Math 299 **Recitation Worksheet #6** **October 9, 2014**

Proof by Contradiction

To prove a statement P you can show that $\sim P$ implies a false statement.

Exercise 1. Use a truth table to show that P is equivalent to $\sim P \implies F$ where F is a false statement.

Exercise 2. Prove the statement: *If a and b are natural numbers, then $a^2 - 4b^2 \neq 1, 2, 3$ or 4.* (Hint: if $a^2 = j + 4b^2$ with $j \in \mathbb{N}$, then $a^2 > 4b^2$.)

Exercise 3. Prove that there is no *natural* number n such that $2n < n^2 < 3n$.

Exercise 4. Prove that there is no rational solution to the equation $x^3 + x + 1 = 0$. (Hint: Suppose $x = {}^{n}/_{m}$ is a rational solution in reduced form, with n and m integers and $m \neq 0$. Show that $n^3 + nm^2 + m^3 = 0$. Derive a contradiction by considering the three cases n and m odd, n even and m odd, and n odd and m even. Why isn't it possible that n is even and m is even?)

Existence proofs

Existence proofs: The most straightforward way to prove a statement of the form $\exists x \in S, P(x)$ is to give a constructive. In a constructive proof, one proves the statement by exhibiting a specific $x \in S$ such that $P(x)$ is true.
Example of a constructive proof: Suppose we are to prove

$$\exists n \in \mathbb{N}, n \text{ is equal to the sum of its proper divisors.}$$

Proof. Let $n = 6$. The proper divisors of 6 are $1, 2$, and 3. Since $1 + 2 + 3 = 6$, we have proved the statement. $\square$

Exercise 5. Give another proof of this statement by finding a different example. (Hint: The smallest example larger than 6 happens to be a number between 25 and 30.)

Another type of existence proof is a *non-constructive* proof, which derives $\exists x \in S,\ P(x)$ without exhibiting a specific example of x such that $P(x)$ holds. One common type of non-constructive proof uses the
Intermediate Value Theorem. *Suppose that $f(x)$ is a continuous function on an interval $[a, b]$. If y is a real number between $f(a)$ and $f(b)$, then there exists $c \in (a, b)$ such that $f(c) = y$.*

Exercise 6. Prove that there is $x \in \mathbb{R}$ such that $x^3 + x + 1 = 0$. (You may assume that $f(x) = x^3 + x + 1$ is continuous.)

Another type of non-constructive proof is based on the

Pigeon hole principle. *If n items are divided into m collections and $n > m$, then there is at least one collection with more than one item.*

Exercise 7. Without asking everyone in the room what month they were born in, prove that there are at least two people in this class with birthdays in the same month.

Exercise 8. Show that among any 5 numbers there are 2 such that their difference is divisible by 4. (Hint: what are the possible remainders?)

Exercise 9. Suppose we are given $m + 1$ numbers, with $m \geq 2$. Prove that there are at least 2 of the numbers whose difference is divisible by m.