

1. Describe the elements of the set $(\mathbb{Z} \times \mathbb{Q}) \cap \mathbb{R} \times \mathbb{N}$. Is this set countable or uncountable?
2. Let $A = \{\emptyset, \{\emptyset\}\}$. What is the cardinality of A ? Is $\emptyset \subset A$? Is $\emptyset \in A$? Is $\{\emptyset\} \subset A$? Is $\{\emptyset\} \in A$? Is $\{\emptyset, \{\emptyset\}\} \in A$?
3. List the elements of the set $A \times B$ where A is the set in the previous question and $B = \{1, 2\}$.
4. Suppose that A , B , and C are sets. Which of the following statements is true for all sets A , B , and C ? For each, either prove the statement or give a counterexample: $(A \cap B) \cup C = A \cap (B \cup C)$, $A \cap B \subseteq A \cup B$, if $A \subset B$ then $A \times A \subset B \times B$, $\overline{A \cap B \cap C} = \overline{A} \cup \overline{B} \cup \overline{C}$.
5. State the negation of each of the following statements:
 - There exists a natural number m such that $m^3 - m$ is not divisible by 3.
 - $\sqrt{3}$ is a rational number.
 - 1 is a negative integer.
 - 57 is a prime number.
6. Verify the following laws:
 - (a) Let P, Q and R are statements. Then, $P \wedge (Q \vee R)$ and $(P \wedge Q) \vee (P \wedge R)$ are logically equivalent.
 - (b) Let P and Q are statements. Then, $P \Rightarrow Q$ and $(\sim Q) \Rightarrow (\sim P)$ are logically equivalent.
7. Write the open statement $P(x, y)$: "**for all real x and y the value $(x - 1)^2 + (y - 3)^2$ is positive**" using quantifiers. Is the quantified statement true or false? Explain.
8. Prove that $3x + 7$ is odd if and only if x is even.
9. Prove that if a and b are positive numbers, the $\sqrt{ab} \leq \frac{a+b}{2}$. This is referred to as "Inequality between geometric and arithmetic mean."
10. Let A, B , and C be sets. Prove that $A \times (B \cap C) = (A \times B) \cap (A \times C)$.
11. Let A, B , and C be sets. Prove that $(A - B) \cap (A - C) = A - (B \cup C)$.
12. Suppose that x and y are real numbers. Prove that if $x + y$ is irrational, then x is irrational or y is irrational.
13. Let x be an irrational number. Prove that x^4 or x^5 is irrational.
14. Use a proof by contradiction to prove the following.
There exist no natural numbers m such that $m^2 + m + 3$ is divisible by 4.
15. Let a, b be distinct primes. Then $\log_a(b)$ is irrational.
16. Prove or disprove the statement: there exists an integer n such that $n^2 - 3 = 2n$.
17. Prove or disprove the statement: there exists a real number x such that $x^4 + 2 = 2x^2$.
18. Prove that there exists a unique real number x such that $x^3 + 2 = 2x$.
19. Disprove that statement: There exists integers a and b such that $a^2 + b^2 \equiv 3 \pmod{4}$.
20. Use induction to prove that $6|(n^3 + 5n)$ for all $n \geq 0$.
21. Use induction to prove that $1 \cdot 4 + 2 \cdot 7 + \cdots + n(3n + 1) = n(n + 1)^2$ for all $n \in \mathbb{N}$.
22. Use the Strong Principle of Mathematical Induction to prove that for each integer $n \geq 11$, there are nonnegative integers x and y such that $n = 4x + 5y$.
23. A sequence $\{a_n\}$ is defined recursively by $a_0 = 1$, $a_1 = -2$ and for $n \geq 1$,
$$a_{n+1} = 5a_n - 6a_{n-1}.$$
Prove that for $n \geq 0$,
$$a_n = 5 \times 2^n - 4 \times 3^n.$$
24. Suppose R is an equivalence relation on a set A . Prove or disprove that R^{-1} is an equivalence relation on A .
25. Consider the set $A = \{a, b, c, d\}$, and suppose R is an equivalence relation on A . If R contains the elements (a, b) and (b, d) , what other elements must it contain?
26. Let $A = \{a_1, a_2, a_3\}$ and $B = \{b_1, b_2\}$. Find a relation on $A \times B$ that is transitive and symmetric, but not reflexive.
27. Suppose A is a finite set and R is an equivalence relation on A .

- (a) Prove that $|A| \leq |R|$.
 (b) If $|A| = |R|$, what can you conclude about R ?
28. Consider the relation $R \subset \mathbb{Z}_4 \times \mathbb{Z}_6$ defined by
- $$R = \{(x \bmod 4, 3x \bmod 6) \mid x \in \mathbb{Z}\}.$$
- Prove that R is a function from \mathbb{Z}_4 to \mathbb{Z}_6 . Is R a bijective function?
29. Consider the relation $S \subset \mathbb{Z}_4 \times \mathbb{Z}_6$ defined by
- $$S = \{(x \bmod 4, 2x \bmod 6) \mid x \in \mathbb{Z}\}.$$
- Prove that S is not a function from \mathbb{Z}_4 to \mathbb{Z}_6 .
30. Suppose $f : A \rightarrow B$ and $g : X \rightarrow Y$ are bijective functions. Define a new function $h : A \times X \rightarrow B \times Y$ by $h(a, x) = (f(a), g(x))$. Prove that h is bijective.
31. Prove or disprove: Suppose $f : A \rightarrow B$ and $g : B \rightarrow C$ are functions. Then $g \circ f$ is bijective if and only if f is injective and g is surjective.
32. (*X points*) Let \mathbb{R}^+ denote the set of positive real numbers and let A and B be denumerable subsets of \mathbb{R}^+ . Define $C = \{x \in \mathbb{R} : -x/2 \in B\}$. Show that $A \cup C$ is denumerable.
33. Prove that the interval $(0, 1)$ is numerically equivalent to the interval $(0, +\infty)$.
34. Prove the following statement: A nonempty set S is **countable** if and only if there exists an injective function $g : S \rightarrow \mathbb{N}$.
35. Compute the greatest common divisor of 42 and 13 and then express the greatest common divisor as a linear combination of 42 and 13.
36. Let $a, b, c \in \mathbb{Z}$. Prove that if c is a common divisor of a and b , then c divides any linear combination of a and b .

37. Define the term “ p is a prime”. Then prove that if $a, p \in \mathbb{Z}$, p is prime, and p does not divide a , then $\gcd(a, p) = 1$.
38. The greatest common divisor of three integers a, b, c is the largest positive integer which divides all three. We denote this greatest common divisor by $\gcd(a, b, c)$. Assume that a and b are not both zero. Prove the following equation:

$$\gcd(a, b, c) = \gcd(\gcd(a, b), c).$$

39. By using the formal definition of the limit of the sequence, without assuming any propositions about limits, prove the following:

$$\lim_{n \rightarrow \infty} \frac{3n + 1}{n - 2} = 3.$$

40. By using the formal definition of the limit of the sequence, without assuming any propositions about limits, prove that

$$\lim_{n \rightarrow \infty} \frac{(-1)^n 3n + 1}{n - 2}$$

does not exist.

41. Let (a_n) be a sequence with positive terms such that $\lim_{n \rightarrow \infty} a_n = 1$. By using the formal definition of the limit of the sequence, prove the following:

$$\lim_{n \rightarrow \infty} \frac{3a_n + 1}{2} = 2.$$

42. (a) Use induction to prove

$$\frac{1}{2 \cdot 4} + \frac{1}{4 \cdot 6} + \cdots + \frac{1}{2n(2n+2)} = \frac{n}{4(n+1)}$$

for all $n \in \mathbb{N}$.

- (b) Prove $\sum_{k=1}^{\infty} \frac{1}{2k(2k+2)} = \frac{1}{4}$.