Math 396 - Homework 4

Due 2/9/15

1. Let C denote the set of matrices of the form

$$\begin{pmatrix} a & -b \\ b & a \end{pmatrix}$$

where $a, b \in \mathbb{R}$.

(a) Define a map $\Phi : \mathbb{C} \to C$ by

$$\Phi(a + ib) = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}.$$

Show that Φ is a ring isomorphism.

(b) It follows that C is a field isomorphic to $\mathbb{C}$. Which element of C corresponds to $i \in \mathbb{C}$?

(c) Let $\theta \in \mathbb{R}$. Use the power series definition of the exponential to compute e^A, where

$$A = \begin{pmatrix} 0 & -\theta \\ \theta & 0 \end{pmatrix}.$$

Hint: Think carefully about your answers to parts (a) and (b). You can probably guess what the answer to (c) should be.

2. Consider the matrix

$$A = \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}$$

and the vector

$$v_0 = \begin{pmatrix} 0 \\ 3 \end{pmatrix}.$$

Here we are interested in the differential equation

$$\frac{d}{dt}v(t) = Av(t), \qquad v(0) = v_0, \tag{1}$$

where $v : \mathbb{R} \to \mathbb{R}^2$ is a path of vectors.

(a) Use the power series definition of e^{tA} to show that

$$v(t) := e^{tA}v_0$$

is the solution to (1).

(b) The matrix A has two eigenvectors e_1, e_2. Find them, as well as their eigenvalues.

(c) Find an invertible 2×2 matrix S with the property that

$$D := SAS^{-1}$$

is diagonal. *Hint: Use part (b).*

(d) Compute the matrix e^{tD}.

(e) Compute the matrix e^{tA}. *Hint: First show* $e^{tA} = S^{-1}e^{tD}S$.

(f) Compute $e^{tA}v_0$ and show explicitly that this solves (1).

3. Suppose V is an n-dimensional vector space with an inner product $(\cdot, \cdot)$. Assume $A : V \to V$ is a linear map that satisfies

$$(Av, w) = (v, Aw)$$

for all $v, w \in V$; such a linear map is called *symmetric*. Since A is symmetric, it has a basis $e_1, \ldots, e_n$ of eigenvectors (you do not need to prove this, but you can for extra credit). Let $\lambda_1, \ldots, \lambda_n$ be the corresponding eigenvalues.

(a) For each $i \in \{1, \ldots, n\}$, solve the differential equation

$$\frac{d}{dt}w_i(t) = Aw_i(t), \qquad w_i(0) = e_i$$

for $w_i : \mathbb{R} \to V$. *Hint: Use the fact that e_i is an eigenvector. You can probably guess what the solution should be.*

(b) Let $v_0 \in V$, and solve

$$\frac{d}{dt}v(t) = Av(t), \qquad v(0) = v_0, \tag{2}$$

for $v : \mathbb{R} \to V$. *Hint: Write v_0 as a linear combination of $e_1, \ldots, e_n$, and then use part (a).*

(c) Using the basis $e_1, \ldots, e_n$, we can write A as an $n \times n$ matrix M. Determine the matrix M.

(d) Let M be as in (c). Determine the matrix e^M.

(e) Briefly explain why $e^{tM}v_0$ solves (2), and compare this solution with the solution you found in part (b).