WHAT IS...

a Bubble Tree?

Thomas H. Parker

Some of the most important equations of physics and geometry are conformally invariant. One example is Laplace's equation

$$\Delta u = 0 \tag{1}$$

for functions u on a domain in $\mathbb{R}^2$, which arises as the variational equation of the energy

$$E(u) = \int |du|^2 \, dvol. \tag{2}$$

Conformal invariance can be seen by explicitly writing the integrand, as geometers do, in terms of the Riemannian metric g_{ij} (and its inverse g^{ij}). Thus we write $|du|^2$ as $\sum_{ij} g^{ij}\partial_i u \partial_j u$ and write the volume form as $\sqrt{\det g_{ij}}\, dx\, dy$. One then sees that, for any positive function φ, the conformal change of metric $g_{ij} \mapsto \varphi\, g_{ij}$ leaves $E(u)$ invariant, and therefore solutions of the variational equation (1) remain solutions. That is the meaning of the phrase "conformally invariant".

The holomorphic map equation—which is a nonlinear generalization of the Cauchy-Riemann equation that applies to maps from a Riemann surface to a complex manifold X—also is conformally invariant. When X has a Kähler metric, solutions minimize the energy integral (2).

Other nonlinear conformally invariant equations, such as the equations for Yang-Mills fields, harmonic maps, and constant-mean-curvature hypersurfaces in $\mathbb{R}^3$, also have an associated energy. For each, standard methods of partial differential equations imply that there is an $\varepsilon_0 > 0$ such that:

(A) (Energy Gap) Any positive-energy solution u on the n-sphere S^n has $E(u) \geq \varepsilon_0$.

(B) (Uniform Convergence) Any bounded sequence $\{u_k\}$ of solutions defined on a ball with $E(u_k) < \varepsilon_0$ for all n has a subsequence converging in the C^∞ topology on compact subsets.

(C) (Removable Singularities) Any smooth finite-energy solution on a punctured ball $B \setminus \{0\}$ extends to a smooth solution on B.

Solutions of conformally invariant equations on $\mathbb{R}^n$ pull back to solutions under any conformal transformations, including translations, rescalings $x \mapsto \lambda x$, and stereographic projection $\sigma : S^n \to \mathbb{R}^n$. Thus, given a positive-energy solution on S^n, we can pull back by σ^{-1}, rescale, and then translate to get a solution on $\mathbb{R}^n$ concentrated near any desired point. Such a solution is called an *instanton*. Superpositions of instantons concentrated at different points are not solutions—these are nonlinear equations!—but are nearly solutions. It is a general theme, which began with C. Taubes's work on Yang-Mills fields and now runs across the study of all these conformally invariant equations, that under certain conditions one can perturb such superpositions to obtain "multi-instanton" solutions on $\mathbb{R}^n$ (and also on manifolds, as illustrated below).

Additional rescalings of $\mathbb{R}^n$ yield a sequence of multi-instanton solutions that concentrate at the origin and converge to a trivial solution pointwise on $\mathbb{R}^n \setminus \{0\}$. That limit loses energy. A bubble tree is a way of recovering the lost energy by keeping track of the part of the solution that is squeezed into the origin.

The Bubble Tree Construction

Given a sequence $\{u_k\}$ of solutions of some conformally invariant equation, we would like to find a convergent subsequence. For that we assume that the images are uniformly bounded, that $E(u_k) < E$ for all k, and that the domain is a closed manifold M. The key is to look at the energy densities $e(u_k) = |du_k|^2\, dvol$ and use the "bubbling" idea of K. Uhlenbeck [3] as modified in [2].

Cover M with balls of radius ρ so that no point is in more than, say, ten balls. Passing to a subsequence, we can ensure that (B) applies on all but at most $10E/\varepsilon_0$ "bad" balls. Doing that for a

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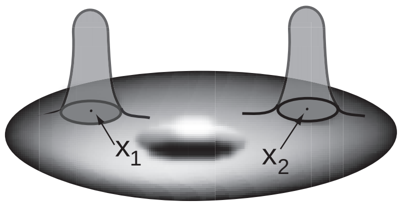
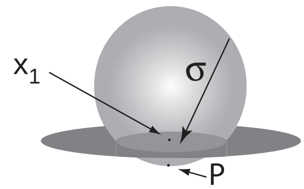
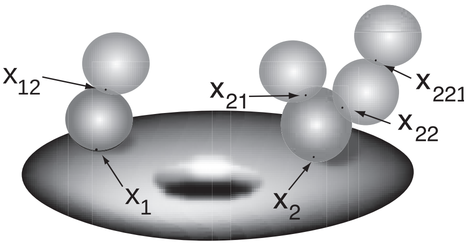


Energy density of a 2-instanton on a torus



Forming a bubble



A bubble tree domain

sequence $\rho_m \to 0$, we can pass to a diagonal subsequence (still denoted u_k) for which the bad balls converge to points $x_1, \ldots, x_\ell$. By facts (B) and (C), these u_k converge in C^∞ to a limit u_0 on $M \setminus \{x_1, \ldots, x_\ell\}$, and the energy densities $e(u_k)$ converge as measures to $e(u_0)$ plus a sum of point measures at the x_i with mass $m_i \geq \varepsilon_0$:

$$e(u_k) \to e(u_0) + \sum_i m_i \, \delta_{x_i}.$$

While this limit u_0 is a solution, it is not the full story, because some energy (an amount $m_i > 0$) concentrates at each x_i and is not accounted for by u_0. We can recover the lost energy by renormalizing the u_k to "catch a bubble".

Fix a ball $B(x_i, \varepsilon)$ around one x_i. For each k translate the center of mass of the measure $e(u_k)$ to the origin, then dilate by the (smallest) factor λ_k which pushes mass (at least) $\varepsilon_0/4$ outside the unit ball. It follows that $\lambda_k \to \infty$. Pulling the u_k back by stereographic projection from the south pole $P \in S^n$ gives renormalized maps $\tilde{u}_k$ defined on larger and larger sets in $S^n \setminus \{P\}$. We can then let $\varepsilon \to 0$ and pass to a subsequence to conclude that $\tilde{u}_k \to \tilde{u}_0$ on $S^n \setminus \{x_{i1}, \cdots, x_{im}, P\}$. This $\tilde{u}_0$ is the *bubble map* at x_i.

Convergence near the pole P must be carefully studied. The issue is whether energy accumulates in the "neck"—the annulus $B(x_i, \varepsilon) \setminus \sigma(B(P, \varepsilon))$ that connects the original domain to the domain of the bubble. That does not happen for holomorphic maps, and consequently the bubble map has a removable singularity at P, and the images of the base and the bubble meet at $u_0(x_i) = \tilde{u}_0(P)$. A similar "No Neck Energy" lemma holds for some—but not all—other conformally invariant equations (see [1]).

We can now iterate the construction, renormalizing around each x_{ij} and repeating, constructing bubbles on bubbles (see [2]). The end result is a *bubble tree domain* consisting of the original domain M with an attached tree of spheres S^n_α and a limit map u_∞ with component maps u_0 on M and $\tilde{u}_{0,\alpha}$ on each bubble S^n_α. Facts (A)–(C) and a "No Neck Energy" lemma imply two key properties:

- *Stability*: Each bubble map $\tilde{u}_0$ either has $E(\tilde{u}_0) \geq \varepsilon_0$ or has at least two higher bubbles attached to its domain.

Because the total energy is bounded, stability implies that the iteration process ends.

- *Bubble Tree Convergence Theorem*: After passing to a subsequence, $e(u_k)$ converges to $e(u_\infty)$ as measures, and $\{u_k\} \to u_\infty$ in C^0, and in C^∞ away from the double points of the bubble domain. In particular, no energy is lost in the limit.

Application to Gromov-Witten Invariants

The bubble tree construction generalizes to include (i) marked points p_i on the domain (by adding point masses at the p_i to the measures $e(u_k)$) and (ii) varying conformal structures on the domain. That leads to M. Kontsevich's notion of a stable map: a holomorphic map u, from a genus-g nodal complex curve C with ℓ marked points to a compact Kähler manifold X, is a *stable map* if $2g + \ell \geq 3$ and $E(u) > 0$ on each genus-0 component with fewer than three marked or double points. Let $\overline{\mathcal{M}}_{g,\ell}(X, \beta)$ be the space of all such stable maps whose image represents the homology class $\beta \in H_2(X)$, topologized by bubble tree convergence. The above analysis then implies the fact, first pointed out by M. Gromov, that *the space $\overline{\mathcal{M}}_{g,\ell}(X, \beta)$ of stable maps is compact*. After a perturbation of the equations, evaluation at the ℓ marked points gives a map

$$\overline{\mathcal{M}}_{g,\ell}(X, \beta) \to X^\ell$$

whose image, thought of as a homology class, is a *Gromov-Witten invariant* of X. In that sense, Gromov-Witten invariants arise naturally from the bubble tree construction.

References

[1] T. Parker, Bubble tree convergence for harmonic maps, *J. Differential Geom.* **44** (1996), 595–633.

[2] T. Parker and J. Wolfson, Pseudo-holomorphic maps and bubble trees, *J. Geom. Anal.* **3** (1993), 63–98.

[3] J. Sacks and K. Uhlenbeck, The existence of minimal immersions of 2-spheres, *Ann. of Math.* (2) **113** (1981), 1–24.

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