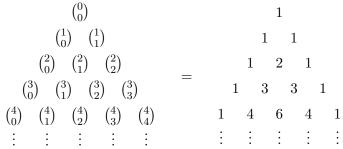
## Math 299

## Homework 9/9

In Supplement 9/9, we defined the choose number (or binomial coefficient)  $\binom{n}{k}$  to be the number of possible k-element subsets  $S \subset [n]$ , where  $[n] = \{1, 2, ..., n\}$ . We can put these numbers into an array called Pascal's Triangle (in China, Yang Hui's Triangle; in Persia, Khayyam's Triangle):



For example,  $\binom{4}{2} = 6$  counts the subsets  $S = \{1,2\}, \{1,3\}, \{1,4\}, \{2,3\}, \{2,4\}, \{3,4\}$ . We can compute the entries by the formula  $\binom{n}{k} = \frac{n(n-1)\cdots(n-k+1)}{k!}$ , but there is an easier way. It is a remarkable fact that each entry in the triangle is the sum of the two entries immediately above it (except for the edges  $\binom{n}{0} = \binom{n}{n} = 1$ ). For example, the next row will be:

$$\binom{5}{0} = 1, \quad \binom{5}{1} = \binom{4}{0} + \binom{4}{1} = 5, \quad \binom{5}{2} = \binom{4}{1} + \binom{4}{2} = 10, \quad \binom{5}{3} = \binom{4}{2} + \binom{4}{3} = 10, \quad \dots$$

In general, the recurrence formula is:

$$(*) \qquad \binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}.$$

**Problem 1.** Use the above recurrence to compute the  $\binom{6}{k}$  and  $\binom{7}{k}$  rows of the table.

**Problem 2.** Find the sum of each row:  $\binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{n}$ , for  $n = 0, 1, \dots, 7$ . Guess a general formula for this sum.

**Problem 3.** Prove your formula using a proposition from Supplement 9/9.

We can prove recurrence formula (\*) through the Bijection Principle:

- $\binom{n}{k} = |\mathcal{A}|$ , where  $\mathcal{A}$  is the set of all k-element subsets of [n].
- $\binom{n-1}{k-1} = |\mathcal{B}_1|$ , where  $\mathcal{B}_1$  is the set of all (k-1)-element subsets of [n-1].
- $\binom{n-1}{k} = |\mathcal{B}_2|$ , where  $\mathcal{B}_2$  is the set of all k-element subsets of [n-1].

Now if we find a bijection  $\phi : \mathcal{A} \to \mathcal{B}_1 \cup \mathcal{B}_2$ , then (since  $\mathcal{B}_1$  and  $\mathcal{B}_2$  are disjoint) this will show that  $|\mathcal{A}| = |\mathcal{B}_1| + |\mathcal{B}_2|$ , which is precisely the recurrence formula  $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$ .

The mapping is defined on k-element subsets of [n] by:  $\phi(S) = S' = S \setminus \{n\}$ , meaning we remove n from S if it is present, and leave S' = S otherwise. The result S' is a subset of [n-1] with either k-1 or k elements.

**Problem 4.** Illustrate the mapping  $\phi$  in the case of  $\binom{5}{3} = \binom{4}{2} + \binom{4}{3}$ . Make a left column of sets S in  $\mathcal{A}$ , a right column of sets S' in  $\mathcal{B}_1$  and  $\mathcal{B}_2$ , and draw an arrow from each S to the corresponding  $S' = \phi(S)$ .

**Problem 5.** Formally define the inverse mapping  $\psi : \mathcal{B}_1 \cup \mathcal{B}_2 \to \mathcal{A}$ . That is, given a subset  $S' \subset [n-1]$  with either k-1 or k elements, define the corresponding k-element  $S \subset [n]$ . Take several examples of S, S' that  $\psi$  and  $\phi$  are inverses (that they undo each other):-  $\psi(\phi(S)) = S$  and  $\phi(\psi(S')) = S'$ .