Math 153H **Recitation Worksheet #4** **Sep 30, 2014**

Problem 1. (a) Use u-substitution to evaluate $\int \frac{x}{x^2-4}\,dx$.

(b) Use partial fractions to evaluate $\int \frac{x}{x^2-4}\,dx$. Is your answer the same as in part (a)?

(c) Repeat (a) and (b) with x^2-4 replaced by x^2+4.

Problem 2. Let n be a positive integer.

(a) Find a polynomial $P(x)$ and a constant C so that

$$\frac{x^{100}}{x+1} = P(x) + \frac{C}{x+1}.$$

(b) Evaluate $\int \frac{x^{100}}{x+1}\,dx$.

Problem 3. Let S denote the set consisting of rational functions of the form

$$\frac{P(x)}{(x-1)(x-2)(x-3)},$$

where $P(x)$ is a polynomial of degree at most 2.

(a) Show that if $\frac{P(x)}{(x-1)(x-2)(x-3)}$ and $\frac{Q(x)}{(x-1)(x-2)(x-3)}$ are in S, then so is

$$a\frac{P(x)}{(x-1)(x-2)(x-3)}+b\frac{Q(x)}{(x-1)(x-2)(x-3)}$$

for any real numbers a, b.

(b) Calculate the dimension of S. *Hint: This is the number of degrees of freedom you have in writing down an element of S; the variable x does not count as a degree of freedom here.*

(c) Show that if A, B, C are real numbers, then

$$\frac{A}{x-1}+\frac{B}{x-2}+\frac{C}{x-3}$$

is an element of S.

(d) Let T denote the set of rational functions of the form $\frac{A}{x-1}+\frac{B}{x-2}+\frac{C}{x-3}$, where A, B, C are real numbers. Calculate the dimension of T.

Part (c) says that every element of T lies in S. You have also shown that both T and S have the same dimension. When you take linear algebra you will learn that this forces T and S to actually be equal. It follows that every rational function of the form $\frac{P(x)}{(x-1)(x-2)(x-3)}$, with the degree of P at most 2 can be written as $\frac{A}{x-1}+\frac{B}{x-2}+\frac{C}{x-3}$ for some real numbers A, B, C.

Problem 4. Suppose A and B are two solid 3-dimensional objects with base lying on the xy-plane, and with the same height. Let A_z denote the 2-dimensional region obtained by intersecting A with the plane of height z parallel to the xy-plane; define B_z similarly. Assume that A and B have the property that the area of A_z is equal to the area of B_z for all z. *Cavalieri's principle* states that A and B have the same volume.

(a) Use integration to prove Cavalieri's principle. *Hint: It may help to draw a picture.*

(b) Let C be a cone of radius r and height h. Let P be a square pyramid with height h and base of area πr^2. Use Cavalieri's principle to show that the cone and the pyramid have the same volume.

(c) Notice that in part (b) you can take P to have vertex lying directly over one of the vertices of the square base. There are two other square pyramids P' and P'' with the same volume as P, so that P, P' and P'' fit together to form a box with height h and base having area πr^2. Try to visualize P', P'' and the box (use P to form the obvious box, and then see what P', P'' have to be). Then use this to calculate the area of the cone from part (b). *(You don't need to actually be able to visualize P', P'' or the box to do this calculation.)*

The above is how people used to calculate the volume of a cone. Now we will work through how they calculated the volume of a sphere.

(d) Consider a cylinder of height $2r$ and with base of radius r. Stick two cones of height r and base of radius r inside of it so their apexes touch in the middle (you would get an hour glass). Let B be the region inside of the cylinder, but outside of the cones. Calculate the volume of B. *Hint: Use part (c).*

(e) Let B_z be the cross-section of B with a plane perpendicular to the axis of the cylinder and a distance z away from the point where the apexes touch. Calculate the area of B_z.

(f) Let A be a sphere of radius r, and let A_z be the cross-section of A with a plane a distance z away from the center. Calculate the area of A_z (it should be the same as part (e)). Use parts (d), (e) and Cavalieri's principle to obtain the formula for the volume of the sphere.