Math 330 - Quiz 2 Study Guide

Choose any *three* of the following seven problems to hand in on Tuesday, June 16.

1. Consider a triangle ΔABC, and let x be a positive real number. There is a point $L \in \overline{BC}$ with the property that

$$BL = x(LC)$$

Similarly, there are points $M \in \overline{CA}$ and $N \in \overline{AB}$ such that

$$CM = x(MA), \qquad AN = x(NB).$$

Show that the lines $\overleftrightarrow{AL}, \overleftrightarrow{BM}, \overleftrightarrow{CN}$ are concurrent if and only if $x = 1$.

2. Suppose I is an isometry. Show that I is injective.

3. (a) Find two rotations R_1, R_2 whose composition $R_1 \circ R_2$ is a rotation.

(b) Find two rotations R_1, R_2 whose composition is a translation.

(c) Can you find rotations R_1, R_2 whose composition is a reflection?

(d) Does the set of all rotations form a group under composition?

4. Suppose that ℓ, ℓ' are two parallel lines in the Euclidean plane, and let d be the distance between them. Let R_ℓ and $R_{\ell'}$ be the reflections over these lines.

(a) Show the following: (i) the composition $R_\ell \circ R_{\ell'}$ is a translation, (ii) the distance of the translation is $2d$, and (iii) the direction of the translation is along the line perpendicular to ℓ' and ℓ.

(b) Is the translation $R_\ell \circ R_{\ell'}$ the same as $R_{\ell'} \circ R_\ell$?

5. Suppose ΔABC and $\Delta A'B'C'$ are two congruent triangles in the plane. Explain how to find lines $\ell_1, \ell_2, \ldots, \ell_n$ so that the isometry given by

$$R_{\ell_n} \circ \ldots \circ R_{\ell_2} \circ R_{\ell_1}$$

takes ΔABC to $\Delta A'B'C'$. Here R_ℓ is the reflection over ℓ.

6. (a) Prove the following angle addition formulas for sin and cos. Here $\alpha, \beta \in \mathbb{R}$.

$$\begin{aligned}\cos(\alpha+\beta) &= \cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta)\\ \sin(\alpha+\beta) &= \cos(\alpha)\sin(\beta) + \cos(\alpha)\sin(\beta)\end{aligned}$$

(b) Evaluate the following and use geometry to justify your answers. All angles are measured in radians.

- $\sin(\pi/3)$
- $\sin(\pi/4)$
- $\sin(\pi/6)$

(c) Evaluate $\sin(\pi/24)$.

7. Suppose $\theta \in \mathbb{R}$. Explain why $e^{i\theta}$ lies on the unit circle in S^1. That is, you should explain why the distance from $e^{i\theta}$ to the origin is 1.

Extra Credit: Suppose A is an $n \times n$ matrix, and define

$$e^A := \sum_{n=0}^{\infty} \frac{1}{n!} A^n.$$

It turns out this series converges to an $n \times n$ matrix.

(a) Suppose

$$A = \begin{pmatrix} \alpha & -\beta \\ \beta & \alpha \end{pmatrix}$$

where $\alpha, \beta \in \mathbb{R}$. Express the matrix e^A in components. *Hint: Write $A = \alpha M_1 + \beta M_2$ for some matrices M_1 and M_2 that do not depend on α, β.*

(b) Suppose

$$A = \begin{pmatrix} a_1 & 0 & \dots & 0 \\ 0 & a_2 & \dots & 0 \\ \vdots & & & \vdots \\ 0 & 0 & \dots & a_n \end{pmatrix}$$

is a diagonal matrix. Express the matrix e^A in components.

(c) Suppose A is a diagonal matrix. Show

$$\det(e^A) = e^{tr(A)}, \tag{1}$$

where det is the determinant and tr is the trace.

(d) Show that (1) holds for all $n \times n$ matrices A.