V \;=\; \pi_\varphi^* J \;=\; (\pi_\varphi^{-1})_* \circ J \circ (\pi_\varphi)_*$$

Then J_V is an almost complex structure on a neighborhood of V which preserves N_V, and therefore preserves V, and $|J - J_V| \le C\varepsilon$ with C_α independent of ε and V.

This defines J_V in a neighborhood of V. Statement (a) follows because we can use the method of Theorem A.2 of [IP1] to merge J_V into J, maintaining the bound $|J - J_V| \le C_\alpha \varepsilon$.

Finally, we can also achieve statement (b) by successively modifying J_V, beginning with the deepest stratum V_{all}. The needed V-compatibility condition along a stratum V_I requires that the Nijenhuis tensor $\mathcal{N}$ of J_V at $p \in V_I$ satisfy $\mathcal{N}(v, \xi) \in TV_i$ for every $v \in TV_I$ and every ξ in the normal bundle to V_I, which is $N_I = \oplus_{i\in I} N_i$. This can be achieved by applying the proof of Theorem A.2 of [IP2], parallel transporting in directions inside N_I along V_I and merging into the existing J. This yields a new J_V that is now V-compatible in a neighborhood of V_I (and preserves the fact that (V, N_V) is J_V-holomorphic everywhere). One then repeats the process along the lower strata inductively to construct the required V-compatible J_V. $\square$

Proposition A.9 has the following corollary that was used repeatedly in Section 8.

Corollary A.10. *Suppose that V is a J-holomorphic normal crossing divisor in (X, ω, J, g) and fix $\eta > 0$. Then there exists constants $\varepsilon, C_\eta > 0$ with the following property: for each Donaldson divisor D that is ε-holomorphic for $\varepsilon < \varepsilon_0$ and η-transverse to V (in the sense of* [A2]*), $V \cup D$ is a symplectic normal crossing divisor for some J' with $|J - J'| \le C_\eta \varepsilon$.*

Proof. By compactness, V is α-general for some $\alpha > 0$. Similarly, because D is η-transverse to V, $V \cup D$ is a topological normal crossing divisor that is α-general for some $\alpha = \alpha(\eta) > 0$ independent of D. Lemma A.8 applies pointwise to produce an adapted splitting $N_{D\cup V}$ for $V \cup D$ that is c_η-general and $C'_\eta\varepsilon$-holomorphic with constants c_η and C'_η independent of ε and D.

Proposition A.9 then applies provided ε was sufficiently small (less than an ε_0 depending on η), yielding a $V \cup D$ compatible almost complex structure $J' = J_{V\cup D}$ with $|J - J'| \le C_\eta \varepsilon$ for a constant C_η independent of ε and D. $\square$

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