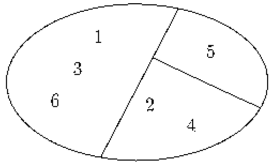

Figure 1.7 A partition of a set

the number 0. In addition, **R** can be partitioned into the set **Q** of rational numbers and the set **I** of irrational numbers.

Example 1.20 *Let* $A = \{1, 2, \ldots, 12\}$.

(a) Give an example of a partition S of A such that $|S| = 5$.
(b) Give an example of a subset T of the partition S in (a) such that $|T| = 3$.
(c) List all those elements B in the partition S in (a) such that $|B| = 2$.

Solution (a) We are seeking a partition S of A consisting of five subsets. One such example is

$$S = \{\{1, 2\}, \{3, 4\}, \{5, 6\}, \{7, 8, 9\}, \{10, 11, 12\}\}.$$

(b) We are seeking a subset T of S (given in (a)) consisting of three elements. One such example is

$$T = \{\{1, 2\}, \{3, 4\}, \{7, 8, 9\}\}.$$

(c) We have been asked to list all those elements of S (given in (a)) consisting of two elements of A. These elements are: $\{1, 2\}, \{3, 4\}, \{5, 6\}$. ♦

1.6 Cartesian Products of Sets

We've already mentioned that when a set A is described by listing its elements, the order in which the elements of A are listed doesn't matter. That is, if the set A consists of two elements x and y, then $A = \{x, y\} = \{y, x\}$. When we speak of the **ordered pair** (x, y), however, this is another story. The ordered pair (x, y) is a single element consisting of a pair of elements in which x is the first element (or first coordinate) of the ordered pair (x, y) and y is the second element (or second coordinate). Moreover, for two ordered pairs (x, y) and (w, z) to be equal, that is, $(x, y) = (w, z)$, we must have $x = w$ and $y = z$. So, if $x \neq y$, then $(x, y) \neq (y, x)$.

The **Cartesian product** (or simply the product) $A \times B$ of two sets A and B is the set consisting of all ordered pairs whose first coordinate belongs to A and whose second coordinate belongs to B. In other words,

$$A \times B = \{(a, b) : a \in A \text{ and } b \in B\}.$$

Example 1.21 *If* $A = \{x, y\}$ *and* $B = \{1, 2, 3\}$, *then*

$$A \times B = \{(x, 1), (x, 2), (x, 3), (y, 1), (y, 2), (y, 3)\},$$

while

$$B \times A = \{(1, x), (1, y), (2, x), (2, y), (3, x), (3, y)\}.$$

Since, for example, $(x, 1) \in A \times B$ *and* $(x, 1) \notin B \times A$, *these two sets do not contain the same elements; so* $A \times B \neq B \times A$. *Also,*

$$A \times A = \{(x, x), (x, y), (y, x), (y, y)\}$$

and

$$B \times B = \{(1, 1), (1, 2), (1, 3), (2, 1), (2, 2), (2, 3), (3, 1), (3, 2), (3, 3)\}.$$ ♦

We also note that if $A = \emptyset$ or $B = \emptyset$, then $A \times B = \emptyset$.

The Cartesian product $\mathbf{R} \times \mathbf{R}$ is the set of all points in the Euclidean plane. For example, the graph of the straight line $y = 2x + 3$ is the set

$$\{(x, y) \in \mathbf{R} \times \mathbf{R} : y = 2x + 3\}.$$

For the sets $A = \{x, y\}$ and $B = \{1, 2, 3\}$ given in Example 1.21, $|A| = 2$ and $|B| = 3$; while $|A \times B| = 6$. Indeed, for all finite sets A and B,

$$|A \times B| = |A| \cdot |B|.$$

Cartesian products will be explored in more detail in Chapter 7.

EXERCISES FOR CHAPTER 1

Section 1.1: Describing a Set

1.1. Which of the following are sets?
(a) $1, 2, 3$
(b) $\{1, 2\}, 3$
(c) $\{\{1\}, 2\}, 3$
(d) $\{1, \{2\}, 3\}$
(e) $\{1, 2, a, b\}$

1.2. Let $S = \{-2, -1, 0, 1, 2, 3\}$. Describe each of the following sets as $\{x \in S : p(x)\}$, where $p(x)$ is some condition on x.
(a) $A = \{1, 2, 3\}$
(b) $B = \{0, 1, 2, 3\}$
(c) $C = \{-2, -1\}$.
(d) $D = \{-2, 2, 3\}$.

1.3. Determine the cardinality of each of the following sets:
(a) $A = \{1, 2, 3, 4, 5\}$
(b) $B = \{0, 2, 4, \ldots, 20\}$
(c) $C = \{25, 26, 27, \ldots, 75\}$

(d) $D = \{\{1,2\}, \{1,2,3,4\}\}$
(e) $E = \{\emptyset\}$
(f) $F = \{2, \{2,3,4\}\}$

1.4. Write each of the following sets by listing its elements within braces.
(a) $A = \{n \in \mathbf{Z} : -4 < n \leq 4\}$
(b) $B = \{n \in \mathbf{Z} : n^2 < 5\}$
(c) $C = \{n \in \mathbf{N} : n^3 < 100\}$
(d) $D = \{x \in \mathbf{R} : x^2 - x = 0\}$
(e) $E = \{x \in \mathbf{R} : x^2 + 1 = 0\}$

1.5. Write each of the following sets in the form $\{x \in \mathbf{Z} : p(x)\}$, where $p(x)$ is a property concerning x.
(a) $A = \{-1, -2, -3, \ldots\}$
(b) $B = \{-3, -2, \ldots, 3\}$
(c) $C = \{-2, -1, 1, 2\}$

1.6. The set $E = \{2x : x \in \mathbf{Z}\}$ can be described by listing its elements, namely $E = \{\ldots, -4, -2, 0, 2, 4, \ldots\}$. List the elements of the following sets in a similar manner.
(a) $A = \{2x + 1 : x \in \mathbf{Z}\}$
(b) $B = \{4n : n \in \mathbf{Z}\}$
(c) $C = \{3q + 1 : q \in \mathbf{Z}\}$

1.7. The set $E = \{\ldots, -4, -2, 0, 2, 4, \ldots\}$ of even integers can be described by means of a defining condition by $E = \{y = 2x : x \in \mathbf{Z}\} = \{2x : x \in \mathbf{Z}\}$. Describe the following sets in a similar manner.
(a) $A = \{\ldots, -4, -1, 2, 5, 8, \ldots\}$
(b) $B = \{\ldots, -10, -5, 0, 5, 10, \ldots\}$
(c) $C = \{1, 8, 27, 64, 125, \ldots\}$

Section 1.2: Subsets

1.8. Give examples of three sets A, B, and C such that
(a) $A \subseteq B \subset C$.
(b) $A \in B$, $B \in C$, and $A \notin C$.
(c) $A \in B$ and $A \subset C$.

1.9. Let (a, b) be an open interval of real numbers and let $c \in (a, b)$. Describe an open interval I centered at c such that $I \subseteq (a, b)$.

1.10. Which of the following sets are equal?
$A = \{n \in \mathbf{Z} : |n| < 2\}$ $D = \{n \in \mathbf{Z} : n^2 \leq 1\}$
$B = \{n \in \mathbf{Z} : n^3 = n\}$ $E = \{-1, 0, 1\}$
$C = \{n \in \mathbf{Z} : n^2 \leq n\}$

1.11. For a universal set $U = \{1, 2, \ldots, 8\}$ and two sets $A = \{1, 3, 4, 7\}$ and $B = \{4, 5, 8\}$, draw a Venn diagram that represents these sets.

1.12. Find $\mathcal{P}(A)$ and $|\mathcal{P}(A)|$ for
(a) $A = \{1, 2\}$.
(b) $A = \{\emptyset, 1, \{a\}\}$.

1.13. Find $\mathcal{P}(A)$ for $A = \{0, \{0\}\}$.

1.14. Find $\mathcal{P}(\mathcal{P}(\{1\}))$ and its cardinality.

1.15. Find $\mathcal{P}(A)$ and $|\mathcal{P}(A)|$ for $A = \{0, \emptyset, \{\emptyset\}\}$.

1.16. Give an example of a set S such that
(a) $S \subseteq \mathcal{P}(\mathbf{N})$
(b) $S \in \mathcal{P}(\mathbf{N})$
(c) $S \subseteq \mathcal{P}(\mathbf{N})$ and $|S| = 5$.
(d) $S \in \mathcal{P}(\mathbf{N})$ and $|S| = 5$.

Section 1.3: Set Operations

1.17. Let $U = \{1, 3, \ldots, 15\}$ be the universal set, $A = \{1, 5, 9, 13\}$, and $B = \{3, 9, 15\}$. Determine the following:
(a) $A \cup B$, (b) $A \cap B$, (c) $A - B$, (d) $B - A$, (e) $\overline{A}$, (f) $A \cap \overline{B}$.

1.18. Give examples of three sets A, B, and C such that
(a) $A \in B$, $A \subseteq C$, and $B \nsubseteq C$.
(b) $B \in A$, $B \subset C$, and $A \cap C \neq \emptyset$.
(c) $A \in B$, $B \subseteq C$, and $A \nsubseteq C$.

1.19. Give examples of three sets A, B, and C such that $B \neq C$ but $B - A = C - A$.

1.20. Give examples of two sets A and B such that $|A - B| = |A \cap B| = |B - A| = 3$. Draw the accompanying Venn diagram.

1.21. Let U be a universal set and let A and B be two subsets of U. Draw a Venn diagram for each of the following sets.
(a) $\overline{A \cup B}$ (b) $\overline{A} \cap \overline{B}$ (c) $\overline{A \cap B}$ (d) $\overline{A} \cup \overline{B}$
What can you say about parts (a) and (b)? parts (c) and (d)?

1.22. Give an example of a universal set U, two sets A and B, and an accompanying Venn diagram such that $|A \cap B| = |A - B| = |B - A| = |\overline{A \cup B}| = 2$.

1.23. Let A, B, and C be nonempty subsets of a universal set U. Draw a Venn diagram for each of the following set operations.
(a) $(C - B) \cup A$
(b) $C \cap (A - B)$

1.24. Let $A = \{\emptyset, \{\emptyset\}, \{\{\emptyset\}\}\}$.
(a) Determine which of the following are elements of A: $\emptyset$, $\{\emptyset\}$, $\{\emptyset, \{\emptyset\}\}$.
(b) Determine $|A|$.
(c) Determine which of the following are subsets of A: $\emptyset$, $\{\emptyset\}$, $\{\emptyset, \{\emptyset\}\}$.
For (d)–(i), determine the indicated sets.
(d) $\emptyset \cap A$
(e) $\{\emptyset\} \cap A$
(f) $\{\emptyset, \{\emptyset\}\} \cap A$
(g) $\emptyset \cup A$
(h) $\{\emptyset\} \cup A$
(i) $\{\emptyset, \{\emptyset\}\} \cup A$.

Section 1.4: Indexed Collections of Sets

1.25. Give examples of a universal set U and sets A, B, and C such that each of the following sets contains exactly one element: $A \cap B \cap C$, $(A \cap B) - C$, $(A \cap C) - B$, $(B \cap C) - A$, $A - (B \cup C)$, $B - (A \cup C)$, $C - (A \cup B)$, $\overline{A \cup B \cup C}$. Draw the accompanying Venn diagram.

1.26. For a real number r, define $A_r = \{r^2\}$, B_r as the closed interval $[r-1, r+1]$, and C_r as the interval (r, ∞). For $S = \{1, 2, 4\}$, determine

(a) $\bigcup_{\alpha \in S} A_\alpha$ and $\bigcap_{\alpha \in S} A_\alpha$
(b) $\bigcup_{\alpha \in S} B_\alpha$ and $\bigcap_{\alpha \in S} B_\alpha$
(c) $\bigcup_{\alpha \in S} C_\alpha$ and $\bigcap_{\alpha \in S} C_\alpha$.

1.27. Let $A = \{1, 2, 5\}$, $B = \{0, 2, 4\}$, $C = \{2, 3, 4\}$, and $S = \{A, B, C\}$. Determine $\bigcup_{X \in S} X$ and $\bigcap_{X \in S} X$.

1.28. For a real number r, define S_r to be the interval $[r-1, r+2]$. Let $A = \{1, 3, 4\}$. Determine $\bigcup_{\alpha \in A} S_\alpha$ and $\bigcap_{\alpha \in A} S_\alpha$.

1.29. Let $A = \{a, b, \ldots, z\}$ be the set consisting of the letters of the alphabet. For $\alpha \in A$, let A_α consist of α and the two letters that follow it, where $A_y = \{y, z, a\}$ and $A_z = \{z, a, b\}$. Find a set $S \subseteq A$ of smallest cardinality such that $\bigcup_{\alpha \in S} A_\alpha = A$. Explain why your set S has the required properties.

1.30. For each of the following collections of sets, define a set A_n for each $n \in \mathbf{N}$ such that the indexed collection $\{A_n\}_{n \in \mathbf{N}}$ is precisely the given collection of sets. Then find both the union and intersection of the indexed collection of sets.

(a) $\{[1, 2+1), [1, 2+1/2), [1, 2+1/3), \ldots\}$
(b) $\{(-1, 2), (-3/2, 4), (-5/3, 6), (-7/4, 8), \ldots\}$

1.31. For each of the following, find an indexed collection $\{A_n\}_{n \in \mathbf{N}}$ of distinct sets (that is, no two sets are equal) satisfying the given conditions.

(a) $\bigcap_{n=1}^{\infty} A_n = \{0\}$ and $\bigcup_{n=1}^{\infty} A_n = [0, 1]$.
(b) $\bigcap_{n=1}^{\infty} A_n = \{-1, 0, 1\}$ and $\bigcup_{n=1}^{\infty} A_n = \mathbf{Z}$.

Section 1.5: Partitions of Sets

1.32. Which of the following are partitions of $A = \{a, b, c, d, e, f, g\}$? For each collection of subsets that is not a partition of A, explain your answer.

(a) $S_1 = \{\{a, c, e, g\}, \{b, f\}, \{d\}\}$
(b) $S_2 = \{\{a, b, c, d\}, \{e, f\}\}$
(c) $S_3 = \{A\}$
(d) $S_4 = \{\{a\}, \emptyset, \{b, c, d\}, \{e, f, g\}\}$
(e) $S_5 = \{\{a, c, d\}, \{b, g\}, \{e\}, \{b, f\}\}$

1.33. Which of the following sets are partitions of $A = \{1, 2, 3, 4, 5\}$?

(a) $S_1 = \{\{1, 3\}, \{2, 5\}\}$
(b) $S_2 = \{\{1, 2\}, \{3, 4, 5\}, \{2, 1\}\}$
(c) $S_3 = \{\{1, 2\}, \{2, 3\}, \{3, 4\}, \{4, 5\}\}$
(d) $S_4 = A$

1.34. Let $A = \{1, 2, 3, 4, 5, 6\}$. Give an example of a partition S of A such that $|S| = 3$.

1.35. Give an example of a set A with $|A| = 4$ and two disjoint partitions S_1 and S_2 of A with $|S_1| = |S_2| = 3$.

1.36. Give an example of three sets A, S_1, and S_2 such that S_1 is a partition of A, S_2 is a partition of S_1, and $|S_2| < |S_1| < |A|$.

1.37. Give an example of a partition of $\mathbf{Q}$ into three subsets.

1.38. Give an example of a partition of $\mathbf{N}$ into three subsets.

1.39. Give an example of a partition of $\mathbf{Z}$ into four subsets.

1.40. Let $A = \{1, 2, \ldots, 12\}$. Give an example of a partition S of A satisfying the following requirements: (i) $|S| = 5$, (ii) T is a subset of S such that $|T| = 4$ and $|\cup_{X \in T} X| = 10$, and (iii) there is no element $B \in S$ such that $|B| = 3$.

Section 1.6: Cartesian Products of Sets

1.41. Let $A = \{x, y, z\}$ and $B = \{x, y\}$. Determine $A \times B$.

1.42. Let $A = \{1, \{1\}, \{\{1\}\}\}$. Determine $A \times A$.

1.43. For $A = \{a, b\}$. Determine $A \times \mathcal{P}(A)$.

1.44. For $A = \{\emptyset, \{\emptyset\}\}$. Determine $A \times \mathcal{P}(A)$.

1.45. For $A = \{1, 2\}$ and $B = \{\emptyset\}$, determine $A \times B$ and $\mathcal{P}(A) \times \mathcal{P}(B)$.

1.46. Describe the graph of the circle whose equation is $x^2 + y^2 = 4$ as a subset of $\mathbf{R} \times \mathbf{R}$.

1.47. List the elements of the set $S = \{(x, y) \in \mathbf{Z} \times \mathbf{Z} : |x| + |y| = 3\}$. Plot the corresponding points in the Euclidean x-y plane.

ADDITIONAL EXERCISES FOR CHAPTER 1

1.48. Let $S = \{-10, -9, \ldots, 9, 10\}$. Describe each of the following sets as $\{x \in S : p(x)\}$, where $p(x)$ is some condition on x.

(a) $A = \{-10, -9, \ldots, -1, 1, \ldots, 9, 10\}$
(b) $B = \{-10, -9, \ldots, -1, 0\}$
(c) $C = \{-5, -4, \ldots, 7\}$
(d) $D = \{-10, -9, \ldots, -4, 4, 6, 7, \ldots, 10\}$

1.49. Describe each of the following sets by listing its elements within braces.

(a) $\{x \in \mathbf{Z} : x^3 - 4x = 0\}$
(b) $\{x \in \mathbf{R} : |x| = -1\}$
(c) $\{m \in \mathbf{N} : 2 < m \le 5\}$
(d) $\{n \in \mathbf{N} : 0 \le n \le 3\}$
(e) $\{k \in \mathbf{Q} : k^2 - 4 = 0\}$
(f) $\{k \in \mathbf{Z} : 9k^2 - 3 = 0\}$
(g) $\{k \in \mathbf{Z} : 1 \le k^2 \le 10\}$

1.50. Determine the cardinality of each of the following sets.

(a) $A = \{1, 2, 3, \{1, 2, 3\}, 4, \{4\}\}$
(b) $B = \{x \in \mathbf{R} : |x| = -1\}$
(c) $C = \{m \in \mathbf{N} : 2 < m \le 5\}$
(d) $D = \{n \in \mathbf{N} : n < 0\}$
(e) $E = \{k \in \mathbf{N} : 1 \le k^2 \le 100\}$
(f) $F = \{k \in \mathbf{Z} : 1 \le k^2 \le 100\}$

1.51. For $A = \{-1, 0, 1\}$ and $B = \{x, y\}$, determine $A \times B$.

1.52. Let $U = \{1, 2, 3\}$ be the universal set, and let $A = \{1, 2\}$, $B = \{2, 3\}$, and $C = \{1, 3\}$. Determine the following.

(a) $(A \cup B) - (B \cap C)$
(b) $\overline{A}$
(c) $\overline{B \cup C}$
(d) $A \times B$

1.53. Let $A = \{1, 2, \ldots, 10\}$. Give an example of two sets S and B such that $S \subseteq \mathcal{P}(A)$, $|S| = 4$, $B \in S$, and $|B| = 2$.

1.54. For $A = \{1\}$ and $C = \{1, 2\}$, give an example of a set B such that $\mathcal{P}(A) \subset B \subset \mathcal{P}(C)$.

1.55. Give examples of two sets A and B such that

(a) $A \cap \mathcal{P}(A) \in B$
(b) $\mathcal{P}(A) \subseteq A \cup B$.

1.56. Which of the following sets are equal?

$A = \{n \in \mathbf{Z} : -4 \le n \le 4\}$ $\quad D = \{x \in \mathbf{Z} : x^3 = 4x\}$
$B = \{x \in \mathbf{N} : 2x + 2 = 0\}$ $\quad E = \{-2, 0, 2\}$
$C = \{x \in \mathbf{Z} : 3x - 2 = 0\}$

1.57. Let A and B be sets in some unknown universal set U. Suppose that $\overline{A} = \{3, 8, 9\}$, $A - B = \{1, 2\}$, $B - A = \{8\}$, and $A \cap B = \{5, 7\}$. Determine U, A, and B.

1.58. Let I denote the interval $[0, \infty)$. For each $r \in I$, define

$$\begin{aligned} A_r &= \{(x, y) \in \mathbf{R} \times \mathbf{R} : x^2 + y^2 = r^2\}, \\ B_r &= \{(x, y) \in \mathbf{R} \times \mathbf{R} : x^2 + y^2 \le r^2\}, \\ C_r &= \{(x, y) \in \mathbf{R} \times \mathbf{R} : x^2 + y^2 > r^2\}. \end{aligned}$$

(a) Determine $\bigcup_{r \in I} A_r$ and $\bigcap_{r \in I} A_r$.
(b) Determine $\bigcup_{r \in I} B_r$ and $\bigcap_{r \in I} B_r$.
(c) Determine $\bigcup_{r \in I} C_r$ and $\bigcap_{r \in I} C_r$.

1.59. Give an example of four sets A_1, A_2, A_3, A_4 such that $|A_i \cap A_j| = |i - j|$ for every two integers i and j with $1 \le i < j \le 4$.

1.60. (a) Give an example of two problems suggested by Exercise 1.59 (above).
(b) Solve one of the problems in (a).

1.61. Let $A = \{1, 2, 3\}$, $B = \{1, 2, 3, 4\}$, and $C = \{1, 2, 3, 4, 5\}$. For the sets S and T described below, explain whether $|S| < |T|$, $|S| > |T|$, or $|S| = |T|$.

(a) Let B be the universal set and let S be the set of all subsets X of B for which $|X| \ne |\overline{X}|$. Let T be the set of 2-element subsets of C.
(b) Let S be the set of all partitions of the set A and let T be the set of 4-element subsets of C.
(c) Let $S = \{(b, a) : b \in B, a \in A, a + b \text{ is odd}\}$ and let T be the set of all nonempty proper subsets of A.

1.62. Give an example of a set $A = \{1, 2, \ldots, k\}$ for a smallest $k \in \mathbf{N}$ having subsets A_1, A_2, A_3 such that $|A_i - A_j| = |A_j - A_i| = |i - j|$ for every two integers i and j with $1 \le i < j \le 3$.

2 Logic

In mathematics our goal is to seek the truth. Are there connections between two given mathematical concepts? If so, what are they? Under what conditions does an object possess a particular property? Finding answers to questions such as these is important, but we cannot be satisfied only with this. We must be certain that we are right and that our explanation for why we believe we are correct is convincing to others. The reasoning we use as we proceed from what we know to what we wish to show must be logical. It must make sense to others, not just to ourselves.

There is joint responsibility here, however. It is the writer's responsibility to use the rules of logic to give a valid and clear argument with enough details provided to allow the reader to understand what we have written and to be convinced. It is the reader's responsibility to know the basics of logic and to study the concepts involved sufficiently well so that he or she will not only be able to understand a well-presented argument but can decide as well whether it is valid. Consequently, both writer and reader must have some familiarity with logic.

Although it is possible to spend a great deal of time studying logic, we will present only what we actually need and will instead use the majority of our time putting what we learn into practice.

2.1 Statements

In mathematics we are constantly dealing with statements. By a **statement** we mean a declarative sentence or assertion that is true or false (but not both). Statements therefore declare or assert the truth of something. Of course, the statements in which we will be primarily interested deal with mathematics. For example, the sentences

The integer 3 is odd.
The integer 57 is prime.

are statements (only the first of which is true).

Every statement has a **truth value**, namely **true** (denoted by T) or **false** (denoted by F). We often use P, Q, and R to denote statements, or perhaps $P_1, P_2, \ldots, P_n$ if