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1. (a) Apply A3 with $c = a$ to get $a^2 \leq ab$. Since $0 \leq a$ and $a \leq b$, A1 implies that $0 \leq b$. This means that we can apply A3 again but with $c = b$ to get $ab \leq b^2$. By A1 we know that $a^2 \leq ab$ and $ab \leq b^2$ implies $a^2 \leq b^2$, as desired. □

   (b) We have $a \leq 0$, so by A2 we can add $-a$ to both sides to get $0 \leq -a$. □

   (c) By the previous problem we have $0 \leq -a$, so we can apply A3 with $c = -a$ to get $-b^2 \leq -ba$. By A1 we know that $b \leq 0$, so by the previous problem we get $0 \leq -b$. Now apply A3 with $c = -b$ to get $-ba \leq -a^2$, and hence $-b^2 \leq -a^2$. By A2 we can add $a^2$ and $b^2$ to both sides to get $a^2 \leq b^2$, as desired. □

   (d) We prove this by cases. If $0 \leq b$, then apply the Problem (a) to get $0^2 \leq b^2$. Since $0^2 = 0$, we are done in this case. If $b \leq 0$, then apply the third problem to get $0 \leq b^2$. □

   (e) By the previous problem we know $0 \leq (a - b)^2$. Expanding out the right-hand side gives $0 \leq a^2 + b^2 - 2ab$. Apply A2 with $c = 2ab$ to get $2ab \leq a^2 + b^2$. Now apply A3 with $c = 1/2$. □

   (f) Apply the previous problem replacing $a$ with $\delta a$ and $b$ with $\delta^{-1} b$. Notice that since $\delta \neq 0$, we have that $\delta^{-1}$ is a real number, and so the previous problem applies. □

   (g) If $a = b$, then $ab = a^2$ and $\frac{1}{2}(a^2 + b^2) = a^2$ as well. Conversely, suppose $ab = \frac{1}{2}(a^2 + b^2)$. This gives $2ab = a^2 + b^2$, and hence $0 = a^2 - 2ab + b^2$. The right-hand side factors to give $0 = (a - b)^2$. Taking the square root gives $0 = a - b$, which implies $a = b$. □

   (h) Apply Problem (e), but replace $a$ with $\sqrt{a}$ and $b$ with $\sqrt{b}$. Notice that since $a$ and $b$ are non-negative, their square roots are real numbers and so Problem (e) does indeed apply. □

2. We prove this using contrapositive. So assume that $x$ and $y$ do not have the same parity, and we want to show that $3x + 5y$ is odd. Since $x$ and $y$ do not have the same parity, there are two cases to consider.

   *Case 1: $x$ is odd and $y$ is even.*

In this case we can write $x = 2n + 1$ and $y = 2m$ for some integers $n$ and $m$. Then

$$\begin{aligned} 3x + 5y &= 3(2n + 1) + 5(2m) \\ &= 6n + 3 + 10m \\ &= 2(3n + 5m + 1) + 1. \end{aligned}$$

This is odd, so we are done in this case.

*Case 2: $x$ is even and $y$ is odd.*

In this case we can write $x = 2n$ and $y = 2m + 1$ for some integers $n$ and $m$. Then

$$\begin{aligned} 3x + 5y &= 3(2n) + 5(2m + 1) \\ &= 6n + 10m + 5 \\ &= 2(3n + 5m + 2) + 1. \end{aligned}$$

This is odd, so we are done in this case as well. □

3. To prove the statement, we will use the following lemma.

**Lemma.** If $z \in \mathbb{R}$, then $-z \leq |z|$.

*Proof.* We prove the lemma using cases. If $z \geq 0$, then $-z \leq 0$ and $z = |z|$. Then $-z \leq 0 \leq z = |z|$, so we are done in this case. If $z \leq 0$, then $|z| = -z$ and so we are done in this case as well. □

Now to prove the statement, assume $x + y < 0$. Then $|x + y| = -x + (-y)$. By the lemma we have $-x \leq |x|$ and $-y \leq |y|$, so

$$|x + y| \leq |x| + |y|,$$

as desired. □