Review for MTH 234 Exam 1.

- Sections 12.1-12.6.
- 50 minutes.
- Problems similar to homework problems.
- No calculators, no notes, no books, no phones.
- The exam covers:
  - Cartesian coordinates in space (12.1).
  - Vectors in space (12.2).
  - The dot product (12.3).
  - The cross product (12.4).
  - Lines and planes in space (12.5).
  - Cylinders and quadratic surfaces (12.6).

Section 12.1

Example

Find the equation and describe the region given by the intersection of the radius 5 sphere centered at the origin and the horizontal plane containing the point $P = (1, 1, 3)$.

Solution: The equation of the sphere, $x^2 + y^2 + z^2 = 25$.

Horizontal plane, $z = z_0$. Contains $P = (1, 1, 3)$, that is, $z = 3$.

The intersection is:

$$x^2 + y^2 + 3^2 = 25 \quad \Rightarrow \quad x^2 + y^2 = 25 - 9 = 16 = 4^2.$$

This is a circle radius $r = 4$ centered at $x = 0$, $y = 0$, $z = 3$, contained in the horizontal plane $z = 3$. $\triangleleft$

Section 12.3

Example

Consider the vectors $\mathbf{v} = 2\,\mathbf{i} - 2\,\mathbf{j} + \mathbf{k}$ and $\mathbf{w} = \mathbf{i} + 2\,\mathbf{j} - \mathbf{k}$.

(a) Compute $\mathbf{v} \cdot \mathbf{w}$.

Solution: Recall: $\mathbf{v} \cdot \mathbf{w} = v_x w_x + v_y w_y + v_z w_z$.

$$\mathbf{v} \cdot \mathbf{w} = \langle 2, -2, 1 \rangle \cdot \langle 1, 2, -1 \rangle = 2 - 4 - 1 \quad \Rightarrow \quad \mathbf{v} \cdot \mathbf{w} = -3.$$

$\lhd$

(b) Find the cosine of the angle between $\mathbf{v}$ and $\mathbf{w}$.

Solution: Recall: $\mathbf{v} \cdot \mathbf{w} = |\mathbf{v}|\,|\mathbf{w}|\,\cos(\theta)$.

$$|\mathbf{v}| = \sqrt{4 + 4 + 1} = 3, \quad |\mathbf{w}| = \sqrt{1 + 4 + 1} = \sqrt{6}.$$

$$\cos(\theta) = \frac{\mathbf{v} \cdot \mathbf{w}}{|\mathbf{v}|\,|\mathbf{w}|} = \frac{-3}{3\sqrt{6}} \quad \Rightarrow \quad \cos(\theta) = -\frac{1}{\sqrt{6}}. \qquad \lhd$$

Section 12.3

Example

(a) Find a unit vector opposite to $\mathbf{v} = \mathbf{i} - 2\,\mathbf{j} + \mathbf{k}$.

Solution: The vector is $\mathbf{u} = -\dfrac{\mathbf{v}}{|\mathbf{v}|}$. Since,

$$|\mathbf{v}| = \sqrt{1 + 4 + 1} = \sqrt{6}, \quad \Rightarrow \quad \mathbf{u} = -\frac{1}{\sqrt{6}}\langle 1, -2, 1 \rangle.$$

(b) Find $|\mathbf{u} - 2\mathbf{v}|$, where $\mathbf{u} = 3\,\mathbf{i} + 2\,\mathbf{j} + \mathbf{k}$, and $\mathbf{v} = \mathbf{i} - 2\,\mathbf{j} + \mathbf{k}$.

Solution: First find $\mathbf{u} - 2\mathbf{v}$, then find $|\mathbf{u} - 2\mathbf{v}|$.

$$\mathbf{u} - 2\mathbf{v} = \langle 3, 2, 1 \rangle - 2\langle 1, -2, 1 \rangle = \langle 1, 6, -1 \rangle$$

$$|\mathbf{u} - 2\mathbf{v}| = \sqrt{1 + 36 + 1} \quad \Rightarrow \quad |\mathbf{u} - 2\mathbf{v}| = \sqrt{38}. \qquad \lhd$$

Section 12.3

Example

Find the vector projection of vector $\mathbf{v} = -\mathbf{i} + 3\mathbf{j} - 3\mathbf{k}$ onto vector $\mathbf{u} = \mathbf{i} - \mathbf{j} + 2\mathbf{k}$.

Solution: Recall: $\mathbf{P}_u(\mathbf{v}) = \left(\frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{u}|}\right) \frac{\mathbf{u}}{|\mathbf{u}|}$.

$$\mathbf{u} \cdot \mathbf{v} = \langle -1, 3, -3 \rangle \cdot \langle 1, -1, 2 \rangle = -1 - 3 - 6 \quad \Rightarrow \quad \mathbf{u} \cdot \mathbf{v} = -10.$$

Since $|\mathbf{u}| = \sqrt{1^2 + (-1)^2 + 2^2} = \sqrt{6}$, we obtain that

$$\mathbf{P}_u(\mathbf{v}) = \left(\frac{-10}{\sqrt{6}}\right) \frac{1}{\sqrt{6}} \langle 1, -1, 2 \rangle.$$

We conclude that $\mathbf{P}_u(\mathbf{v}) = -\frac{5}{3} \langle 1, -1, 2 \rangle$. $\triangleleft$

Section 12.4

Example

Find a unit vector $\mathbf{u}$ normal to both $\mathbf{v} = \langle 6, 2, -3 \rangle$ and $\mathbf{w} = \langle -2, 2, 1 \rangle$.

Solution: A solution is a vector proportional to $\mathbf{v} \times \mathbf{w}$.

$$\mathbf{v} \times \mathbf{w} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 6 & 2 & -3 \\ -2 & 2 & 1 \end{vmatrix} = (2+6)\mathbf{i} - (6-6)\mathbf{j} + (12+4)\mathbf{k} = \langle 8, 0, 16 \rangle.$$

Since we look for a unit vector, the calculation is simpler with $\langle 1, 0, 2 \rangle$ instead of $\langle 8, 0, 16 \rangle$.

$$\mathbf{u} = \frac{\langle 1, 0, 2 \rangle}{|\langle 1, 0, 2 \rangle|} \quad \Rightarrow \quad \mathbf{u} = \frac{1}{\sqrt{5}} \langle 1, 0, 2 \rangle. \quad \triangleleft$$

Section 12.4

Example

Find the area of the parallelogram formed by $\mathbf{v} = \langle 6, 2, -3 \rangle$ and $\mathbf{w} = \langle -2, 2, 1 \rangle$, given in the example above.

Solution:

Recall: The area of the parallelogram formed by the vectors $\mathbf{v}$ and $\mathbf{w}$ is $A = |\mathbf{v} \times \mathbf{w}|$.

Since $\mathbf{v} \times \mathbf{w} = \langle 8, 0, 16 \rangle$, then

$$A = |\mathbf{v} \times \mathbf{w}| = \sqrt{8^2 + 16^2} = \sqrt{8^2 + 8^2\, 2^2} = \sqrt{8^2(1 + 4)}.$$

We conclude that

$$A = 8\sqrt{5}. \qquad \triangleleft$$

Triple product (Only if covered by your instructor.)

Example

Find the volume of the parallelepiped determined by the vectors $\mathbf{u} = \langle 6, 3, -1 \rangle$, $\mathbf{v} = \langle 0, 1, 2 \rangle$, and $\mathbf{w} = \langle 4, -2, 5 \rangle$.

Solution: We need to compute the triple product $\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})$. We must start with the cross product.

$$\mathbf{v} \times \mathbf{w} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 1 & 2 \\ 4 & -2 & 5 \end{vmatrix} = \langle (5 + 4), -(0 - 8), (0 - 4) \rangle.$$

We obtain $\mathbf{v} \times \mathbf{w} = \langle 9, 8, -4 \rangle$. The triple product is

$$\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = \langle 6, 3, -1 \rangle \cdot \langle 9, 8, -4 \rangle = 54 + 24 + 4 = 82.$$

Since $V = |\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})|$, we obtain $V = 82$. $\triangleleft$

Section 12.5

Example

Does the line given by $\mathbf{r}(t) = \langle 0, 1, 1\rangle + \langle 1, 2, 3\rangle t$ intersects the plane $2x + y - z = 1$? If "yes", then find the intersection point.

Solution: First, find the parametric equation of the line,

$$x(t) = t, \quad y(t) = 1 + 2t, \quad z(t) = 1 + 3t.$$

Then replace $x(t)$, $y(t)$, and $z(t)$ above in the equation of the plane $2x + y - z = 1$.

If there is a solution for t, then there is an intersection between the line and the plane. Let us find that out,

$$2t + (1 + 2t) - (1 + 3t) = 1 \quad \Rightarrow \quad t = 1.$$

So, the intersection of the line and the plane is the point with coordinates $x = 1$, $y = 3$, $z = 4$, that is, $P = (1, 3, 4)$. $\lhd$

Section 12.5

Example

Does the line given by $\mathbf{r}(t) = \langle 0, 1, 1\rangle + \langle 1, 2, 3\rangle t$ intersects the plane $7x - 2y - z = 1$? If "yes", then find the intersection point.

Solution: First, find the parametric equation of the line,

$$x(t) = t, \quad y(t) = 1 + 2t, \quad z(t) = 1 + 3t.$$

Then replace $x(t)$, $y(t)$, and $z(t)$ above in the equation of the plane $7x - 2y - z = 1$. In this case we get

$$7t - 2(1 + 2t) - (1 + 3t) = 1 \quad \Rightarrow \quad -3 = 1, \quad \text{No solution.}$$

So, there is no intersection between the line and the plane. $\lhd$

Notice: The line is parallel and not contained in the plane.

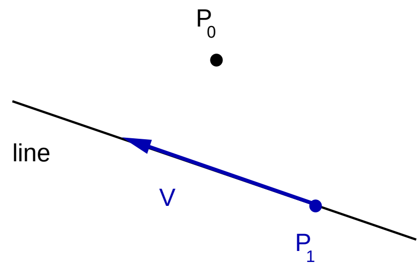
$$\langle 1, 2, 3\rangle \cdot \langle 7, -2, -3\rangle = 0, \quad \text{and} \quad P = (0, 1, 1) \notin \text{The Plane}.$$

Section 12.5

Example

Find the equation for the plane that contains the point $P_0 = (1, 2, 3)$ and the line $x = -2 + t$, $y = t$, $z = -1 + 2t$.

Solution:



The vector equation of the line is $\mathbf{r}(t) = \langle -2, 0, -1 \rangle + t\,\langle 1, 1, 2 \rangle$.
A vector tangent to the line, and so to the plane, is $\mathbf{v} = \langle 1, 1, 2 \rangle$.

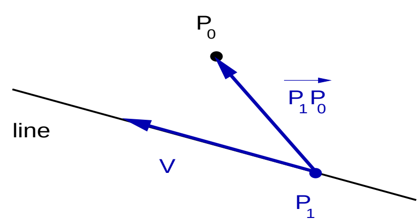
The point $P_0 = (1, 2, 3)$ is in the plane. The line is in the plane, hence $P_1 = (-2, 0, -1)$ is in the plane.
Then a second vector tangent to the plane is $\overrightarrow{P_1P_0} = \langle 3, 2, 4 \rangle$.

Section 12.5

Example

Find the equation for the plane that contains the point $P_0 = (1, 2, 3)$ and the line $x = -2 + t$, $y = t$, $z = -1 + 2t$.

Solution:



Recall: $\mathbf{v} = \langle 1, 1, 2 \rangle$, $\overrightarrow{P_1P_0} = \langle 3, 2, 4 \rangle$.
The normal to the plane is
$\mathbf{n} = \mathbf{v} \times \overrightarrow{P_1P_0}$.

$$\mathbf{n} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & 2 \\ 3 & 2 & 4 \end{vmatrix} = \langle (4-4), -(4-6), (2-3) \rangle \quad \Rightarrow \quad \mathbf{n} = \langle 0, 2, -1 \rangle.$$

The plane normal to $\langle 0, 2, -1 \rangle$ containing $P_0 = (1, 2, 3)$ is given by

$$0\,(x-1) + 2(y-2) - (z-3) = 0 \quad \Rightarrow \quad 2y - z = 1. \qquad \triangleleft$$

Section 12.5

Example

Find the equation of the plane that containing the points $P = (1,1,1)$, $Q = (1,-1,1)$, and $R = (0,0,2)$.

Solution: Find two vectors tangent to the plane: $\overrightarrow{PQ}$, $\overrightarrow{PR}$.

$$\overrightarrow{PQ} = \langle 0,-2,0 \rangle, \qquad \overrightarrow{PR} = \langle -1,-1,1 \rangle.$$

The normal vector to the plane is $\mathbf{n} = \overrightarrow{PQ} \times \overrightarrow{PR}$.

$$\vec{PQ} \times \vec{PR} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & -2 & 0 \\ -1 & -1 & 1 \end{vmatrix} = (-2-0)\mathbf{i} - (0-0)\mathbf{j} + (0-2)\mathbf{k},$$

that is, $\mathbf{n} = \langle -2, 0, -2 \rangle$. A point in the plane is $R = (0,0,2)$. The equation of the plane is

$$-2(x-0) + 0(y-0) - 2(z-2) = 0 \quad \Rightarrow \quad x + z = 2. \quad \triangleleft$$

Section 12.5

Example

Find the equation of the plane parallel to $x - 2y + 3z = 1$ and containing the center of the sphere $x^2 + 2x + y^2 + z^2 - 2z = 0$.

Solution: Recall: Planes are parallel iff their normal are parallel.
We choose the normal vector $\mathbf{n} = \langle 1, -2, 3 \rangle$.
We need to find the center of the sphere. We complete squares:

$$0 = x^2 + 2x + y^2 + z^2 - 2z = (x^2 + 2x + 1) - 1 + y^2 + (z^2 - 2z + 1) - 1$$

$$0 = (x+1)^2 + y^2 + (z-1)^2 - 2 \quad \Rightarrow \quad (x+1)^2 + y^2 + (z-1)^2 = 2.$$

Therefore, the center of the sphere is at $P_0 = (-1, 0, 1)$.
The equation of the plane is

$$(x+1) - 2(y-0) + 3(z-1) = 0 \quad \Rightarrow \quad x - 2y + 3z = 2. \quad \triangleleft$$

Example

Find the angle between the planes $2x - 3y + 2z = 1$ and $x + 2y + 2z = 5$.

Solution: The angle between planes is the angle between their normal vectors. The normal vectors are

$$\mathbf{n} = \langle 2, -3, 2 \rangle, \qquad \mathbf{N} = \langle 1, 2, 2 \rangle.$$

Use the dot dot product to find the cosine of the angle θ between these vectors;

$$\cos(\theta) = \frac{\mathbf{n} \cdot \mathbf{N}}{|\mathbf{n}|\,|\mathbf{N}|}.$$

But $\mathbf{n} \cdot \mathbf{N} = 2 - 6 + 4 = 0$, we conclude that $\mathbf{n} \perp \mathbf{N}$.

The planes are perpendicular, the angle is $\theta = \pi/2$. $\triangleleft$

Example

Find the vector equation for the line of intersection of the planes $2x - 3y + 2z = 1$ and $x + 2y + 2z = 5$.

Solution: First, find a vector $\mathbf{v}$ tangent to both planes. Then, find a point in the intersection.

Since vector $\mathbf{v}$ must belong to both planes, $\mathbf{v} \perp \mathbf{n} = \langle 2, -3, 2 \rangle$ and $\mathbf{v} \perp \mathbf{N} = \langle 1, 2, 2 \rangle$. We choose

$$\mathbf{v} = \mathbf{n} \times \mathbf{N} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -3 & 2 \\ 1 & 2 & 2 \end{vmatrix} = \langle (-6 - 4), -(4 - 2), (4 + 3) \rangle.$$

So, $\mathbf{v} = \langle -10, -2, 7 \rangle$.

Section 12.5

Example

Find the vector equation for the line of intersection of the planes $2x - 3y + 2z = 1$ and $x + 2y + 2z = 5$.

Solution: Recall $\mathbf{v} = \langle -10, -2, 7 \rangle$. Now find a point in the intersection of the planes.

From the first plane we compute z as follows: $2z = 1 - 2x + 3y$. Introduce this equation for $2z$ into the second plane:

$$x + 2y + (1 - 2x + 3y) = 5 \quad \Rightarrow \quad -x + 5y = 4.$$

We need just one solution. Choose: $y = 0$, then $x = -4$, and this implies $z = 9/2$. A point in the intersection of the planes is $P_0 = (-4, 0, 9/2)$. The vector equation of the line is:

$$\mathbf{r}(t) = \langle -4, -0, 9/2 \rangle + t \langle -10, -2, 7 \rangle.$$

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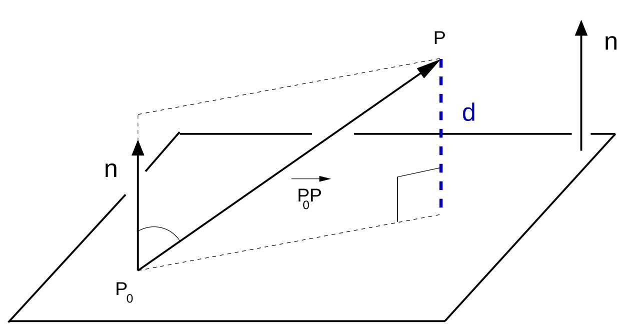
Distance formula from a point to a plane

Theorem

The distance d from a point P to a plane containing P_0 with normal vector $\mathbf{n}$ is the shortest distance from P to any point in the plane, and is given by the expression

$$d = \frac{|(\overrightarrow{P_0P}) \cdot \mathbf{n}|}{|\mathbf{n}|}.$$



Distance formula from a point to a plane

Example

Find the distance from the point $P = (1, 2, 3)$ to the plane $x - 3y + 2z = 4$.

Solution: We need to find a point P_0 on the plane and its normal vector $\mathbf{n}$. Then, use the formula $d = |(\overrightarrow{P_0P}) \cdot \mathbf{n}|/|\mathbf{n}|$.

To find a point on the plane: for example, if $y = 0$, $z = 0$, then $x = 4$. That is, $P_0 = (4, 0, 0)$.

The normal vector is in the plane equation: $\mathbf{n} = \langle 1, -3, 2 \rangle$.

We now compute $\overrightarrow{P_0P} = \langle -3, 2, 3 \rangle$. Then,

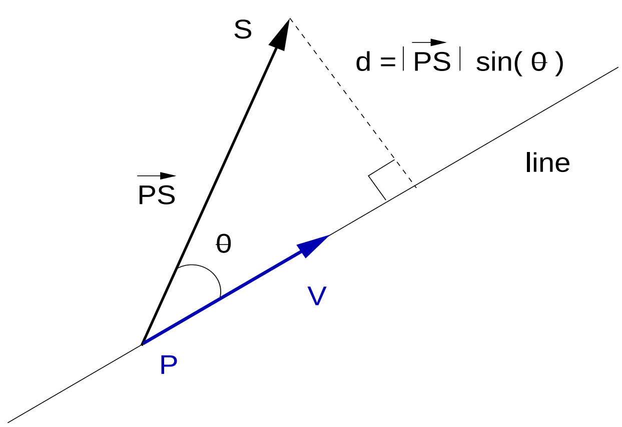
$$d = \frac{|-3 - 6 + 6|}{\sqrt{1 + 9 + 4}} \quad \Rightarrow \quad d = \frac{3}{\sqrt{14}}. \qquad \triangleleft$$

Distance from a point to a line in space

Theorem

The distance from a point S in space to a line through the point P with tangent vector $\mathbf{v}$ is given by

$$d = \frac{|\overrightarrow{PS} \times \mathbf{v}|}{|\mathbf{v}|}.$$

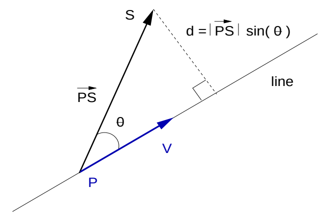


Distance from a point to a line in space

Example

Find the distance from the point $S = (1, 2, 1)$ to the line

$x = 2 - t, \quad y = -1 + 2t, \quad z = 2 + 2t.$



Solution:
First we need to compute the vector equation of the line above.

The vector components are the numbers that multiply t.
This line has tangent vector $\mathbf{v} = \langle -1, 2, 2 \rangle$.

To find a point in the line, just evaluate it at $t = 0$.
This line contains the vector $P = (2, -1, 2)$.

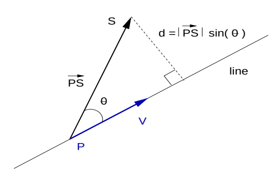
Therefore, $\overrightarrow{PS} = \langle -1, 3, -1 \rangle$.

Distance from a point to a line in space

Example

Find the distance from the point $S = (1, 2, 1)$ to the line

$x = 2 - t, \quad y = -1 + 2t, \quad z = 2 + 2t.$



Solution: $P = (2, -1, 2)$, $\mathbf{v} = \langle -1, 2, 2 \rangle$, and $\overrightarrow{PS} = \langle -1, 3, -1 \rangle$.
Since $d = |\overrightarrow{PS} \times \mathbf{v}| / |\mathbf{v}|$, we need to compute:

$$\overrightarrow{PS} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -1 & 3 & -1 \\ -1 & 2 & 2 \end{vmatrix} = (6 + 2)\,\mathbf{i} - (-2 - 1)\,\mathbf{j} + (-2 + 3)\,\mathbf{k},$$

that is, $\overrightarrow{PS} \times \mathbf{v} = \langle 8, 3, 1 \rangle$. We then compute the lengths:

$$|\overrightarrow{PS} \times \mathbf{v}| = \sqrt{64 + 9 + 1} = \sqrt{74}, \quad |\mathbf{v}| = \sqrt{1 + 4 + 4} = 3.$$

The distance from S to the line is $d = \sqrt{74}/3$. $\triangleleft$

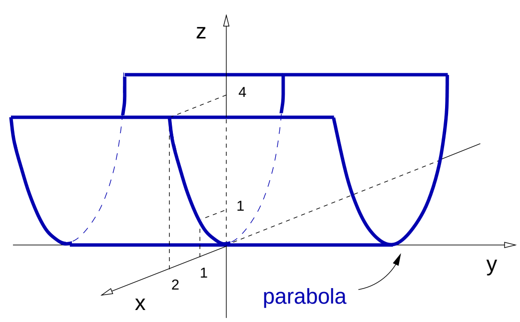
Section 12.6

- Cylinders.
- Quadratic surfaces:
  - Spheres, $\frac{x^2}{r^2} + \frac{y^2}{r^2} + \frac{z^2}{r^2} = 1.$
  - Ellipsoids, $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1.$
  - Cones, $\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 0.$
  - Hyperboloids, $\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1, \qquad -\frac{x^2}{a^2} - \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1.$
  - Paraboloids, $\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z}{c} = 0.$
  - Saddles, $\frac{x^2}{a^2} - \frac{y^2}{b^2} - \frac{z}{c} = 0.$

Section 12.6

Example

Find the equation of the cylinder given in the picture.



Solution:

The generating curve is a parabola on planes with constant y.

This parabola contains the points $(0,0,0)$, $(1,0,1)$, and $(2,0,4)$.

Since three points determine a unique parabola and $z = x^2$ contains these points, then at $y = 0$ the generating curve is $z = x^2$.

The cylinder equation does not contain the coordinate y. Hence,

$$z = x^2, \qquad y \in \mathbb{R}.$$

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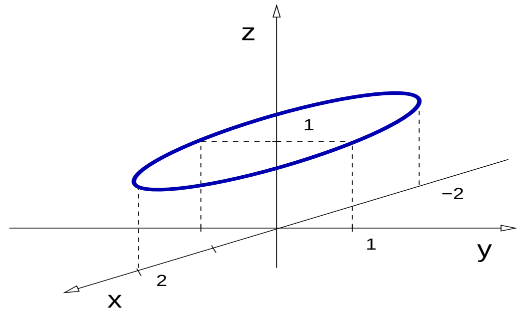
Section 12.6

Example

Graph the cone, $z = +\sqrt{\frac{x^2}{2^2} + y^2}$.

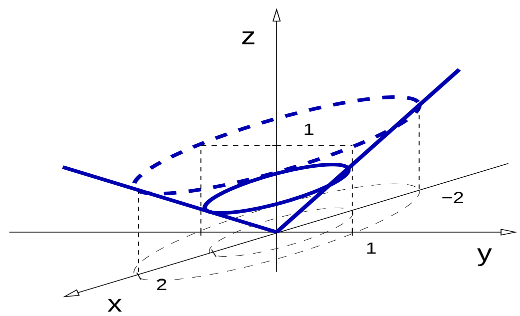
Solution:

On the plane $z = 1$ we have the ellipse $\dfrac{x^2}{2^2} + y^2 = 1$.



On the plane $z = z_0 > 0$ we have the ellipse $\dfrac{x^2}{2^2} + y^2 = z_0^2$, that is,

$$\frac{x^2}{2^2 z_0^2} + \frac{y^2}{z_0^2} = 1. \qquad \triangleleft$$



Section 12.6

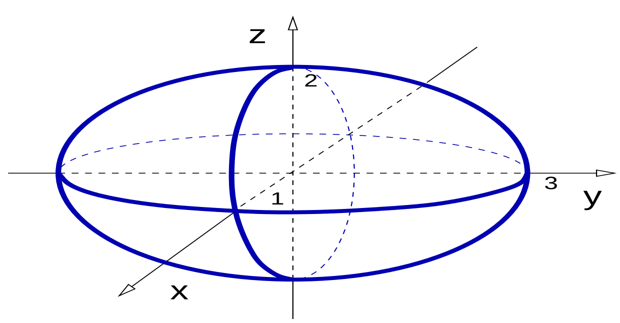
Example

Sketch the surface $36x^2 + 4y^2 + 9z^2 = 36$.

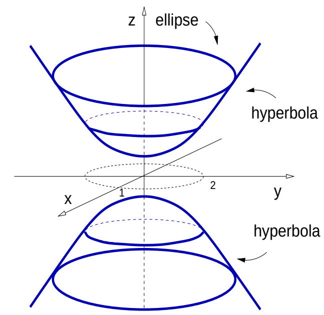
Solution: We first rewrite the equation above in the standard form

$$x^2 + \frac{4}{36}\,y^2 + \frac{9}{36}\,z^2 = 1 \quad \Leftrightarrow \quad x^2 + \frac{y^2}{3^2} + \frac{z^2}{2^2} = 1.$$

This is the equation of an ellipsoid with principal radius of length 1, 3, and 2 on the x, y and z axis, respectively. Therefore



Hyperboloids



Example

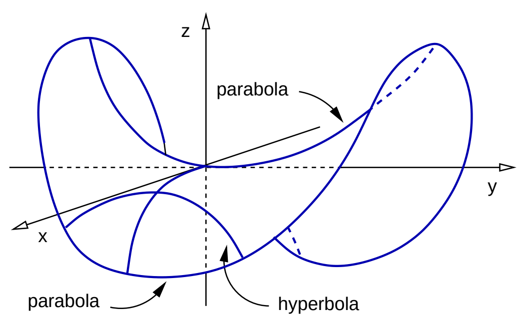
Graph the hyperboloid $-x^2 - \frac{y^2}{2^2} + z^2 = 1.$

Solution:

Find the intersection of the surface with horizontal and vertical planes. Then combine all these results into a qualitative graph.

- On horizontal planes, $z = z_0$, with $|z_0| > 1$, we obtain ellipses $x^2 + \frac{y^2}{2^2} = -1 + z_0^2.$
- On vertical planes, $y = y_0$, we obtain hyperbolas $-x^2 + z^2 = 1 + \frac{y_0^2}{2^2}.$
- On vertical planes, $x = x_0$, we obtain hyperbolas $-\frac{y^2}{2^2} + z^2 = 1 + x_0^2.$ ◁

Saddles



Example

Graph the saddle

$$z = -x^2 + \frac{y^2}{2^2}.$$

Solution:

Find the intersection of the surface with horizontal and vertical planes. Then combine all these results into a qualitative graph.

- On planes, $z = z_0$, we obtain hyperbolas $-x^2 + \frac{y^2}{2^2} = z_0.$
- On planes, $y = y_0$, we obtain parabolas $z = -x^2 + \frac{y_0^2}{2^2}.$
- On planes, $x = x_0$, we obtain parabolas $z = -x_0^2 + \frac{y^2}{2^2}.$