In Supplement 9/9, we defined the choose number (or binomial coefficient) $\binom{n}{k}$ to be the number of possible k-element subsets $S \subset [n]$, where $[n] = \{1, 2, \ldots, n\}$. We can put these numbers into an array called Pascal's Triangle (in China, Yang Hui's Triangle; in Persia, Khayyam's Triangle):

$$\begin{array}{ccccccccc} & & & & \binom{0}{0} & & & & \\ & & & \binom{1}{0} & & \binom{1}{1} & & & \\ & & \binom{2}{0} & & \binom{2}{1} & & \binom{2}{2} & & \\ & \binom{3}{0} & & \binom{3}{1} & & \binom{3}{2} & & \binom{3}{3} & \\ \binom{4}{0} & & \binom{4}{1} & & \binom{4}{2} & & \binom{4}{3} & & \binom{4}{4} \\ \vdots & & \vdots & & \vdots & & \vdots & & \vdots \end{array} \quad = \quad \begin{array}{ccccccccc} & & & & 1 & & & & \\ & & & 1 & & 1 & & & \\ & & 1 & & 2 & & 1 & & \\ & 1 & & 3 & & 3 & & 1 & \\ 1 & & 4 & & 6 & & 4 & & 1 \\ \vdots & & \vdots & & \vdots & & \vdots & & \vdots \end{array}$$

For example, $\binom{4}{2} = 6$ counts the subsets $S = \{1,2\}, \{1,3\}, \{1,4\}, \{2,3\}, \{2,4\}, \{3,4\}$. We can compute the entries by the formula $\binom{n}{k} = \frac{n(n-1)\cdots(n-k+1)}{k!}$, but there is an easier way. It is a remarkable fact that each entry in the triangle is the sum of the two entries immediately above it (except for the edges $\binom{n}{0} = \binom{n}{n} = 1$). For example, the next row will be:

$$\binom{5}{0} = 1, \quad \binom{5}{1} = \binom{4}{0} + \binom{4}{1} = 5, \quad \binom{5}{2} = \binom{4}{1} + \binom{4}{2} = 10, \quad \binom{5}{3} = \binom{4}{2} + \binom{4}{3} = 10, \ \ldots$$

In general, the recurrence formula is:

$$(*) \qquad \binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}.$$

Problem 1. Use the above recurrence to compute the $\binom{6}{k}$ and $\binom{7}{k}$ rows of the table.

Problem 2. Find the sum of each row: $\binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{n}$, for $n = 0, 1, \ldots, 7$. Guess a general formula for this sum.

Problem 3. Prove your formula using a proposition from Supplement 9/9.

The rest of this worksheet is not required for the homework. However, you may hand in solutions to Problems 4 and 5 as an Extra Credit assignment, if you wish.

We can prove recurrence formula $(*)$ through the Bijection Principle:

- $\binom{n}{k} = |\mathcal{A}|$, where $\mathcal{A}$ is the set of all k-element subsets of $[n]$.
- $\binom{n-1}{k-1} = |\mathcal{B}_1|$, where $\mathcal{B}_1$ is the set of all $(k-1)$-element subsets of $[n-1]$.
- $\binom{n-1}{k} = |\mathcal{B}_2|$, where $\mathcal{B}_2$ is the set of all k-element subsets of $[n-1]$.

Now if we find a bijection $\phi : \mathcal{A} \to \mathcal{B}_1 \cup \mathcal{B}_2$, then (since $\mathcal{B}_1$ and $\mathcal{B}_2$ are disjoint) this will show that $|\mathcal{A}| = |\mathcal{B}_1| + |\mathcal{B}_2|$, which is precisely the recurrence formula $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$.

The mapping is defined on k-element subsets of $[n]$ by: $\phi(S) = S' = S \setminus \{n\}$, meaning we remove n from S if it is present, and leave $S' = S$ otherwise. The result S' is a subset of $[n-1]$ with either $k-1$ or k elements.

Problem 4. Illustrate the mapping ϕ in the case of $\binom{5}{3} = \binom{4}{2} + \binom{4}{3}$. Make a left column of sets S in $\mathcal{A}$, a right column of sets S' in $\mathcal{B}_1$ and $\mathcal{B}_2$, and draw an arrow from each S to the corresponding $S' = \phi(S)$.

Problem 5. Formally define the inverse mapping $\psi : \mathcal{B}_1 \cup \mathcal{B}_2 \to \mathcal{A}$. That is, given a subset $S' \subset [n{-}1]$ with either $k{-}1$ or k elements, define the corresponding k-element $S \subset [n]$. Take several examples of S, S' that ψ and ϕ are iinverses (that they undo each other):- $\psi(\phi(S)) = S$ and $\phi(\psi(S')) = S'$.