The mass of spacelike hypersurfaces in asymptotically anti-de Sitter space-times

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Abstract

We give a Hamiltonian definition of mass for spacelike hypersurfaces in space-times with metrics which are asymptotic to the anti-de Sitter one, or to a class of generalizations thereof. We show that our definition provides a geometric invariant for a spacelike hypersurface embedded in a space-time.

Contents

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1 Introduction

Let $\mathscr{S}$ be an n-dimensional spacelike hypersurface in a $n+1$-dimensional Lorentzian space-time $(\mathscr{M}, g)$, $n \geq 2$. Suppose that $\mathscr{M}$ contains an open set $\mathscr{U}$ with a global time coordinate t (with range not necessarily equal to $\mathbb{R}$), as well as a global "radial" coordinate $r \in [R, \infty)$, leading to local coordinate systems (t, r, v^A), with (v^A) — local coordinates on some compact $n-1$ dimensional manifold M. We further require that $\mathscr{S} \cap \mathscr{U} = \{t = 0\}$. Assume that the metric g approaches (as r tends to infinity, in a sense which is made precise in Section 2 below) a background metric b of the form

$$b = -\Big(\tfrac{r^2}{\ell^2} + k\Big)dt^2 + \frac{1}{\frac{r^2}{\ell^2} + k}dr^2 + r^2 h \ , \tag{1.1}$$

where h is an r-independent Riemannian metric on M, while k and ℓ are constants[1]. Suppose further that g satisfies the vacuum Einstein equations with a cosmological constant

$$R_{\mu\nu} - \frac{g^{\alpha\beta}R_{\alpha\beta}}{2}g_{\mu\nu} = -\Lambda g_{\mu\nu} \ , \qquad \Lambda = -\tfrac{n(n-1)}{2\ell^2} \ , \tag{1.2}$$

similarly for b. (The existence of a large family of such g's follows from the work in [18, 24].) A Hamiltonian analysis (following [10], and discussed in some more detail in Appendix A; see also [16, Section 5]) leads to the following expression for the Hamiltonian associated to the flow of a vector field X, assumed to be a Killing vector field of the background b:[2]

$$m(\mathscr{S}, g, b, X) \ = \ \frac{1}{2}\int_{\partial\mathscr{S}} \mathbb{U}^{\alpha\beta} dS_{\alpha\beta} \ , \tag{1.3}$$

$$\mathbb{U}^{\nu\lambda} \ = \ \mathbb{U}^{\nu\lambda}{}_{\beta}X^{\beta} + \frac{1}{8\pi}\left(\sqrt{|\det g_{\rho\sigma}|}\ g^{\alpha[\nu} - \sqrt{|\det b_{\rho\sigma}|}\ b^{\alpha[\nu}\right) X^{\lambda]}{}_{;\alpha} \ , \tag{1.4}$$

$$\mathbb{U}^{\nu\lambda}{}_{\beta} \ = \ \frac{2|\det b_{\mu\nu}|}{16\pi\sqrt{|\det g_{\rho\sigma}|}} g_{\beta\gamma}(e^2 g^{\gamma[\nu}g^{\lambda]\kappa})_{;\kappa} \ , \tag{1.5}$$

$$e \ = \ \sqrt{|\det g_{\rho\sigma}|}/\sqrt{|\det b_{\mu\nu}|} \ . \tag{1.6}$$

[1] A warped product form of the metric, with the factor in front of h not being constant, together with the Einstein equations (1.2), force g_{rr} and g_{tt} to have the form (1.1) in an appropriate coordinate system [8], with k being a function of r which approaches a constant as r tends to infinity. Further h itself has to satisfy the Einstein equation (1.2) with Λ replaced by an appropriate constant. Some metrics slightly more general than (1.1) will be considered in the body of the paper.

[2] The integral over $\partial\mathscr{S}$ should be understood by a limiting process, as the limit as R tends to infinity of integrals over the sets $t = 0$, $r = R$. $dS_{\alpha\beta}$ is defined as $\frac{\partial}{\partial x^\alpha}\rfloor\frac{\partial}{\partial x^\beta}\rfloor dx^0 \wedge \cdots \wedge dx^n$, with $\rfloor$ denoting contraction; g stands for the space-time metric unless explicitly indicated otherwise. Further, a semicolon denotes covariant differentiation *with respect to the background metric b*.

(The question of convergence of the right-hand-side of (1.3) is considered in Section 2 below.) The hypersurface $\mathscr{S}$ singles out a set of Killing vectors X for the metric b which are normal to $\mathscr{S}$,

$$X\Big|_{\mathscr{S}} = Nn\ , \tag{1.7}$$

where N is a function and $n = e_0 = (\frac{r^2}{\ell^2}+k)^{-1/2}\partial_t$ is the future-directed b-unit normal to $\mathscr{S}$. We shall use the symbol $\mathscr{K}_{\mathscr{S}^\perp}$ to denote this set of Killing vectors. The question then arises whether one can extract out of (1.3), with $X \in \mathscr{K}_{\mathscr{S}^\perp}$, one or more geometric invariants associated to g along $\mathscr{S}$. Another way of stating this question is, essentially, whether the integrals (1.3) are background independent. As discussed in more detail below, every metric g asymptotes many different backgrounds of the form (1.1) whenever it asymptotes one, and it is not at all clear how these backgrounds relate to each other: if the geometry of space-time does not sufficiently constrain the set of allowed backgrounds (1.1), then the numbers obtained from (1.3) could be completely unrelated to each other when different backgrounds are chosen. If this were the case, it would appear questionable to associate physical meaning to the integrals (1.3). The purpose of this paper is to prove that, in several cases of interest, geometric invariants can indeed be extracted out of the integrals (1.3).

The model problem of interest are space-times which are asymptotic to anti-de Sitter space-time. In this context there exist several alternative methods of defining mass — using coordinate systems [7, 20], preferred foliations [19], generalized Komar integrals [31], conformal techniques [2–4], or *ad-hoc* methods [1]; an extended discussion can be found in [16, Section 5]. We wish to stress that the key advantage of the Hamiltonian approach is the uniqueness of the candidate expression for the energy (which follows from the fact that Hamiltonians are uniquely defined up to a constant on each path connected component of the phase space), and that no such uniqueness properties are known in the alternative approaches mentioned above (*cf.*, however, [23, 32] for some partial results in the "Noether charges" framework). Now, independently of the question of what is the "correct" candidate expression for the energy, each of the expressions proposed in the existing literature suffers from some ambiguities, so that the question of well-posedness of the definition of mass as defined in those papers arises as well. For instance, the Abbott-Deser mass [1], or the Hamiltonian mass of [21], both suffer from precisely the same potential ambiguities as the Hamiltonian mass analyzed in this paper. As shown in Appendix C, under the asymptotic conditions considered in our well-posedness results, the Hamiltonian mass defined by (1.3) coincides with the Abbott-Deser one. Thus, one way of interpreting our results is that we prove the existence of a geometric invariant which can be calculated using Abbott-Deser type integrals. As another example, we note the potential ambiguity in the mass defined by the conformal methods in [2, 3], related to the possibility of existence of conformal completions which are *not* smoothly conformally equivalent. The results proved here can be used to show [13] that no such completions exist, establishing the invariant character of the definitions of [2, 3].

We note that a similar problem for the ADM mass of asymptotically flat initial data sets has been solved in [5, 11] (see also [12]). Our treatment here is a non-trivial adaptation to the current setup of the methods of [11]. Some of the results proved here have been independently observed in [33].

The detailed statements of our results in a general context are to be found in the sections below, and will not be reproduced here. We shall, instead, discuss the application of our results to two families of examples:

1. Let M be the $n-1$ dimensional sphere ${}^{n-1}S$ with the round metric h of scalar curvature $(n-1)(n-2)$, while b is the $n+1$ dimensional anti-de Sitter metric:

$$b = -\left(\frac{r^2}{\ell^2} + 1\right) dt^2 + \frac{1}{\frac{r^2}{\ell^2} + 1} dr^2 + r^2 h \,, \tag{1.8}$$

   The space $\mathscr{K}_{\mathscr{S}^\perp}$ of b-Killing vector fields normal to $\mathscr{S} \cap \mathscr{U}$ consists of vector fields $X(\lambda)$, $\lambda = (\lambda^{(\mu)}) \in \mathbb{R}^{n+1}$, which[3] on $\mathscr{S}$ take the form (1.7) with $N = \lambda^{(\mu)} N_{(\mu)}$, where

$$N_{(0)} = \sqrt{\frac{r^2}{\ell^2} + 1} \,, \qquad N_{(i)} = \frac{x^i}{\ell} \,,$$

   and $x^i = rn^i$, r being the coordinate which appears in (1.1), while $n^i \in {}^{n-1}S \subset \mathbb{R}^n$. The group $Iso(\mathscr{S}, b)$ of isometries Φ of b which map $\mathscr{S}$ into $\mathscr{S}$ acts on $\mathscr{K}_{\mathscr{S}^\perp}$ by push-forward; in Appendix B.1 we show for completeness the well known fact that for every such Φ there exists a Lorentz transformation $M : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ so that we have

$$\Phi_* X(\lambda) = X(M\lambda) \,.$$

   Letting $\mathfrak{g}_{(\mu)}$ be the μ'th basis vector of $\mathscr{K}_{\mathscr{S}^\perp} \approx \mathbb{R}^{n+1}$, $\mathfrak{g}_{(\mu)} := X(\lambda^{(\alpha)} = \delta^\alpha_\mu)$, we set

$$m_{(\mu)} \equiv m\left(\mathscr{S}, g, b, \mathfrak{g}_{(\mu)}\right) \;;$$

   it follows that the number

$$m^2(\mathscr{S}, g) = |\eta^{(\mu)(\nu)} m_{(\mu)} m_{(\nu)}| \,,$$

   where $\eta^{(\mu)(\nu)} = diag(-1, +1, \cdots, +1)$ is the Minkowski metric on $\mathbb{R}^{n+1}$, is an invariant of the action of $Iso(\mathscr{S}, b)$.[4] Further, if we define $m(\mathscr{S}, g)$ to be positive if m^μ is spacelike, while we take the sign of $m(\mathscr{S}, g)$ to coincide with that of $m_0 = -m^0$ if m^μ is timelike or null, then $m(\mathscr{S}, g)$ so defined is invariant under the action of the connected component $Iso_0(\mathscr{S}, b)$ of the identity in $Iso(\mathscr{S}, b)$. We show in detail in Section 4 that $m(\mathscr{S}, g)$ is independent of the background metric chosen to calculate the integrals

[3] We stress that the index (μ) on λ does not have anything to do with space-time; $\lambda^{(\mu)}$ is simply a coordinate on the $n+1$ dimensional vector space $\mathscr{K}_{\mathscr{S}^\perp}$. Similarly the Lorentz metric $\eta_{(\mu)(\nu)}$, which arises naturally on $\mathscr{K}_{\mathscr{S}^\perp}$ from the construction here, does not have anything to do with the space-time metric g. To emphasize this we put brackets around the μ's.

[4] This has been observed independently by X. Wang [33] in, however, a considerably less general setting.

(1.3), provided that the fall-off conditions of Theorem 2.1 and Theorem 2.3 hold, which justifies the notation. The number $m(\{t=0\},g)$ so defined coincides with the mass parameter m of the Kottler metrics[5]

$$g \;=\; -\left(1-\frac{2m}{r}+\frac{r^2}{\ell^2}\right)dt^2+\left(1-\frac{2m}{r}+\frac{r^2}{\ell^2}\right)^{-1}dr^2+r^2h\;, \quad (1.9)$$

where $h = d\theta^2+\sin^2\theta d\varphi^2$. Similarly $m(\{t=0\},g)$ is proportional to the parameter η which occurs in the $(n+1)$-dimensional generalizations of the Kottler metrics (*cf., e.g.,* [22])

$$g \;=\; -\left(1-\frac{2\eta}{r^{n-2}}+\frac{r^2}{\ell^2}\right)dt^2+\left(1-\frac{2\eta}{r^{n-2}}+\frac{r^2}{\ell^2}\right)^{-1}dr^2+r^2h\;, \quad (1.10)$$

with h — a round metric on a $n-1$ dimensional sphere of scalar curvature $(n-1)(n-2)$.

Some further global geometric invariants of the metrics asymptotic to the backgrounds (1.10) are discussed in Section 4.

2. Let M be a compact $n-1$ dimensional manifold with a metric h of constant scalar curvature and with non-positive Ricci tensor, and let b take the form

$$b = -\frac{1}{a^2(r)}dt^2+a^2(r)dr^2+r^2h\;, \quad (1.11)$$

h being r-independent, as before. We show (see Proposition B.2, Appendix B.2) that for such metrics the space of b-Killing vector fields normal to $\mathscr{S}$ consists of vector fields of the form

$$X(\lambda)=\lambda\partial_t\;,\quad \lambda\in\mathbb{R}\;. \quad (1.12)$$

The discussion in Section 4 shows that

$$m(\mathscr{S},g)\equiv m(\mathscr{S},g,b,X(1))$$

is background independent, hence a geometric invariant. Some other geometric invariants can be obtained from the integrals (1.3) when Killing vectors which are not necessarily normal to $\mathscr{S}$ exist, using invariants of the action of the isometry group of b on the space of Killing vectors. If the Ricci tensor of M is strictly negative no other Killing vectors exist, *cf.* Appendix B.2. On the other hand, if h is a flat torus, then each h–Killing vector provides a geometric invariant via the integrals (1.3), provided that those converge and that the fall-off conditions of Theorem 2.3 are met (this will be the case if, *e.g.,* Equations (2.9)-(2.10) hold).

The number $m(\mathscr{S},g)$ defined in each case above is our proposal for the geometric definition of total mass of $\mathscr{S}$ in $(\mathscr{M},g)$.

[5]The Kottler metrics, published in 1918 [29], are also known as the "Schwarzschild – de Sitter metrics".

The results described above can be reformulated in a purely Riemannian context, this will be discussed elsewhere [13]. The extension of the results proved here to hyperboloidal hypersurfaces in Minkowski space-time, that leads to a geometric definition of the Trautman-Bondi mass, requires a considerable amount of work and will be discussed elsewhere [14]. Let us simply mention that if the metric of a hyperboloidal hypersurface in asymptotically Minkowskian space-times satisfies the fall-off conditions here then its Trautman-Bondi mass coincides with the Hamiltonian one. More general statements require care.

It is natural to study the behaviour of the mass when $\mathscr{S}$ is allowed to move in $\mathscr{M}$. A partial answer to this question is given in Theorem 2.3 below. A complete answer would require establishing an equivalent of Theorem 3.3 in a space-time setting. The difficulties that arise in the corresponding problem for asymptotically Minkowskian metrics [12] suggest that this might be a considerably more delicate problem, which we plan to analyze in the future. It should be stressed that this problem mixes two different issues, one being the potential background dependence of (1.3), another one being the possibility of energy flowing in or out through the timelike conformal boundary of space-time.

This paper is organized as follows: In Section 2 we present conditions which guarantee convergence of the mass integrals (1.3), see Theorem 2.1. We also show that the integrals (1.3) are invariant (Theorem 2.3) or covariant (Lemma 2.4) under a class of well-controlled coordinate transformations, consisting of symmetries of the background, and of certain generalizations of the usual "supertranslations" that occur in the asymptotically flat case. Section 3 contains the proof of the asymptotic symmetries theorem, Theorem 3.3, which is the key result in this work. In this theorem we show that the coordinate transformations allowed by our conditions are compositions of those considered in Section 2. In Section 4 we apply the previous results to the construction of global geometric invariants in a reasonably general setting. In Appendix A the Hamiltonian approach to the definition of mass is examined in our context. Appendix B contains some results on Killing vectors which are needed in the body of the paper.

2 Convergence, covariance under well behaved coordinate transformations

Let us start by establishing convergence of the mass integrals (1.3) — this involves setting up appropriate boundary conditions on g. Let, thus, g and b be two metrics on a set $\{R_0 \le r < \infty\ ,\ (v^A) \in M\}$, let e_a be an orthonormal frame for b, set

$$e^{\mu\nu} \equiv g^{\mu\nu} - b^{\mu\nu}\ , \tag{2.1}$$

and let $e^{ab} \equiv g(\theta^a, \theta^b) - \eta^{ab}$ denote the coefficients of $e^{\mu\nu}$ with respect to the frame θ^a dual to the e_a's:

$$e^{\mu\nu}\partial_\mu \otimes \partial_\nu = e^{ac} e_a \otimes e_c\ .$$

Here $\eta^{ab} = \mathrm{diag}(-1, +1, \cdots, +1)$. We stress that we do *not* assume existence of global frames on the asymptotic region: when M is not parallelizable, then any conditions on the e^{ab}'s, *etc.* assumed below should be understood as the requirement of *existence of a covering of M by a finite number of open sets $\mathscr{O}_i$ together with frames defined on $[R_0,\infty)\times\mathscr{O}_i$ satisfying the conditions spelled out above.* The "matter energy-momentum tensor" $T^\lambda{}_\kappa$ is defined as

$$8\pi T^\lambda{}_\kappa := R^\lambda{}_\kappa - \frac{1}{2}g^{\alpha\beta}R_{\alpha\beta}\delta^\lambda_\kappa + \Lambda\delta^\lambda_\kappa\ . \qquad (2.2)$$

In our first result we assume for simplicity[6] that b is Einstein, that is, b satisfies Equation (2.2) with $T^\lambda{}_\kappa = 0$, with a cosmological constant Λ the sign of which is irrelevant for the theorem that follows:

Theorem 2.1 *Let X be a Killing vector of an Einstein metric b, set*

$$|X|^2 \equiv \sum_a |X^a|^2\ ,\quad |\mathring{\nabla}X|^2 \equiv \sum_{a,b}|\mathring{\nabla}_b X^a|^2\ ,\quad |J|^2 = \sum_b |T^0{}_b|^2 \qquad (2.3)$$

where $\mathring{\nabla}$ is the covariant derivative of b; the indices here refer to a b-orthonormal frame such that e_0 is normal to the hypersurface $t=0$. Suppose that $\lim_{r\to\infty} e^{ab}=0$ and that

$$\int_{\{r\ge R_0\}}\Bigg\{|X|\Bigg(|J| + |\Lambda||e^{ab}b_{ab}| + \sum_{a,b,c} e_a(e^{bc})^2 + \sum_{b,c}|e^{bc}|^2\Bigg) + |\mathring{\nabla}X|\sum_{a,b,c,d,e}|e_a(e^{bc})|\ |e^{de}|\Bigg\}\ d\mu_b < \infty\ , \qquad (2.4)$$

where $d\mu_b \equiv \sqrt{\det b_{ij}}dr\ dv^2\dots dv^n$ is the Riemannian measure induced on $\{t=0\}$ by b. Then the right-hand-side of Equation (1.3), understood as the limit as $R\to\infty$ of integrals over the sets $\{r=R, t=0\}$, exists and is finite.

Remark: We note that the somewhat unexpected restriction on integrability (for $\Lambda\ne 0$) of $e^{ab}b_{ab}$ arises also in the requirement of a well defined generalized Komar mass for static asymptotically anti-de Sitter metrics [16, 31].

Proof: We have

$$\int_{\{r=R\}}\mathbb{U}^{\alpha\beta}dS_{\alpha\beta} = 2\int_{\{R_0\le r\le R\}}\mathring{\nabla}_\beta\mathbb{U}^{\alpha\beta}dS_\alpha + \int_{\{r=R_0\}}\mathbb{U}^{\alpha\beta}dS_{\alpha\beta}\ . \qquad (2.5)$$

A formula for the volume integrand in Equation (2.5) is given in Equation (A.27), Appendix A. Clearly conditions (2.4) guarantee convergence of that volume integral to a finite value when R tends to infinity. □

[6] Using Equation (A.27), Appendix A, it is straightforward to obtain results similar to Theorem 2.1 without the hypothesis that b is Einstein. Similarly, the hypothesis that X is a Killing vector field can be relaxed using the calculations of [10, Appendix B]; *cf.* also [15, Section 5.1].

In the remainder of the paper we will only consider background metrics of the form

$$b = -a^{-2}(r)dt^2 + a^2(r)dr^2 + r^2 h \;, \qquad h = h_{AB}(v^C)dv^A dv^B \;, \tag{2.6}$$

where h is a Riemannian metric on a $(n-1)$-dimensional compact manifold M. The condition that b is Einstein will not be made unless explicitly stated otherwise. Let

$$\theta^0 = \frac{1}{a}dt \;, \quad \theta^1 = adr \;, \quad \theta^A = r\alpha^A \;,$$

where α^A is an h-orthonormal coframe. We let e_a be the frame dual to θ^a,

$$e_0 = a\partial_t \;, \qquad e_1 = \frac{1}{a}\partial_r \quad e_A = \frac{1}{r}\beta_A \;, \tag{2.7}$$

so that β_A is a h-orthonormal frame on (M,h). As an application of Theorem 2.1, consider, first, the Killing vector field $X = \partial_t = (1/a)e_0$ and suppose that

$$a(r) = \frac{\ell}{r} + o(r) \;, \quad e_1(a) = -\frac{1}{r} + o(r) \;, \tag{2.8}$$

for some constant ℓ. We then have $|X| \approx \ell|\mathring{\nabla} X|/\sqrt{2} \approx r/\ell$ and $d\mu_b = r^{n-2}\sqrt{\det h}\, dr\, dv^2 \dots dv^n$. When g and b are Einstein, the condition (2.4) will be satisfied if

$$e^{ab} = O(r^{-\beta}) \;, \quad e_a(e^{bc}) = O(r^{-\beta}) \;, \quad b_{ab}e^{ab} = O(r^{-\gamma}) \;, \tag{2.9}$$

with

$$\beta > n/2 \;, \qquad \gamma > n \;. \tag{2.10}$$

(We note that the generalized $n+1$ dimensional Kottler metrics (1.10) satisfy (2.9) with $\beta = n$, and with $\gamma = 2n$.) An identical convergence analysis applies for all "rotational" b-Killing vector fields $X^A(v^B)\partial_A$ (whenever occurring) for all the metrics (2.6)-(2.8), as well as all the remaining Killing vectors for the $(n+1)$-dimensional anti-de Sitter metric (listed in Appendix B.1), showing that all the corresponding charges are finite when the conditions (2.9)-(2.10) hold. Surprisingly, in retrospect the analysis in the case $\Lambda \neq 0$ turns out to be simpler than that for the asymptotically Minkowskian case, where $\Lambda = 0$: in the latter case the requirement of convergence of angular momentum or of boost integrals imposes more stringent conditions on the metric than that of convergence of the energy-momentum integrals.

The conditions presented above are sufficient, but certainly not necessary, for convergence of the integrals (1.3): indeed, the metric considered in Proposition 2.2 below has a convergent mass integral, but the conditions of Theorem 2.1 fail to hold. However, there is a potential essential ambiguity in the definition of the integrals (1.3), which we will describe now. Proposition 2.2 below then shows that (2.9)-(2.10) are essentially sharp, if one requires that the integrals (1.3) are convergent and *background-independent*.

The ambiguity in the integrals (1.3) arises as follows: to define those integrals we have fixed a model background metric b with the corresponding coordinate system (t, r, v^A) as in (2.6), as well as an orthonormal frame as in (2.7). Once this has been done, let g be any metric such that the frame components g^{ab} of g tend to η^{ab} as r tends to infinity in such a way that the integrals $m(\mathscr{S}, g, b, X)$ given by (1.3) (labeled by the background Killing vector fields X or perhaps by a subset thereof) converge. Consider another coordinate system $(\hat{t}, \hat{r}, \hat{v}^A)$ with the associated background metric $\hat{b}$:

$$\hat{b} = -\frac{1}{a^2(\hat{r})} d\hat{t}^2 + a^2(\hat{r}) d\hat{r}^2 + \hat{r}^2 \hat{h} , \qquad \hat{h} = h_{AB}(\hat{v}^C) d\hat{v}^A d\hat{v}^B , \tag{2.11}$$

together with an associated frame $\hat{e}^a$,

$$\hat{e}_0 = a(\hat{r}) \partial_{\hat{t}} , \qquad \hat{e}_1 = \frac{1}{a(\hat{r})} \partial_{\hat{r}} \quad \hat{e}_A = \frac{1}{\hat{r}} \hat{\beta}_A , \tag{2.12}$$

and suppose that in the new hatted coordinates the integrals defining the charges $m(\widehat{\mathscr{S}}, g, \hat{b}, \hat{X})$ converge again. An obvious way of obtaining such coordinate systems is to make a coordinate transformation

$$t \to \hat{t} = t + \delta t , \qquad r \to \hat{r} = r + \delta r , \qquad v^A \to \hat{v}^A = v^A + \delta v^A , \tag{2.13}$$

with $(\delta t, \delta r, \delta v^A)$ decaying sufficiently fast, as *e.g.* in the statement of Theorem 2.3 below. (However, we do not know *a priori* that the hatted coordinates are related to the unhatted one by the simple coordinate transformation (2.13) with $(\delta t, \delta r, \delta v^A)$ decaying as $r \to \infty$, or behaving in some controlled way — the behaviour of $(\delta t, \delta r, \delta v^A)$ could be very wild.) The question then arises, how do the $m(\widehat{\mathscr{S}}, g, \hat{b}, \hat{X})$'s relate to the $m(\mathscr{S}, g, b, X)$'s. A geometric definition of mass should be coordinate-independent, therefore one would like to have a simple relation between those integrals.

At this point it is worth recalling that there exist several expressions for mass alternative to (1.3), which might or might not coincide with each other when the decay of the metric is too slow. For example, we show in Appendix C that (1.3) coincides with the Abbott-Deser [1] mass for all metrics satisfying the decay conditions (2.9)-(2.10) for Killing vectors such that $|X| = O(r)$ with, say, $a(r)$ as in Equation (1.1). Now, if $X = \partial_t$, for background metrics of the form (1.1), in space-times of dimension 4, the integral defining the Abbott-Deser m_{AD} can be written in a particularly simple form [16]

$$m_{AD}(\{t=0\}, g, b, \partial_t) = \lim_{R\to\infty} \frac{R^3}{16\pi\ell^2} \int_{\Sigma\cap\{r=R\}} \left(r \sum_A \frac{\partial e^{AA}}{\partial r} - 2e^{11} \right) d^2\mu_h \tag{2.14}$$

Generalizing an argument of [17], we show that if the decay conditions in (2.9) are too weak then one can obtain essentially any value of $m_{AD}(\widehat{\mathscr{S}}, g, \hat{b}, \hat{X})$ by performing coordinate transformations of the form (2.13). We do this explicitly in $n = 3$, the same argument applies in any dimension n:

Proposition 2.2 *Let the physical metric g equal the background metric b, and let $\{r, v^A\}$ be coordinates so that b takes the form (1.8). Consider a new set of coordinates defined as*

$$\hat{r} = r + \frac{\zeta}{r^{1/2}} , \quad \hat{v}^A = v^A , \tag{2.15}$$

where ζ is a constant. (This leads to $\hat{e}^{ab} = O(r^{-3/2})$.) If $\zeta \neq 0$ then the mass $m_{AD}(\{t=0\}, g, \hat{b}, \partial_t)$ of g with respect to the background metric $\hat{b}$ defined by the coordinates $\{\hat{r}, \hat{v}^A\}$ does not vanish.

Proof: First notice that this transformation satisfies $\frac{\delta r}{r} = O\left(r^{-3/2}\right)$. Then, by straightforward computations one has, assuming without loss of generality that $\ell = 1$,

$$\hat{e}_1 = \left[1 + \frac{3\zeta}{2r^{3/2}} + \frac{3\zeta^2}{4r^3} + O(r^{-7/2})\right] e_1 ,$$

$$\hat{e}_A = \left[1 - \frac{\zeta}{r^{3/2}} + \frac{\zeta^2}{r^3} + O(r^{-7/2})\right] e_A ,$$

and so,

$$\hat{e}^{11} = \frac{3\zeta}{r^{3/2}} + \frac{15\zeta^2}{4r^3} + O(r^{-7/2}) ,$$

$$\sum_A e^{AA} = -\frac{4\zeta}{r^{3/2}} + \frac{6\zeta^2}{r^3} + O(r^{-9/2}) .$$

Hence

$$r\sum_A \partial_r(e^{AA}) - 2e^{11} = -\frac{51\zeta^2}{2r^3} + O(r^{-7/2}) ,$$

and the result follows. □

While the above shows that the Abbott-Deser mass ceases to be well defined below the threshold $o(r^{-3/2})$ in dimension 3+1, this still leaves open the unlikely possibility that the Hamiltonian mass (1.3) could be well defined. In order to see that this is not the case let us, first, calculate the mass integrand for metrics of the form

$$g = -\mu\left(\frac{r^2}{\ell^2} + k\right)dt^2 + \frac{\nu}{\frac{r^2}{\ell^2} + k}dr^2 + \sigma r^2 h_{AB} dv^A dv^B , \tag{2.16}$$

where μ, ν and σ are arbitrary functions. One finds

$$\begin{aligned}\mathbb{U}^{tr} &= \frac{r^n \sigma^{(n-1)/2}\sqrt{\det h_{AB}}}{16\pi\ell^2\sqrt{\mu\nu}}\left\{\frac{2\sqrt{\mu\nu}}{\sigma^{(n-1)/2}} - (\mu+\nu)\right. \\ &\quad \left. + (n-1)\left(1 + \frac{\ell^2 k}{r^2}\right)\mu\left[\frac{1}{\sigma}\left(r\frac{\partial\sigma}{\partial r} - \nu\right) + 1\right]\right\} .\end{aligned} \tag{2.17}$$

Suppose that —in space-time dimension $n+1$ — g is the metric b expressed in a hatted coordinate system $(\hat{r}, \hat{v}^A)$, and consider the coordinate transformation

$$\hat{r} = r + \frac{\zeta}{r^{n/2-1}} , \quad \hat{v}^A = v^A ,$$

where ζ is a constant. The metric g, when expressed in the unhatted coordinates (r, v^A), satisfies (2.9) with $\beta = \gamma = n/2$, and is of the form (2.16) so that (2.17) applies. A MATHEMATICA calculation then shows that g has a mass integral (1.3) with respect to the unhatted background b equal to

$$\frac{\mathrm{Vol}_h(M)}{8\pi\ell^2}\left(n+\frac{n^2}{8}\right)\zeta^2 ,$$

which is non-zero for any $\zeta \neq 0$ and for any $n \in \mathbb{N}$. Here $\mathrm{Vol}_h(M)$ is the $n-1$-dimensional volume of M — area if $n-1=2$ — with respect to the metric h.

The coordinate transformation (2.15) is not yet as good as one would wish, because it leads — in space dimension three – to coordinates in which the trace of the metric $e^{ab}b_{ab}$ is $O(r^{-n/2})$, quite a bit above the threshold r^{-n} set forth in Equations (2.9)-(2.10). We note that the change of coordinates (2.15) accompanied by a further time redefinition (which clearly does not change the mass as given by Equation (2.14))

$$\bar{t} = t(1+cr^{-3/2}) ,$$

with an appropriate choice of the constant c, will lead to a metric which at $\bar{t}=t=0$ satisfies

$$e^{ij} = O(r^{-3/2}) , \quad e_k(e^{ij}) = O(r^{-3/2}) , \quad b_{ab}e^{ab} = O(r^{-3}) , \tag{2.18}$$

where the indices i, j run from 0 to 3. Note that the above fall-off conditions will not hold for some of the e^{0a}'s, and for some e_0 derivatives of the e_{ab}'s, but this turns out irrelevant for the problem at hand: the new hypersurface $\bar{t}=0$ coincides with the previous one, therefore its extrinsic curvature will not change. One can check [13] that — similarly to the ADM case — conditions on the induced metric on the surface $t=0$ and on its extrinsic curvature are sufficient for a well defined notion of mass, so that the result in [13] complete the proof of sharpness of the condition on γ in (2.10).

Let us show that the decay rates (2.9)-(2.10) guarantee non-occurrence of the behaviour exhibited in Proposition 2.2:

Theorem 2.3 *Consider an $n+1$ dimensional space-time $(\mathscr{M}, g)$, and let b and $\hat{b}$ be two background metrics of the form (2.6) and (2.11), with $a(r)$ as in Equation (1.1), in coordinates $\{t, r, v^A\}$ and $\{\hat{t}, \hat{r}, \hat{v}^A\}$ with ranges covering $\{r \geq R_0\} \times M$ and $\{\hat{r} \geq \hat{R}_0\} \times M$ for some $R_0, \hat{R}_0 \in \mathbb{R}$. Suppose that b satisfies the vacuum Einstein equations with a negative cosmological constant, that the conditions of Theorem 2.1 hold both for the hatted and unhatted coordinates, and that we have*

$$e^{ab} = o(r^{-n/2}) , \quad e_c(e^{ab}) = o(r^{-n/2}) . \tag{2.19}$$

Let $X = X^a(t, r, v^A)e_a \in \mathscr{K}$ be a Killing vector field of the metric b satisfying

$$|X| + |\mathring{\nabla} X| = O(r) , \tag{2.20}$$

and let $\hat{X} = X^a(\hat{t}, \hat{r}, \hat{v}^A)\hat{e}_a \in \widehat{\mathscr{K}}$ be its hatted counterpart (with the $\hat{e}_a$ components of $\hat{X}$ given by the same *functions X^a of the hatted variables as the e_a*

components of X in the unhatted variables). Let $\mathscr{S}$ and $\widehat{\mathscr{S}}$ be the hypersurfaces given by $t=0$ and $\hat{t}=0$ respectively. If the coordinate transformation satisfies

$$\hat{t} = t + o(r^{-(1+n/2)}) \,, \qquad e_a(\hat{t}) = \ell\, \delta^0_a + o(r^{-(1+n/2)}) \,,$$

$$\hat{r} = r + o(r^{1-n/2}) \,, \qquad e_a(\hat{r}) = \frac{\delta^1_a}{\ell} + o(r^{1-n/2}) \,,$$

$$\hat{v}^A = v^A + o(r^{-(1+n/2)}) \,, \qquad e_a(\hat{v}^A) = \delta^A_a + o(r^{-(1+n/2)}) \,, \tag{2.21}$$

then

$$m(\mathscr{S}, g, b, X) = m(\widehat{\mathscr{S}}, g, \hat{b}, \hat{X}) \,.$$

Proof: The idea of the proof is to compute the background metric $\hat{b}$ in a frame related to the unhatted coordinates, obtaining an expression in terms of b plus correction terms. Then, we compute (1.4) for $\hat{b}$, similarly obtaining an expression in terms of (1.4) for b plus additional terms. We show that these terms integrate to zero, up to terms vanishing in the limit as r tends to infinity, keeping thus the mass invariant.

In terms of δt, δr, and δv^A defined as in Equation (2.13), the decay conditions (2.21) imply

$$\sqrt{k + r^2/\ell^2}\, \delta t = o(r^{-n/2}) \,, \quad \frac{\delta r}{\sqrt{k + r^2/\ell^2}} = o(r^{-n/2}) \,, \quad r \delta v^A = o(r^{-n/2}) \,. \tag{2.22}$$

From Equation (2.12) one finds the following relation between the hatted and non-hatted coframes

$$\hat{\theta}^0 = \left(1 + \frac{r\delta r}{r^2 + k\ell^2}\right)\theta^0 + \sqrt{k + r^2/\ell^2} d(\delta t) + o(r^{-n}) \,,$$

$$\hat{\theta}^1 = \left(1 - \frac{r\delta r}{r^2 + k\ell^2}\right)\theta^1 + \frac{1}{\sqrt{k + r^2/\ell^2}} d(\delta r) + o(r^{-n}) \,,$$

$$\hat{\theta}^A = \theta^A + \frac{\delta r}{r}\theta^A + r\left\{\frac{\partial \alpha^A{}_B}{\partial v^C}\,\delta v^C\, dv^B + \alpha^A{}_B\, d(\delta v^B)\right\} + o(r^{-n}) \,. \tag{2.23}$$

where α^A ia a co-frame on M dual to β_A, and we have introduced the notation $\alpha^A = \alpha^A{}_B dv^B$, with the index A being a tetrad index, while the index B is a coordinate index. All the terms denoted by $o(r^{-n})$ above have $o(r^{-n})$ coefficients when expressed in terms of the θ^a frame. Actually, the term in the curly brackets in the right hand side of the last equation gives a clue to a convenient way of writing these equations, since it can be rewritten as $\pounds_{\delta v^B \partial/\partial v^B}\alpha^A$, where $\pounds$ denotes a Lie derivative; in order to justify such a procedure, we use the following artifact: As explained in [9, Section 4], embedding M in $\mathbb{R}^{2(n-1)}$ and extending the metric appropriately, we can without loss of generality assume that the coordinates v^A and the frames θ^A are globally defined on M. We then set

$$\zeta = \delta t \frac{\partial}{\partial t} + \delta r \frac{\partial}{\partial r} + \delta v^A \frac{\partial}{\partial v^A} \,;$$

one sees from (2.22) that the components of ζ in the e_a frame are all of the same order $o(r^{-n/2})$, and one can check that Equation (2.23) for the hatted tetrad reduces to

$$\hat{\theta}^a = \theta^a + \pounds_\zeta \theta^a + o(r^{-n}) \ ;$$

we note that $\pounds_\zeta \theta^a = o(r^{-n/2})$. Let us write

$$\hat{e}_a = e_a + \delta e_a \ ;$$

one verifies that the tetrad components of δe_a are $o(r^{-n/2})$, and from the condition $\theta^a(e_b) = \delta^a_b$ and its hatted analogue one has

$$\begin{aligned} \theta^a(\delta e_b) &= -\pounds_\zeta \theta^a(e_b) + o(r^{-n}) \\ &= -\underbrace{\pounds_\zeta \left(\theta^a(e_b)\right)}_{0} + \theta^a(\pounds_\zeta e_b) + o(r^{-n}) \\ &= \theta^a(\pounds_\zeta e_b) + o(r^{-n}) \ , \end{aligned}$$

which shows that

$$\hat{e}_a = e_a + \pounds_\zeta e_a + o(r^{-n}) \ ,$$

with $\pounds_\zeta e_a = o(r^{-n/2})$. This looks to leading order like a change of tetrad under an infinitesimal transformation, but we emphasize that *we are not* assuming that the transformation is infinitesimal. Denoting $\{x^\mu\} = \{t, r, v^A\}$, and using $\hat{b}^{\mu\nu} = \eta^{ab}\hat{e}_a{}^\mu \hat{e}_b{}^\nu$, and $b^{\mu\nu} = \eta^{ab} e_a{}^\mu e_b{}^\nu$, we obtain

$$\hat{b}^{\mu\nu} = b^{\mu\nu} + \pounds_\zeta b^{\mu\nu} + \delta b^{\mu\nu} \ , \tag{2.24}$$

with

$$\delta b^{ab} \equiv \delta b^{\mu\nu} \theta^a{}_\mu \theta^b{}_\nu = o(r^{-n}) \ .$$

The expression (2.24) is the first step to compute the change in the integrand of (1.3). The next step is to rewrite (1.4) in the following more convenient way

$$\mathbb{U}^{\alpha\beta} = \frac{3}{8\pi}\sqrt{|\mathfrak{g}|}g_{\gamma\sigma}g^{\kappa[\alpha}X^\beta \mathring{\nabla}_\kappa g^{\gamma]\sigma} + \frac{1}{8\pi}\left(\sqrt{|\mathfrak{g}|}g^{\kappa[\alpha}b^{\beta]\gamma} - \sqrt{|\mathfrak{b}|}b^{\kappa[\alpha}b^{\beta]\gamma}\right) b_{\gamma\sigma}\mathring{\nabla}_\kappa X^\sigma \ ,$$

with $\mathfrak{g} = \det(g_{\mu\nu})$, and $\mathfrak{b} = \det(b_{\mu\nu})$. Let $\widehat{\mathbb{U}}^{\alpha\beta}$ be defined as the expression above with b and X replaced with $\hat{b}$ and $\hat{X}$; from Equation (2.24) we obtain

$$\begin{aligned} \widehat{\mathbb{U}}^{\alpha\beta} - \mathbb{U}^{\alpha\beta} \ \ &= \ -\frac{\sqrt{|\mathfrak{b}|}}{8\pi}\left[3\delta^{\alpha\beta\gamma}_{\lambda\nu\mu}X^\mu b^{\kappa\lambda} b_{\gamma\sigma}(\mathring{\nabla}_\kappa \mathring{\nabla}^\nu \zeta^\sigma + \mathring{\nabla}_\kappa \mathring{\nabla}^\sigma \zeta^\nu)\right] \\ &\quad -\frac{\sqrt{|\mathfrak{b}|}}{8\pi}\left[b^{\kappa[\alpha}b^{\beta]\gamma}\mathring{\nabla}_\mu \zeta^\mu - b^{\gamma[\beta}\mathring{\nabla}^{|\kappa|}\zeta^{\alpha]} - b^{\gamma[\beta}\mathring{\nabla}^{\alpha]}\zeta^\kappa\right]\mathring{\nabla}_\kappa X_\gamma + \delta\mathbb{U}^{\alpha\beta} \ , \end{aligned}$$

with

$$\delta\mathbb{U}^{ab} \equiv \delta\mathbb{U}^{\alpha\beta}\theta^a{}_\alpha \theta^b{}_\beta = o(r^{-n}) \ .$$

We have used the fact that $\pounds_\zeta b^{\alpha\beta} = -2\mathring{\nabla}^{(\alpha}\zeta^{\beta)}$, and that $\sqrt{\mathfrak{g}} = \sqrt{\mathfrak{b}}(1 + \mathring{\nabla}_\mu \zeta^\mu - e_\mu{}^\mu/2) + o(r^{-n})$.

The idea now is to write the right hand side above as a total divergence of a totally antisymmetric tensor density. The first term at the right hand side above can be written as

$$3\delta^{\alpha\beta\gamma}_{\lambda\nu\mu}X^\mu b^{\kappa\lambda}b_{\gamma\sigma}(\mathring{\nabla}_\kappa\mathring{\nabla}^\nu\zeta^\sigma+\mathring{\nabla}_\kappa\mathring{\nabla}^\sigma\zeta^\nu)=3\mathring{\nabla}_\gamma\left[\delta^{\alpha\beta\gamma}_{\lambda\nu\mu}X^\mu\mathring{\nabla}^\lambda\zeta^\nu\right]$$
$$+\mathring{R}_{\gamma\rho}{}^{\alpha\beta}X^\gamma\zeta^\rho+2X^{[\beta}\mathring{R}^{\alpha]}{}_\rho\zeta^\rho-(\mathring{\nabla}_\gamma X^\beta)\mathring{\nabla}^{[\gamma}\zeta^{\alpha]}+(\mathring{\nabla}_\gamma X^\alpha)\mathring{\nabla}^{[\gamma}\zeta^{\beta]}\ .$$

Since $\mathring{\nabla}_\alpha X_\beta$ is antisymmetric, one obtains

$$\begin{aligned}\hat{\mathbb{U}}^{\alpha\beta}-\mathbb{U}^{\alpha\beta} &= -\frac{\sqrt{|\mathfrak{b}|}}{8\pi}\Big\{3\mathring{\nabla}_\gamma\left[\delta^{\alpha\beta\gamma}_{\lambda\nu\mu}X^\mu\mathring{\nabla}^\lambda\zeta^\nu\right]+\mathring{R}_{\gamma\rho}{}^{\alpha\beta}X^\gamma\zeta^\rho\\ &\quad +2X^{[\beta}\mathring{R}^{\alpha]}{}_\rho\zeta^\rho+3(\mathring{\nabla}^{[\alpha}X^\beta)(\mathring{\nabla}_\gamma\zeta^{\gamma]})\Big\}+\delta\mathbb{U}^{\alpha\beta}\ .\end{aligned}$$

Finally, with the remaining terms we construct the divergence of a totally antisymmetric tensor, as follows:

$$3(\mathring{\nabla}_\gamma\zeta^{[\gamma})(\mathring{\nabla}^\alpha X^{\beta]})=3\mathring{\nabla}_\gamma\left[\zeta^{[\gamma}(\mathring{\nabla}^\alpha X^{\beta]})\right]-\zeta^\gamma\mathring{\nabla}_\gamma\mathring{\nabla}^\alpha X^\beta-2\zeta^{[\alpha}\mathring{R}^{\beta]}{}_\rho X^\rho.$$

Recalling that a Killing vector satisfies $\mathring{\nabla}_\alpha\mathring{\nabla}_\beta X_\gamma=\mathring{R}^\rho{}_{\alpha\beta\gamma}X_\rho$, and that $\mathring{R}_{\alpha\beta}=2\Lambda b_{\alpha\beta}/(n-1)$, one finds

$$\hat{\mathbb{U}}^{\alpha\beta}-\mathbb{U}^{\alpha\beta}=-\frac{3\sqrt{|\mathfrak{b}|}}{8\pi}\mathring{\nabla}_\gamma\left[X^{[\gamma}\mathring{\nabla}^\alpha\zeta^{\beta]}+\zeta^{[\gamma}\mathring{\nabla}^\alpha X^{\beta]}\right]+\delta\mathbb{U}^{\alpha\beta}\ .$$

(It would suffice that $\mathring{R}_{\alpha\beta}=2\Lambda b_{\alpha\beta}/(n-1)+O(r^{-n/2})$ for the argument to go through.) The first term on the right hand side above integrates out to zero on $(n-1)$ dimensional boundaryless compact submanifolds, and the remainder is order $o(r^{-n})$, so that

$$\int_{r=R}\hat{\mathbb{U}}^{\alpha\beta}dS_{\alpha\beta}=\int_{r=R}\mathbb{U}^{\alpha\beta}dS_{\alpha\beta}+o(1)\ , \tag{2.25}$$

with $o(1)$ tending to zero as r goes to infinity. We also have

$$\int_{r=R}\hat{\mathbb{U}}^{\alpha\beta}dS_{\alpha\beta}=\int_{\hat{r}=\hat{R}}\hat{\mathbb{U}}^{\alpha\beta}dS_{\alpha\beta}+2\int_{V_{R,\hat{R}}}\mathring{\nabla}_\alpha\hat{\mathbb{U}}^{\alpha\beta}dS_\beta\ , \tag{2.26}$$

where $V_{R,\hat{R}}$ is a set the boundary of which is the union of the coordinates sets $\{r=R\}$ and $\{\hat{r}=\hat{R}\}$. Conditions (2.4) guarantee that the volume integral in Equation (2.26) tends to zero when both R and $\hat{R}$ tend to infinity, which together with (2.25) establishes our claims. □

Let us finally show that the integrals (1.3) are *covariant* under isometries of the background. In what follows $\mathscr{S}$ is an arbitrary hypersurface on which the charge integrals (1.3) converge:

Lemma 2.4 *Let $\Phi:\mathscr{M}\to\mathscr{M}$ be an isometric diffeomorphism of $(\mathscr{M},b)$ such that $\Phi(\mathscr{S})=\mathscr{S}$. Then*

$$m(\mathscr{S},\Phi^*g,b,(\Phi_*)^{-1}X)=m(\mathscr{S},g,b,X)\ . \tag{2.27}$$

Proof: Formula (1.3) for mass is invariant under diffeomorphisms, hence

$$m(\Phi(\mathscr{S}), g, b, \Phi_* Y) = m(\mathscr{S}, \Phi^* g, \Phi^* b, Y) ,$$

and the result follows from $\Phi(\mathscr{S}) = \mathscr{S}$, $\Phi^* b = b$. □

3 Asymptotic isometries - the Riemannian problem

Throughout this section, *in contradistinction with the remainder of this paper,* g will denote the Riemannian metric induced by the space-time metric on $\mathscr{S}$. Similarly the letter b will denote the associated Riemannian background metric[7] of the form

$$b = a^2(r)dr^2 + r^2 h , \qquad h = h_{AB}(v^C)dv^A dv^B , \tag{3.1}$$

with the indices A running from 2 to n. We assume that $r \in [R, \infty)$ for some R, while the coordinates v^A are local coordinates on some compact $n-1$ dimensional manifold M. Unless explicitly stated otherwise, we use the symbol $\mathcal{O}(r^\beta)$ to denote either $O(r^\beta)$ *throughout* this section, *or* $o(r^\beta)$ *throughout* this section. We shall mainly be interested in background metrics for which

$$ra(r) = \ell + c(r) , \quad c = O(r^{-m_1}) , \quad \mathbb{R} \ni m_1 > 0 , \tag{3.2a}$$

$$c'(r) = O(r^{-1-m_1}) . \tag{3.2b}$$

for some constants $m_1, \ell > 0$.[8] When the vacuum Einstein equations (1.2) are satisfied by the metric (3.1) we have[1] $a(r) = 1/\sqrt{r^2/\ell^2 + k}$, where k is a constant, which can be written in the form (3.2) with $m_1 = 2$, as well as in the form of footnote 8 (with m_2 of that footnote as large as desired). However, the hypothesis that the vacuum Einstein equations hold plays no role in this section, therefore in (3.2) any $m_1 > 0$ will be allowed. Let us mention that (3.2b) is equivalent to the condition

$$c(\hat{r}) - c(r) = O(r^{-1-m_1})(\hat{r} - r) \tag{3.3}$$

for r large enough, and for $|\hat{r} - r| \le r/2$. Indeed, under (3.2b) we have

$$c(\hat{r}) - c(r) = \left(\int_0^1 c'(t\hat{r} + (1-t)r)dt \right) (\hat{r} - r) ,$$

and Equation (3.3) follows. The implication the other way round is straightforward using the fact that c is smooth (recall that local smoothness of the metric is assumed throughout). Condition (3.3) is actually the one which is needed in the arguments below.

[7] One could also consider background metrics of the form $b = a^2(\bar{r})d\bar{r}^2 + q^2(\bar{r})h$; however, if q is sufficiently differentiable, then b can be brought to the form (3.1) in the asymptotic region by a change of coordinates $r = q(\bar{r})$ provided that $dq/d\bar{r}$ has no zeros for large $\bar{r}$'s.

[8] A typical example is $a(r) = \frac{\ell}{r}\left(1 + \sum_{i=m_1}^{m_2} \frac{a_i}{r^i} + \mathcal{O}(r^{-\alpha})\right)$ for some constants $m_1 \ge 1$, $m_2 \ge m_1$, a_i and α.

Let θ^i, $i = 1, \ldots, n$ be an orthonormal coframe for b, with $\theta^1 = a(r)dr$, and let e_i be the dual frame; we denote by

$$\mathring{\omega}^i{}_{jk} \equiv \theta^i(\mathring{\nabla}_{e_k} e_j)$$

the associated connection coefficients, where $\mathring{\nabla}$ is the Levi-Civita connection of b. One easily finds

$$\mathring{\omega}^A{}_{1B} = \frac{1}{ra(r)} \delta^A_B = -\mathring{\omega}^1{}_{AB} \; . \tag{3.4}$$

If we denote by

$$\alpha^A \equiv \alpha(v^C)^A{}_B dv^B \tag{3.5}$$

an orthonormal frame for the metric h, and by $\beta^A{}_{BC}$ the associated Levi-Civita connection coefficients, then

$$\mathring{\omega}^A{}_{BC} = \frac{1}{r} \beta^A{}_{BC} \; . \tag{3.6}$$

All connection coefficients other than those in (3.4) or (3.6)vanish.

Lemma 3.1 *Let θ^i be an orthonormal coframe for the metric b as in (3.1), let $g = g_{ij}\theta^i \otimes \theta^j$, and suppose that*

$$g_{ij} \to_{r\to\infty} \delta^i_j \; .$$

Denote by σ_R, respectively $\mathring{\sigma}_R$, the g-geodesic distance along $\mathscr{S}$, respectively the b-geodesic distance along $\mathscr{S}$, from the set $\{r = R\}$. There exists a function $C(R) \geq 1$ satisfying $\lim_{R\to\infty} C(R) = 1$ such that

$$C(R)^{-1} \mathring{\sigma}_R \leq \sigma_R \leq C(R) \mathring{\sigma}_R \; . \tag{3.7}$$

Proof: For $s \in [R, r]$ let $\gamma(s) = (s, v^A)$, then

$$\begin{aligned} \sigma_R(r, v^A) &\leq \int_R^r \sqrt{g(\dot\gamma, \dot\gamma)(s, v^A)} \, ds \\ &= \int_R^r \sqrt{g_{ij}\theta^i(\dot\gamma)\theta^j(\dot\gamma)(s, v^A)} \, ds \\ &= \int_R^r \sqrt{g_{11}(s, v^A)} a(s) \, ds \\ &\leq (1 + o(1)) \mathring{\sigma}_R(r, v^A) \; . \end{aligned} \tag{3.8}$$

To obtain the reverse inequality, we note that for points (r, v^A) such that $r \geq R$ it holds that

$$\forall X \quad g(X, X) \geq (1 + o(1)) b(X, X) \; ,$$

with $o(1)$ going to zero as $R \to \infty$, hence for every curve γ from $\{r = R\}$ to (r, v^A) we have

$$\int_\gamma \sqrt{g(\dot\gamma, \dot\gamma)(s)} \, ds \geq (1 + o(1)) \int_\gamma \sqrt{b(\dot\gamma, \dot\gamma)(s)} \, ds \; ,$$

therefore

$$\begin{aligned}\sigma_R &= \inf_\gamma \int_\gamma \sqrt{g(\dot\gamma,\dot\gamma)(s)}\, ds \\ &\geq (1+o(1)) \inf_\gamma \int_\gamma \sqrt{b(\dot\gamma,\dot\gamma)(s)}\, ds \\ &= (1+o(1))\mathring{\sigma}_R\ . \end{aligned} \tag{3.9}$$

□

The proof of Lemma 3.1 uses only the product structure of b, and does not require Equation (3.2) to hold. If, however, that last equation holds, then clearly

$$\mathring{\sigma}_R(r,v^A) = \int_R^r a(s)ds = \ell\ln(r/R) + O(R^{-m_1}) \approx \ell\ln(r/R)$$

for large r, and (3.7) implies that for all $\epsilon > 0$ there exists $R_\epsilon \geq R$ and a constant $\hat{C}(\epsilon)$ such that for all $r \geq R_\epsilon$ we have

$$\hat{C}(\epsilon)^{-1} r^{1-\epsilon} \leq \exp\left(\sigma_R/\ell\right) \leq \hat{C}(\epsilon) r^{1+\epsilon}\ . \tag{3.10}$$

We will need a sharper version of this:

Lemma 3.2 *Under the hypotheses of Lemma 3.1, suppose further that Equation (3.2) holds, and that there exists a constant* $\alpha > 0$ *such that*

$$g_{ij} - \delta^i_j = \mathcal{O}(r^{-\alpha})\ .$$

Then for $r \geq \max[R,1]$ *we have*

$$\exp\left(\sigma_R/\ell\right) = r/R + O(R^{-m_1}) + \mathcal{O}(R^{-\alpha})\ , \tag{3.11}$$

in particular

$$r/C' \leq \exp\left(\sigma_R/\ell\right) \leq C' r\ . \tag{3.12}$$

Proof: Here and elsewhere in this paper the letter C denotes a constant which may vary from line to line; if $\mathcal{O} = o$ the constants in the current proof can be chosen as small as desired by choosing R large enough. In Equation (3.8) we can estimate $\sqrt{g_{11}}$ by $1 + Cs^{-\alpha}$, obtaining thus

$$\begin{aligned}\sigma_R(r,v^A) &\leq \mathring{\sigma}_R(r,v^A) + CR^{-\alpha} \\ &= \ell\ln(r/R) + O(R^{-m_1}) + CR^{-\alpha}\ .\end{aligned}$$

Similarly, Equation (3.9) is rewritten as

$$\begin{aligned}\sigma_R &= \inf_\gamma \int_\gamma \sqrt{g(\dot\gamma,\dot\gamma)(s)}\, ds \\ &\geq \inf_\gamma \int_\gamma (1 - Cs^{-\alpha})\sqrt{b(\dot\gamma,\dot\gamma)(s)}\, ds\ .\end{aligned} \tag{3.13}$$

The last term in Equation (3.13) is the distance from the set $\{r = R\}$ in the metric

$$(1 - Cr^{-\alpha})^2 \left(a^2(r)dr^2 + r^2 h\right) ,$$

which equals

$$\int_R^r (1 - Cs^{-\alpha})a(s)\, ds = \ell \ln(r/R) + O(R^{-m_1}) - O(R^{-\alpha}) ,$$

and our claims immediately follow. □

The key result in our work is the following:

Theorem 3.3 (Asymptotic symmetries) *Let (r, v^A) coordinatize $\Omega \subset \mathscr{S}$ so that $\Omega \approx \{r \in [R,\infty)\} \times M$, and let $(\hat r, \hat v^A)$ be another set of coordinates on Ω so that $\Omega \approx \{(\hat v^A) \in \widehat M, \hat r \in [\hat R(\hat v^A), \infty)\}$ for some continuous function $\hat R$. We further assume that v^A and $\hat v^A$ are consistently oriented, and that (r, v^A) and $(\hat r, \hat v^A)$ are also consistently oriented. Let b, θ^i, e^i,* etc., *be as at the beginning of this section, with a of the form (3.2), and let $\hat b$, $\hat\theta^i$, $\hat e_i$,* etc. *be the hatted equivalents thereof, so that*

$$\hat b = a^2(\hat r)d\hat r^2 + \hat r^2 \hat h = \sum_i \hat\theta^i \otimes \hat\theta^i ,$$

for some Riemannian metric $\hat h$ on $\widehat M$. Suppose that there exists $\alpha > 0$ such that

$$g(e_i, e_j) - \delta_i^j = \mathcal{O}(r^{-\alpha}) , \quad e_k(g(e_i, e_j)) = \mathcal{O}(r^{-\alpha}) , \tag{3.14}$$

$$g(\hat e_i, \hat e_j) - \delta_i^j = \mathcal{O}(\hat r^{-\alpha}) , \quad \hat e_k(g(\hat e_i, \hat e_j)) = \mathcal{O}(\hat r^{-\alpha}) . \tag{3.15}$$

Then:

1. *There exists a C^∞ map $\Psi : M \to \widehat M$ satisfying*

$$\Psi^* \hat h = e^{-2\psi} h$$

for some C^∞ function $\psi : M \to \mathbb{R}$, and

$$\hat r = e^\psi r + O(r^{1-\beta}) , \quad e_i(\hat r) = e_i(e^\psi r) + O(r^{1-\beta}) , \tag{3.16}$$

$$\hat v^A = \psi^A(v^B) + O(r^{-2}) , \quad e_i(\hat v^A) = e_i(\psi^A(v^B)) + O(r^{-2}) , \tag{3.17}$$

in local coordinates with $\Psi = (\psi^A)$, with $\beta = \min(m_1, \alpha, 2)$.

2. *If Ψ is the identity and if $\psi = 0$, then for $\alpha > 1$ we further have*

$$\hat r = r + \mathcal{O}(r^{1-\alpha}) , \quad e_i(\hat r) = e_i(r) + \mathcal{O}(r^{1-\alpha}) , \tag{3.18}$$

$$\hat v^A = v^A + \mathcal{O}(r^{-\alpha-1}) , \quad e_i(\hat v^A) = e_i(v^A) + \mathcal{O}(r^{-\alpha-1}) , \tag{3.19}$$

$$e_i\left(e_j(\hat r - r)\right) = \mathcal{O}(r^{1-\alpha}) , \quad e_i\left(e_j(\hat v^A - v^A)\right) = \mathcal{O}(r^{-\alpha-1}) . \tag{3.20}$$

Remarks: 1. For reference we note the partial derivatives estimates

$$\frac{\partial \hat r}{\partial r} = e^{\psi} + O(r^{-\beta}) \;, \quad \frac{\partial \hat r}{\partial v^A} = \frac{\partial e^{\psi}}{\partial v^A} r + O(r^{1-\beta}) \;, \tag{3.21}$$

$$\frac{\partial \hat v^A}{\partial r} = O(r^{-3}) \;, \quad \frac{\partial \hat v^A}{\partial v^B} = \frac{\partial \psi^A}{\partial v^B} + O(r^{-\beta}) \;, \tag{3.22}$$

with the second estimate in (3.22) being somewhat stronger than its counterpart in (3.17).

2. It should be noted that in point 1. above we do not assume that $\widehat M = M$ and $\hat h = h_{AB}(\hat v^C) d\hat v^A d\hat v^B$; this fact plays a role in [13]. The arguments in that last reference show that Ψ is a diffeomorphism, in particular $\widehat M$ is necessarily diffeomorphic to M. If $\hat h = h_{AB}(\hat v^C) d\hat v^A d\hat v^B$ and Ψ is the identity, then clearly $\psi = 0$ follows.

3. We stress that we do not assume M or $\widehat M$ to be parallelizable, thus Equations (3.16)-(3.20) have to be understood in the sense of finite coverings of M and $\widehat M$, with corresponding frames, on which the claimed estimates hold. However, if M or $\widehat M$ are parallelizable, then the estimates are global.

Proof: Let $\mathscr{O}_p$ be a conditionally compact subset of an open domain of a local coordinate system (v^A) around $p = (v_0^A) \in M$, and let, on a neighbourhood of $\overline{\mathscr{O}_p}$, β_A be a h-orthonormal frame dual to α^A. We note that the connection coefficients $\beta^A{}_{BC}$ are uniformly bounded on $\mathscr{O}_p$. Consider the radial ray

$$\gamma_p(r) := (r, v_0^A) \;,$$

which, in hatted coordinates, can be written as

$$[R,\infty) \ni r \to \gamma_p(r) = \big(\hat r(r, v_0^A), P_{\widehat M}\gamma_p(r) := \big(\hat v^A(r, v_0^A)\big)\big) \in [\hat R, \infty) \times \widehat M \subset \mathscr{S} \;; \tag{3.23}$$

here and in what follows we identify $[R,\infty)\times\mathscr{O}_p$ with the corresponding subset of $\mathscr{S}$, similarly for sets of the form $[\hat R,\infty)\times\widehat{\mathscr{U}}$, where $\widehat{\mathscr{U}} \subset \widehat M$. It should be clear from (3.23) that the operation "$\gamma_p \to P_{\widehat M}\gamma_p$" above is a coordinate projection which consists in forgetting the $\hat r$ coordinate in a coordinate system $(\hat r, \hat v^A)$. We have

$$\begin{aligned}
\hat h_{AB} \frac{\partial \hat v^A}{\partial r}\frac{\partial \hat v^B}{\partial r} &\le \frac{1}{\hat r^2}\left(\hat r^2 \hat h_{AB}\frac{\partial \hat v^A}{\partial r}\frac{\partial \hat v^B}{\partial r} + a^2(\hat r)\left(\frac{\partial \hat r}{\partial r}\right)^2\right)\\
&= \frac{1}{\hat r^2}\hat b(\frac{\partial}{\partial r},\frac{\partial}{\partial r}) \;\le\; C\frac{1}{\hat r^2} g(\frac{\partial}{\partial r},\frac{\partial}{\partial r})\\
&\le C^2 \frac{1}{\hat r^2} b(\frac{\partial}{\partial r},\frac{\partial}{\partial r}) \;=\; C^2\frac{a^2(r)}{\hat r^2} \;.
\end{aligned}$$

Let $d_{\hat h}$ denote the $\hat h$ distance on $\widehat M$, for $r_2 \ge r_1$ we thus obtain

$$\begin{aligned}
d_{\hat h}\left(P_{\widehat M}\gamma_p(r_1), P_{\widehat M}\gamma_p(r_2)\right) &\le \int_{r_1}^{r_2} \sqrt{\hat h_{AB}\frac{\partial \hat v^A}{\partial r}\frac{\partial \hat v^B}{\partial r}(s)}\, ds \\
&\le C \int_{r_1}^{r_2} \frac{a(s)}{\hat r(s, v_o^A)}\, ds \\
&\le C^2 \int_{r_1}^{r_2} \frac{1}{s^2}\, ds = C^2\left(\frac{1}{r_1} - \frac{1}{r_2}\right) \;,
\end{aligned} \tag{3.24}$$

and Lemma 3.2 has been used. It then easily follows that

$$\hat{p} := \lim_{r\to\infty} P_{\widehat{M}}\gamma_p(r)$$

exists, with

$$d_{\hat{h}}\left(P_{\widehat{M}}\gamma_p(r),\hat{p}\right) \le \frac{C^2}{r} \ . \tag{3.25}$$

Let $\widehat{\mathcal{O}}_{\hat{p}}$ be a conditionally compact subset of a domain of local coordinates $\hat{v}^A$ around $\hat{p}$, Equation (3.25) shows that γ_p enters and remains in $[\hat{R},\infty)\times\widehat{\mathcal{O}}_{\hat{p}}$ for r large enough. In what follows only such r's will be considered.

Consider, now, a point $q\in\mathcal{O}_p$; we wish to show that the corresponding ray γ_q will stay within $[\hat{R},\infty)\times\widehat{\mathcal{O}}_{\hat{p}}$ if q is close enough to p. In order to do that, consider an h-geodesic segment $\gamma\subset M$ parameterized by proper length such that $\gamma(0)=p$ and $\gamma(d_h(p,q))=q$. Expressing the path

$$s\to\Gamma(s) := (r,\gamma(s))\in[R,\infty)\times\mathcal{O}_p\subset\mathscr{S}$$

in terms of the barred coordinates we have

$$\begin{aligned}\hat{h}_{AB}\frac{d\hat{v}^A}{ds}\frac{d\hat{v}^B}{ds} &\le \frac{1}{\hat{r}^2}\left(\hat{r}^2\hat{h}_{AB}\frac{d\hat{v}^A}{ds}\frac{d\hat{v}^B}{ds}+a^2(\hat{r})\left(\frac{d\hat{r}}{ds}\right)^2\right)\\ &= \frac{1}{\hat{r}^2}\hat{b}(\frac{d\Gamma}{ds},\frac{d\Gamma}{ds}) \ \le\ C\ \frac{1}{\hat{r}^2}b(\frac{d\Gamma}{ds},\frac{d\Gamma}{ds})\\ &= C\ \frac{r^2}{\hat{r}^2}\ \le\ C^2\ ,\end{aligned}$$

An estimation as in Equation (3.24) gives

$$d_{\hat{h}}\left(P_{\widehat{M}}\gamma_p(r),P_{\widehat{M}}\gamma_q(r)\right) \ \le\ C\ d_h(p,q)\ .$$

Passing to a subset of $\mathcal{O}_p$ if necessary we thus obtain that for all $q\in\mathcal{O}_p$ the rays γ_q enter and remain in $[\hat{R},\infty)\times\widehat{\mathcal{O}}_{\hat{p}}$, for $r\ge R_p$ for some R_p.

Let, on an open neighbourhood of $\overline{\widehat{\mathcal{O}}_{\hat{p}}}$, $\hat{\alpha}^A$ be a $\hat{h}$-ON frame with uniformly bounded connection coefficients $\hat{\beta}^A{}_{BC}$, and let $\hat{\beta}_A$ be a $\hat{h}$-orthonormal frame dual to $\hat{\alpha}^A$. Equations (3.4) and (3.6) show then that all the $\mathring{\omega}^i{}_{kj}$'s and $\widehat{\mathring{\omega}}^i{}_{kj}$'s — the connection coefficients of b and of $\hat{b}$ — are uniformly bounded along the rays γ_q, $q\in\mathcal{O}_p$; the reader will note that the same will be true for the constants controlling various error terms $\mathcal{O}(r^\cdot)$ in the calculations below. The idea of the argument below is then to derive the desired estimates along the γ_q's, $q\in\mathcal{O}_p$; covering the compact manifold M by a finite number of coordinate patches $\mathcal{O}_{p_i}$, $i=1,\ldots,I$, will establish our claims.

Let f_i, respectively $\hat{f}_i$, be a g-orthonormal frame obtained by a Gram-Schmidt orthonormalisation procedure using $\{e_i\}_{i=1}^n$, respectively $\{\hat{e}_i\}_{i=1}^n$. The explicit form of the f_i's and $\hat{f}_i$'s in terms of the e_i's and $\hat{e}_i$'s shows that

$$f_i=e_i+\delta f_i\ ,\quad \delta f_i=\delta f_i{}^je_j\ ,\quad \delta f_i{}^j=\mathcal{O}(r^{-\alpha})\ ,\quad e_k(\delta f_i{}^j)=\mathcal{O}(r^{-\alpha})\ , \tag{3.26}$$

$$\hat{f}_i=\hat{e}_i+\delta\hat{f}_i\ ,\quad \delta\hat{f}_i=\delta\hat{f}_i{}^j\hat{e}_j\ ,\quad \delta\hat{f}_i{}^j=\mathcal{O}(\hat{r}^{-\alpha})\ ,\quad \hat{e}_k(\delta\hat{f}_i{}^j)=\mathcal{O}(\hat{r}^{-\alpha})\ . \tag{3.27}$$

By construction we actually have

$$f_1 = \big(1 + \mathcal{O}(r^{-\alpha})\big)\, e_1 = \big(1 + \mathcal{O}(r^{-\alpha}) + O(r^{-m_1})\big)\, \frac{r}{\ell}\partial_r \ . \tag{3.28}$$

The uniform boundedness of the $\mathring{\omega}^i{}_{kj}$'s further shows that

$$\mathring{\nabla}_{e_i}\delta f_j = \mathcal{O}(r^{-\alpha}) \ , \tag{3.29}$$

similarly for the $\hat{b}$-covariant derivatives of the $\delta\hat{f}_j$'s with respect to the $\hat{e}_i$'s. Recall that

$$\mathring{\omega}^i{}_{kj} = \frac{1}{2}\Big\{\theta^j([e_i,e_k]) - \theta^i([e_k,e_j]) - \theta^k([e_j,e_i])\Big\} \ ; \tag{3.30}$$

Equation (3.30) together with its g-equivalent and (3.26)-(3.29) imply

$$\omega^i{}_{jk} = \mathring{\omega}^i{}_{jk} + \mathcal{O}(r^{-\alpha}) \ , \tag{3.31}$$

similarly

$$\hat{\omega}^i{}_{jk} \equiv \hat{\phi}^i(\nabla_{\hat{f}_k}\hat{f}_j) = \widehat{\mathring{\omega}}^i{}_{jk} + \mathcal{O}(\hat{r}^{-\alpha}) \ . \tag{3.32}$$

We use the symbols ϕ^i and $\hat{\phi}^i$ to denote coframes dual to f_i and $\hat{f}_i$. Now, both the f_i's and $\hat{f}_i$'s are orthonormal frames for g, hence there exists a field of rotation matrices $\Lambda = (\Lambda_i{}^j) \in O(n)$ such that

$$\hat{f}_i = \Lambda_i{}^j f_j \ . \tag{3.33}$$

We recall that for rotation matrices we have[9]

$$\sum_k \Lambda_i{}^k \Lambda_j{}^k = \delta^i_j \ , \tag{3.34}$$

in particular

$$(\Lambda^{-1})_i{}^j = \Lambda_j{}^i \ ,$$

so that $\hat{\phi}^i = \Lambda_j{}^i\phi^j$, and $\phi^j = \sum_i \Lambda_j{}^i\hat{\phi}^i$. Further, from Equation (3.34) we obtain

$$\sum_k \Lambda_i{}^k\Lambda_i{}^k = 1 \quad\Longrightarrow\quad \forall\ i,j \ \ |\Lambda_i{}^j| \le 1 \ . \tag{3.35}$$

From the definition of the connection coefficients we have

$$\begin{aligned}
\hat{\omega}^\ell{}_{ji} &= \langle\hat{\phi}^\ell, \nabla_{\hat{f}_i}\hat{f}_j\rangle \\
&= \langle\Lambda_k{}^\ell\phi^k, \nabla_{\Lambda_i{}^m f_m}(\Lambda_j{}^n f_n)\rangle \\
&= \Lambda_k{}^\ell\Lambda_i{}^m\langle\phi^k, f_m(\Lambda_j{}^n)f_n + \Lambda_j{}^n\nabla_{f_m}f_n\rangle \ ,
\end{aligned}$$

leading to the well known transformation law

$$\hat{\omega}^l{}_{ji}\Lambda_l{}^k = \Lambda_i{}^l f_l(\Lambda_j{}^k) + \Lambda_j{}^l\Lambda_i{}^n\omega^k{}_{ln} \ ,$$

[9] We use the convention summation throughout, so that repeated indices in different positions have to be summed over. We will explicitly indicate the summation only in those equations in which we need to sum over repeated indices which are both subscripts or both superscripts.

which we intepret as an equation for the ${\Lambda_j}^k$'s:

$$f_i({\Lambda_j}^k) = (\Lambda^{-1})_i{}^l \hat{\omega}^n{}_{jl} {\Lambda_n}^k - {\Lambda_j}^l \omega^k{}_{li} \; . \tag{3.36}$$

From Equations (3.2) and (3.26) we obtain

$$\begin{aligned} \phi^m(\partial_r) &= a(r)\left(\delta_1^m + \mathcal{O}(r^{-\alpha})\right) && (3.37a) \\ &= \frac{\ell}{r}\left(\delta_1^m + \mathcal{O}(r^{-\beta})\right) \; , && (3.37b) \end{aligned}$$

with $\beta = \min(\alpha, m_1)$, except if $\mathcal{O} = o$ and $\alpha > m_1$ in which case either β should be taken to be any number smaller than m_1, or $\mathcal{O}$ should be understood as O. Rescaling r and the metric g by a constant conformal factor we may without loss of generality assume that $\ell = 1$; similarly for $\hat{r}$. Equation (3.12) together with Equations (3.31)-(3.37) leads to

$$\begin{aligned} \frac{\partial {\Lambda_i}^j}{\partial r} &= a(r)\left(\sum_k \hat{\hat{\omega}}^\ell{}_{ik} {\Lambda_\ell}^j {\Lambda_k}^1 + \mathcal{O}(r^{-\alpha})\right) && (3.38a) \\ &= \frac{1}{r}\left(\sum_k \hat{\hat{\omega}}^\ell{}_{ik} {\Lambda_\ell}^j {\Lambda_k}^1 + \mathcal{O}(r^{-\beta})\right) \; , && (3.38b) \end{aligned}$$

in particular

$$\begin{aligned} \frac{\partial {\Lambda_1}^j}{\partial r} &= \frac{a(r)}{\hat{r} a(\hat{r})}\left(\sum_A {\Lambda_A}^j {\Lambda_A}^1 + \mathcal{O}(r^{-\alpha})\right) && (3.39a) \\ &= \frac{1}{r}\left(\sum_A {\Lambda_A}^j {\Lambda_A}^1 + \mathcal{O}(r^{-\beta})\right) \; . && (3.39b) \end{aligned}$$

Now, the transpose of a rotation matrix is again a rotation matrix, therefore we also have

$$\sum_k {\Lambda_k}^i {\Lambda_k}^j = \delta^i_j \; ,$$

which gives

$$\sum_A {\Lambda_A}^1 {\Lambda_A}^1 = 1 - ({\Lambda_1}^1)^2 \; , \tag{3.40}$$

and it follows that

$$\begin{aligned} \frac{\partial {\Lambda_1}^1}{\partial r} &= \frac{a(r)}{\hat{r} a(\hat{r})}\left(1 - ({\Lambda_1}^1)^2 + \mathcal{O}(r^{-\alpha})\right) && (3.41a) \\ &= \frac{1}{r}\left(1 - ({\Lambda_1}^1)^2\right) + \mathcal{O}(r^{-\beta-1}) \; . && (3.41b) \end{aligned}$$

We have the following:

Lemma 3.4 *For all $\sigma < \min(m_1, \alpha, 2)$ we have*

$$\begin{aligned} {\Lambda_1}^1 &= 1 + \mathcal{O}(r^{-\sigma}) \; , && (3.42) \\ r\frac{\partial \hat{r}}{\partial r} &= \hat{r} + \mathcal{O}(r^{1-\sigma}) \; . && (3.43) \end{aligned}$$

Proof: Let χ denote the $\mathcal{O}(r^{-\beta-1})$ term in Equation (3.41b), set $g := {\Lambda_1}^1$, and denote by $\phi(r,v^A) = \int_r^\infty \chi(s,v^A)\, ds = \mathcal{O}(r^{-\beta})$; Equation (3.41b) shows that

$$\frac{\partial(g-\phi)}{\partial r} = \frac{1-g^2}{r} \geq 0 \, .$$

It follows that $g-\phi$ is non-decreasing, and therefore has a limit as r goes to infinity; since ϕ tends to zero in this limit we conclude that

$$g_\infty \equiv \lim_{r\to\infty} g$$

exists. Equation (3.41b) shows that

$$\lim_{r\to\infty} r\frac{\partial g}{\partial r} = 1 - g_\infty^2 \, . \tag{3.44}$$

Now Equation (3.35) gives $|g| \leq 1$, while Equation (3.44) implies a logarithmic divergence of g unless $g_\infty = \pm 1$; thus $g_\infty = \pm 1$. Define $f \geq 0$ by the equation

$$g = g_\infty(1-f) \, ,$$

then $f \to_{r\to\infty} 0$ and we have

$$\frac{\partial f}{\partial r} = -\frac{g_\infty f(2-f)}{r} - g_\infty \chi \, .$$

Suppose, first, that $g_\infty = 1$; since $f \to_{r\to\infty} = 0$ it follows that for every $\delta > 0$ there exists r_δ such that $f \leq \delta$ for $r \geq r_\delta$; for such r we then obtain

$$\frac{\partial f}{\partial r} \leq -\frac{f(2-\delta)}{r} - \chi \, , \tag{3.45}$$

and by integration

$$r^{2-\delta} f(r,\cdot) - r_\delta^{2-\delta} f(r_\delta,\cdot) \leq -\int_{r_\delta}^{r} \chi(s,\cdot) s^{2-\delta}\, ds = \mathcal{O}(r^{2-\delta-\beta}) + \mathcal{O}(r_\delta^{2-\delta-\beta}) \, ,$$

so that

$$f(r) = \mathcal{O}(r^{\delta-2}) + \mathcal{O}(r^{-\beta}) \, . \tag{3.46}$$

Choosing δ appropriately we obtain (3.42) with any $\sigma < \min(m_1, \alpha, 2)$, under the assumption that $g_\infty \neq -1$. In the case $g_\infty = -1$ similar, but simpler, manipulations lead again to Equation (3.46) with $\delta = 0$. From Equation (3.40) and from ${\Lambda_1}^1 = g_\infty + \mathcal{O}(r^{-\sigma})$ we obtain

$${\Lambda_A}^1 = \mathcal{O}(r^{-\sigma/2}) \, . \tag{3.47}$$

Equation (3.34) similarly implies

$${\Lambda_1}^A = \mathcal{O}(r^{-\sigma/2}) \, . \tag{3.48}$$

Equation (3.38a) gives

$$\frac{\partial {\Lambda_A}^1}{\partial r} = \frac{1}{r}\left(-{\Lambda_1}^1 {\Lambda_A}^1 + \mathcal{O}(r^{-\sigma})\right) \, ,$$

and integration in r together with (3.47) yields (assuming without loss of generality that $\sigma \neq 1$)

$$\Lambda_A{}^1 = O(r^{-1}) + \mathcal{O}(r^{-\sigma}) \; . \tag{3.49}$$

Integrating in r Equation (3.39b) and using (3.48)-(3.49) we similarly obtain

$$\Lambda_1{}^A = O(r^{-1}) + \mathcal{O}(r^{-\sigma}) \; . \tag{3.50}$$

From the definition of the $\hat{f}_i$'s and from (3.2) (with $\ell = 1$) we have

$$\hat{f}_1(\hat{r}) = \hat{r} + \mathcal{O}(\hat{r}^{1-\beta}) \; , \quad \hat{f}_A(\hat{r}) = \mathcal{O}(\hat{r}^{1-\alpha}) \; ,$$

hence

$$\begin{aligned} f_1(\hat{r}) &= \Lambda_1{}^1 \hat{f}_1(\hat{r}) + \Lambda_1{}^A \hat{f}_A(\hat{r}) \\ &= g_\infty \hat{r} + \mathcal{O}(r^{1-\sigma}) \; , \end{aligned}$$

Inverting Equation (3.28) we have

$$\begin{aligned} \left(1 + O(r^{-m_1})\right) r \frac{\partial \hat{r}}{\partial r} &= e_1(\hat{r}) \\ &= \left(1 + \mathcal{O}(r^{-\alpha})\right) f_1(\hat{r}) \\ &= g_\infty \hat{r} + \mathcal{O}(r^{1-\sigma}) \; . \end{aligned}$$

We have finally obtained

$$r \frac{\partial \hat{r}}{\partial r} = g_\infty \hat{r} + \mathcal{O}(r^{1-\sigma}) \; , \tag{3.51}$$

which is compatible with the fact that the coordinate systems (r, v^A) and $(\hat{r}, \hat{v}^A)$, as well as (v^A) and $(\hat{v}^A)$, carry the same orientations if and only if $g_\infty = 1$, and the lemma is established. □

Returning to the proof of Theorem 3.3, it is useful to introduce new coordinates x and $\hat{x}$ defined as

$$r = e^x \; , \qquad \hat{r} = e^{\hat{x}} \; .$$

In terms of those variables Equation (3.43) can be rewritten as

$$\frac{\partial \hat{x}}{\partial x} = 1 + \phi \; , \qquad \phi = \mathcal{O}(e^{-\sigma x}) \; , \tag{3.52}$$

for an appropriately defined function ϕ: More precisely, if we write

$$e_i = e_i{}^j f_j \; , \quad f_i = f_i{}^j e_j \; , \tag{3.53}$$

with the obvious hatted equivalents, then

$$\phi := \frac{r a(r)}{\hat{r} a(\hat{r})} \sum_{k,j} e_1{}^j \Lambda^j{}_k \hat{f}_k{}^1 - 1 \; . \tag{3.54}$$

Integration of Equation (3.52) gives

$$\begin{aligned}\hat{x}(x,v^A) &= x-\hat{x}(x_0,v^A)+\int_{x_0}^{x}\phi(s,v^A)\,ds\\ &= x-\hat{x}(x_0,v^A)+\int_{x_0}^{\infty}\phi(s,v^A)\,ds-\int_{x}^{\infty}\phi(s,v^A)\,ds\\ &=: x+\psi(v^A)+\mathcal{O}(e^{-\sigma x})\,, \qquad (3.55)\end{aligned}$$

which establishes the existence of a continuous function $\psi : M \to \mathbb{R}$ such that

$$\hat{r}(r,v^A)=e^{\psi(v^A)}r+\mathcal{O}(r^{1-\sigma}) \qquad (3.56)$$

(continuity of ψ follows from the Lebesgue (dominated) theorem on continuity of integrals with parameters; the Lebesgue theorem is used in a similar way without explicit reference at several places below).

Let us write Equation (3.41a) as

$$\frac{\partial({\Lambda_1}^1-1)}{\partial r} = \chi_1({\Lambda_1}^1-1)+\chi_2\,, \qquad (3.57)$$

where

$$\chi_1 := -(1+{\Lambda_1}^1)\frac{a(r)}{\hat{r}a(\hat{r})}=-\frac{2}{r}+\mathcal{O}(r^{-1-\sigma})\,, \qquad \chi_2=\mathcal{O}(r^{-\alpha-1})\,;$$

we obtain

$$\begin{aligned}{\Lambda_1}^1(r,v^A) &= 1+\left({f_1}^1(v^A)+\int_{r_0}^{r}s^2\exp\left\{\int_s^\infty\left(\chi_1(v,v^A)+\frac{2}{v}\right)dv\right\}\chi_2(s,v^A)ds\right)\times\\ & \quad r^{-2}\exp\left\{-\int_r^\infty\left(\chi_1(v,v^A)+\frac{2}{v}\right)dv\right\}\,, \qquad (3.58)\end{aligned}$$

with some continuous function ${f_1}^1 : M \to \mathbb{R}$. In particular

$${\Lambda_1}^1 = 1+O(r^{-2})+\mathcal{O}(r^{-\alpha})+\delta^2_\alpha\mathcal{O}(r^{-\alpha}\ln r)\,. \qquad (3.59)$$

From Equation (3.38a) we obtain

$$\frac{\partial{\Lambda_A}^1}{\partial r} = \frac{a(r)}{\hat{r}a(\hat{r})}\left(-{\Lambda_1}^1{\Lambda_A}^1+O(r^{-2})+\mathcal{O}(r^{-2\sigma})+\mathcal{O}(r^{-\alpha})\right)\,, \qquad (3.60)$$

which integrated in a manner similar to that for Equation (3.57) shows that there exists a continuous function ${f_A}^1 : M \to \mathbb{R}$ such that

$$\begin{aligned}{\Lambda_A}^1(r,v^B) &= \frac{{f_A}^1(v^B)}{r}+\mathcal{O}(r^{-\alpha})+\delta^\alpha_1\mathcal{O}(r^{-\alpha}\ln r)+O(r^{-1-\delta}) \qquad (3.61a)\\ &= O(r^{-1})+\delta^\alpha_1\mathcal{O}(r^{-\alpha}\ln r)+\mathcal{O}(r^{-\alpha})\,, \qquad (3.61b)\end{aligned}$$

with any δ satisfying

$$\delta<\min(1,2\sigma-1)\,. \qquad (3.62)$$

It is easily seen now that the $r^{-2}\ln r$ terms which could potentially be present in Equation (3.59) cannot occur: clearly they are only relevant for $\alpha = 2$; in that last case it immediately follows from Equations (3.40) and (3.61a) that no such terms are allowed. It follows that

$$\Lambda_1{}^1 = 1 + O(r^{-2}) + \mathcal{O}(r^{-\alpha}) . \tag{3.63}$$

Repeating the argument after Equation (3.50) one is led to

$$\frac{\partial \hat r}{\partial r} = \frac{\hat r}{r} + \mathcal{O}(r^{-\alpha}) + O(r^{-m_1}) + O(r^{-2}) , \tag{3.64a}$$

$$\hat r = e^{\psi(v^A)} r + \mathcal{O}(r^{1-\alpha}) + O(r^{1-m_1}) + O(r^{-1}) ; \tag{3.64b}$$

(3.64b) has been obtained from (3.64a) by calculating $\partial\hat x/\partial x$ and integrating the resulting equation, compare Equations (3.52)-(3.55). Equations (3.38a) and (3.64b) yield

$$\begin{aligned}
\frac{\partial \Lambda_A{}^B}{\partial r} &= a(r)\left(\sum_k \hat{\mathring{\omega}}^\ell{}_{Ak}\Lambda_\ell{}^B\Lambda_k{}^1 + \mathcal{O}(r^{-\alpha})\right)\\
&= a(r)\left(\sum_C \hat{\mathring{\omega}}^\ell{}_{AC}\Lambda_\ell{}^B\Lambda_C{}^1 + \mathcal{O}(r^{-\alpha})\right)\\
&= \mathcal{O}(r^{-\alpha-1}) + \delta_1^\alpha \mathcal{O}(r^{-\alpha-1}\ln r) + O(r^{-3}) ,
\end{aligned}$$

and by integration one finds that there exists a continuous matrix valued function $R = (R_A{}^B) : M \to O(n-1)$ such that

$$\Lambda_A{}^B(r, v^C) = R_A{}^B(v^C) + \mathcal{O}(r^{-\alpha}) + \delta_1^\alpha\mathcal{O}(r^{-\alpha}\ln^2 r) + O(r^{-2}) . \tag{3.65}$$

Repeating the argument which led to Equation (3.50) and using (3.65) one finds now that there exists a continuous function $f_1{}^A : M \to \mathbb{R}$ such that

$$\begin{aligned}
\Lambda_1{}^A(r, v^B) &= \frac{f_1{}^A(v^B)}{r}\left(1 + O(r^{-m_1})\right)\\
&\quad + \mathcal{O}(r^{-\alpha}) + \delta_\alpha^1\mathcal{O}(r^{-\alpha}\ln r) + \mathcal{O}(r^{-2\sigma}) ,
\end{aligned} \tag{3.66}$$

compare Equation (3.60); without loss of generality we have assumed that $\sigma \neq 1$. The hatted equivalent of (3.5),

$$\hat\alpha^A \equiv \hat\alpha(\hat v^C)^A{}_B d\hat v^B ,$$

gives

$$\begin{aligned}
d\hat v^A &= \hat\beta^A{}_B\hat\alpha^B = \frac{1}{\hat r}\hat\beta^A{}_B\hat\theta^B\\
&= \frac{1}{\hat r}\hat\beta^A{}_B\left((1 + \mathcal{O}(r^{-\alpha}))\Lambda_C{}^B\theta^C + (1 + \mathcal{O}(r^{-\alpha}))\Lambda_1{}^B\theta^1\right) ,
\end{aligned}$$

where $\beta^A{}_B$ denotes the matrix inverse to $\alpha^A{}_B$, while the symbol $\hat\beta^A{}_B$ is used to denote the the matrix inverse to $\hat\alpha^A{}_B$. This implies

$$\frac{\partial \hat v^A}{\partial r} = O(r^{-3}) , \tag{3.67}$$

$$\frac{\partial \hat v^A}{\partial v^C} = \frac{r}{\hat r}\hat\beta^A{}_B\Lambda_D{}^B\alpha^D{}_C + \mathcal{O}(r^{-\alpha}) . \tag{3.68}$$

Integrating (3.67) in r one obtains that the limits

$$\psi^A \equiv \lim_{r\to\infty} \hat{v}^A$$

exist and are continuous functions, with

$$\hat{v}^A - \psi^A = O(r^{-2})\ . \tag{3.69}$$

Moreover, it follows from (3.68) that the limits as r tends to infinity of $\partial\hat{v}^A/\partial v^B$ exist and are continuous. Passing to the limit $r \to \infty$ in Equation (3.68) one obtains

$$\Psi^*\hat{\alpha}^A = e^{-\psi} R^A{}_B \alpha^B\ , \tag{3.70}$$

hence

$$\begin{aligned}
\Psi^*\hat{h} &= \Psi^*\sum_A \hat{\alpha}^A \otimes \hat{\alpha}^A \\
&= e^{-2\psi}\sum_A R^A{}_B R^A{}_C \alpha^B \otimes \alpha^C \\
&= e^{-2\psi}\sum_A \alpha^A \otimes \alpha^A = e^{-2\psi} h\ ,
\end{aligned}$$

where we have used the fact that $R = (R_A{}^B)$ is a rotation matrix. It follows that the map $\Psi = (\psi^A)$ is a conformal local diffeomorphism from (M, h) to $(\widehat{M}, \hat{h})$. We can thus use a deep result of Lelong-Ferrand [30] to conclude that Ψ is smooth, in particular so is ψ. Equation (3.70) shows then smoothness of $R^A{}_B$. Further

$$\begin{aligned}
\frac{1}{r}\beta^B{}_A \frac{\partial\hat{r}}{\partial v^B} &= e_A(\hat{r}) \\
&= (1 + \mathcal{O}(r^{-\alpha}))\sum_B \Lambda_B{}^A \hat{f}_B(\hat{r}) + (1 + \mathcal{O}(r^{-\alpha}))\Lambda_1{}^A \hat{f}_1(\hat{r}) \\
&= O(1) + \mathcal{O}(r^{1-\alpha})\ ,
\end{aligned} \tag{3.71}$$

hence

$$\frac{\partial\hat{x}}{\partial v^A} = \frac{1}{\hat{r}}\frac{\partial\hat{r}}{\partial v^A} = O(1) + \mathcal{O}(r^{1-\alpha})\ . \tag{3.72}$$

This, together with Equations (3.67)-(3.69), establishes point 1 for $0 < \alpha \le 1$.

If $\alpha > 1$, a closer inspection of the $O(1)$ terms in Equation (3.72), making use of Equation (3.66), shows that the limits $\lim_{r\to\infty} \partial\hat{x}/\partial v^A$ exist, and are continuous functions of the v^A's. Now, in the current range of α's it is easy to show that the function ψ in (3.55) is continuously differentiable without invoking the Lelong-Ferrand theorem, as follows: Let ϕ be the function appearing at the right-hand-side of Equation (3.52), from (3.54) and from what has been said with a little work one finds

$$\frac{\partial\phi}{\partial v^A} = \mathcal{O}(e^{(1-\alpha)x}) + O(e^{-x})\ ;$$

the differentiability of ψ follows now from its definition (3.55) and from Lebesgue's dominated theorem on differentiability of integrals with parameters. The last estimate together with Equation (3.55) also show that

$$\frac{\partial \hat{x}}{\partial v^A} = \frac{\partial \psi}{\partial v^A} + \mathcal{O}(e^{(1-\alpha)x}) + O(e^{-x}) \; . \tag{3.73}$$

and point 1 is established.

To establish point 2, suppose that Ψ is the identity and that $\psi = 0$. The calculation in Equation (3.71) shows that

$$0 = \lim_{r\to\infty} \frac{\partial \hat{x}}{\partial v^A} = \lim_{r\to\infty} \frac{1}{\hat{r}} \frac{\partial \hat{r}}{\partial v^A} = \lim_{r\to\infty} r\alpha^B{}_A e_B(\hat{r}) = \lim_{r\to\infty} \alpha^B{}_A \Lambda_B{}^1 \; ,$$

hence the function $f_A{}^1$ appearing in Equation (3.61b) vanishes. The identity

$$\Lambda_1{}^A \Lambda_1{}^1 + \sum_B \Lambda_B{}^A \Lambda_B{}^1 = 0 \tag{3.74}$$

shows that the function $f_1{}^A$ from Equation (3.66) vanishes as well. If $1 < \alpha < 2$ we thus obtain

$$\Lambda_i{}^j = \delta_i^j + \mathcal{O}(r^{-\alpha}) \; . \tag{3.75}$$

For $\alpha \geq 2$ a closer inspection of Equation (3.39a) is needed:

$$\begin{aligned} \frac{\partial \Lambda_1{}^A}{\partial r} &= \frac{1}{r}\frac{ra(r)}{\hat{r}a(\hat{r})} \left(\sum_C \Lambda_C{}^A \Lambda_C{}^1 + \mathcal{O}(r^{-\alpha}) \right) \\ &= \frac{1}{r}\frac{ra(r)}{\hat{r}a(\hat{r})} \left(-\Lambda_1{}^1 \Lambda_1{}^A + \mathcal{O}(r^{-\alpha}) \right) \; , \end{aligned} \tag{3.76}$$

where we have used Equation (3.74). Integrating this equation in a way somewhat similar to Equation (3.57) shows that

$$\Lambda_1{}^A(r, v^B) = \mathcal{O}(r^{-\alpha}) + \delta_\alpha^2 \mathcal{O}(r^{-\alpha} \ln r) \; . \tag{3.77}$$

If $\alpha = 2$, suppose for the moment that there is no $\ln r$ term in (3.77); it then follows from Equations (3.63) and (3.74) that Equation (3.75) holds. On the other hand, for $\alpha > 2$ Equation (3.40) forces the function $f_1{}^1$ from Equation (3.58) to vanish, which in turn implies that Equation (3.75) holds again. The formula inverse to Equation (3.33) reads

$$f_j = \sum_i \Lambda_i{}^j \hat{f}_i \; , \tag{3.78}$$

in particular

$$f_1(\hat{r}) = \Lambda_1{}^1 \hat{f}_1(\hat{r}) + \sum_A \Lambda_A{}^1 \hat{f}_A(\hat{r}) \; ,$$

which implies

$$\frac{\partial \hat{r}}{\partial r} = \frac{a(r)}{a(\hat{r})} \left(1 + \mathcal{O}(r^{-\alpha})\right) + \mathcal{O}(r^{-\alpha}) \; . \tag{3.79}$$

Equation (3.3) together with the identities

$$\frac{ra(r)}{\hat{r}a(\hat{r})} = \frac{ra(r) - \hat{r}a(\hat{r})}{\hat{r}a(\hat{r})} + 1 = 1 + O(r^{-m_1-1})\delta r \; , \tag{3.80a}$$

$$\frac{a(r)}{a(\hat{r})} = \frac{ra(r)}{\hat{r}a(\hat{r})} \times \frac{\hat{r}}{r} = \frac{ra(r)}{\hat{r}a(\hat{r})}\left(1 + \frac{\delta r}{r}\right) = 1 + \left(\frac{1}{r} + O(r^{-m_1-1})\right)\delta r \; , \tag{3.80b}$$

shows that Equation (3.79) can be rewritten as

$$\frac{\partial \delta r}{\partial r} = \chi_3 \delta r + \chi_4 \; , \tag{3.81}$$

with

$$\chi_3 = \frac{1}{r} + O(r^{-m_1-1}) \; , \qquad \chi_4 = \mathcal{O}(r^{-\alpha}) \; .$$

Hence, for $r_2 > r$ we have

$$\begin{aligned} \frac{\delta r(r_2)}{r_2} &= \left(\frac{\delta r(r)}{r} + \int_r^{r_2} \exp\left\{-\int_t^\infty \left(\chi_3(s) - \frac{1}{s}\right) ds\right\} \frac{\chi_4(t)}{t} dt\right) \times \\ &\quad \exp\left\{\int_r^\infty \left(\chi_3(s) - \frac{1}{s}\right) ds\right\} \; . \end{aligned}$$

Passing with r_2 to infinity and using the fact that $\delta r = o(r)$ shows that

$$\begin{aligned} \delta r &= -r \int_r^\infty \exp\left\{-\int_t^\infty \left(\chi_3(s) - \frac{1}{s}\right) ds\right\} \frac{\chi_4(t)}{t} dt \\ &= \mathcal{O}(r^{1-\alpha}) \; . \end{aligned} \tag{3.82}$$

From Equation (3.79) we obtain

$$\frac{\partial \delta r}{\partial r} = \mathcal{O}(r^{-\alpha}) \; . \tag{3.83}$$

Equation (3.78) implies

$$\begin{aligned} e_j &= \left(\delta_j^k + \mathcal{O}(r^{-\alpha})\right) f_k \\ &= \sum_i \left(\delta_j^k + \mathcal{O}(r^{-\alpha})\right) \Lambda_i{}^k \left(\delta_i^k + \mathcal{O}(r^{-\alpha})\right) \hat{e}_k \\ &= \hat{e}_j + \sum_i \mathcal{O}(r^{-\alpha}) \hat{e}_i \; . \end{aligned} \tag{3.84}$$

It follows that

$$\begin{aligned} e_j(\hat{r}) &= \delta_j^1 \hat{r} + \mathcal{O}(r^{1-\alpha}) \\ &= e_j(r) + \mathcal{O}(r^{1-\alpha}) \; , \end{aligned} \tag{3.85a}$$

$$e_j(\hat{v}^A) = \hat{e}_j(\hat{v}^A) + \mathcal{O}(r^{-1-\alpha}) \; . \tag{3.85b}$$

In particular

$$\frac{\partial \hat{v}^A}{\partial r} = a(r) e_1(\hat{v}^A) = \mathcal{O}(r^{-2-\alpha}) \quad \Longrightarrow \quad \hat{v}^A - v^A = \mathcal{O}(r^{-1-\alpha}) \; . \tag{3.86}$$

Equations (3.82) and (3.86) allow us to rewrite Equation (3.85b) as

$$e_j(\hat{v}^A) \;=\; e_j(v^A) + \mathcal{O}(r^{-1-\alpha}) \; . \qquad (3.87)$$

Equations (3.16)-(3.17) are thus established. A straightforward analysis of the equations

$$\begin{aligned} e_k\left(f_j(\hat{r})\right) &= e_k\left(\sum_i \Lambda_i{}^j \hat{f}_i(\hat{r})\right) \; , \\ e_k\left(f_j(\hat{v}^A)\right) &= e_k\left(\sum_i \Lambda_i{}^j \hat{f}_i(\hat{v}^A)\right) \; , \end{aligned}$$

leads to Equation (3.20), and the theorem is established for $\alpha \neq 2$, or for $\alpha = 2$ provided that no log terms are present in (3.77).

Let us return to the case $\alpha = 2$; then (3.75) holds with any $\alpha < 2$ and therefore the calculations that follow remain valid with any $\alpha < 2$. Further (3.75) holds with $i = j = 1$ and $\alpha = 2$ by Equation (3.63). Equations (3.36) and (3.80) give then

$$\frac{\partial(r\Lambda_A{}^1)}{\partial r} = \mathcal{O}(r^{-2}) \quad \Longrightarrow \quad \Lambda_A{}^1 = \mathcal{O}(r^{-2}) \; ,$$

so that no log terms can occur in $\Lambda_A{}^1$. Equation (3.74) implies then

$$\Lambda_1{}^A = \mathcal{O}(r^{-2}) \; ,$$

and (3.36) establishes (3.75) with $\alpha = 2$,and the theorem follows. □

In the next section we shall need the following:

Corollary 3.5 *Let $\Psi(r, v^A) = (\hat{r}(r, v^A), \hat{v}^B(r, v^A))$ be an isometry of the background metric b:*

$$\Psi^*(a^2(r)dr^2 + r^2 h) = a^2(\hat{r})d\hat{r}^2 + \hat{r}^2\hat{h} \; , \qquad \hat{h} = h_{AB}(\hat{v}^C)d\hat{v}^A d\hat{v}^B \; .$$

If there exists $\alpha > 1$ such that the physical metric g satisfies

$$g(e_i, e_j) - \delta_i^j = \mathcal{O}(r^{-\alpha}) \; , \quad e_k(g(e_i, e_j)) = \mathcal{O}(r^{-\alpha}) \; , \qquad (3.88)$$

where e_i, $i = 1, \dots, n$ is the usual ON frame for the metric b as in (2.7), then

$$g(\hat{e}_i, \hat{e}_j) - \delta_i^j = \mathcal{O}(\hat{r}^{-\alpha}) \; , \quad \hat{e}_k(g(\hat{e}_i, \hat{e}_j)) = \mathcal{O}(\hat{r}^{-\alpha}) \; , \qquad (3.89)$$

where $\hat{e}_i$ is the corresponding hatted frame.

Proof: Applying point 1. of Theorem 3.3 to $g = b$ we obtain that

$$\hat{r} = e^{\psi} r + O(r^{1-\beta}) \; , \qquad (3.90)$$

with $\beta = \min(m_1, 2)$. Since Ψ is an isometry we have $\hat{e}_i = \Lambda_i{}^j e^j$ for some rotation matrix $\Lambda_i{}^j$, which gives

$$\hat{e}^{ij} = (g-b)(\hat{\theta}^i, \hat{\theta}^j) = \Lambda_k{}^i \Lambda_\ell{}^j (g-b)(\theta^k, \theta^\ell) = \mathcal{O}(r^{-\alpha}) = \mathcal{O}(\hat{r}^{-\alpha}) \ .$$

Further,

$$e_r(\hat{e}^{ij}) = e_r\left(\Lambda_k{}^i \Lambda_\ell{}^j e^{kl}\right) \ ,$$

and — since $e_i(\Lambda_k{}^j) = O(1)$ by (3.36) — the result easily follows. □

4 Global charges

Let us give here a general prescription how to assign global geometric invariants to hypersurfaces $\mathscr{S}$ in the class of space-times with metrics asymptotic to backgrounds (2.6): consider such a background metric b and consider a hypersurface $\mathscr{S}$ given by the equation $\{t = 0\}$ in the coordinate system of (2.6). Let $\mathscr{K}$ denote the set of all Killing vector fields of b; the hypersurface $\mathscr{S}$ singles out two subsets of $\mathscr{K}$: a) the set $\mathscr{K}_{\mathscr{S}^\perp}$ of those Killing vector fields of b which are normal to $\mathscr{S}$, and the set $\mathscr{K}_{\mathscr{S}\|}$ of all b-Killing vector fields which are tangent to $\mathscr{S}$. Consider any metric g for which the fall-off hypotheses of Theorem 2.1 are met, with $X \in \mathscr{K}_{\mathscr{S}^\perp}$, or with $X \in \mathscr{K}_{\mathscr{S}\|}$, or perhaps with all $X \in \mathscr{K}$. (In that theorem we have assumed that b satisfies Equation (1.2), but it would be sufficient that (1.2) holds only up to terms which decay sufficiently fast when r tends to infinity, the same for g.) Let $\mathrm{Iso}^\uparrow(\mathscr{S}, b)$ be the group of all time-orientation preserving isometries of b which leave $\mathscr{S}$ invariant.[10] We shall suppose further that the following condition holds:

for every orientation-preserving conformal isometry Ψ *of* (M,h) *there exists* $R_* > 0$ *and a* b*-isometric map* $\Phi : [R_*, \infty) \times M \to [R, \infty) \times M$*, such that*

$$\lim_{r\to\infty} \Phi(r, \cdot) = \Psi(\cdot) \ . \tag{4.1}$$

It follows from [6, Vol. II,Theorem 18.10.4] and from what is said in Appendix B.1 that this condition holds for the $(n+1)$-dimensional anti-de Sitter metrics, $n \geq 3$; the case $n = 2$ is handled by the discussion of toroidal M's below. Further, the above condition obviously holds for those metrics for which every conformal isometry of (M, h) is an isometry, as is the case for the (M, h)'s considered in Appendix B.2: the desired extension Φ is

$$\Phi(r, v^A) = (r, \Psi(v^A)) \ .$$

In fact, it is shown in [13] that condition (4.1) always holds when $a(r) = 1/\sqrt{r^2 + k}$ regardless of the metric h.

[10] Some further invariants can sometimes be obtained by considering the connected component of the identity of $\mathrm{Iso}^\uparrow(\mathscr{S}, b)$, but this seems to require a case by case analysis, so that no general discussion will be given here.

Let $\mathscr{C}$ denote the collection of positively oriented coordinate systems $c = \big(\mathcal{O}, (r, v^A)\big)$, where $\mathcal{O}$ is the domain of definition of the collection of functions (r, v^A), with the associated background metrics and orthonormal tetrads, for which Equations (2.4), (2.19) and (2.20) hold. For each such coordinate system c we can calculate the set of integrals (1.3), where X runs over $\mathscr{K}_{\mathscr{S}^\perp}$, or over $\mathscr{K}_{\mathscr{S}\|}$, or over $\mathscr{K}$, whichever appropriate. By Theorem 3.3 every two coordinate system c_1, c_2 in $\mathscr{C}$ differ by a coordinate transformation, say Υ, the M-part of which asymptotes to an orientation preserving conformal isometry $\Psi : M \to M$. By the hypothesis (4.1) Ψ can be extended to an isometry Φ of b which leaves $\mathscr{S}$ invariant. Writing Υ as

$$\Upsilon = (\Upsilon \circ \Phi^{-1}) \circ \Phi$$

we can decompose Υ into an isometry of b and a map $\Upsilon \circ \Phi^{-1}$ which asymptotes to the identity. By Corollary 3.5 the metric $\Phi^* g$ satisfies the hypotheses of Theorem 3.3, in the new coordinates c_3 as made precise by that Corollary, so that the conclusions of Theorem 3.3 apply to $\Upsilon \circ \Phi^{-1}$. Let b be the background associated with the first coordinate system c_1, and let $\hat{b}$ be that associated with c_2; since Φ is an isometry, the background metric associated with c_3 coincides with that associated with c_1. Now, by Theorem 2.3, the integrals (1.3) are invariant under the change of background which is associated to $\Upsilon \circ \Phi^{-1}$:

$$m(\mathscr{S}, g, b, X) = m(\mathscr{S}, g, \hat{b}, \hat{X}) \; , \tag{4.2}$$

where the $\hat{b}$-Killing vector $\hat{X}$ is associated to the b-Killing vector field X as described in the statement of Theorem 2.3. On the other hand, Lemma 2.4 shows that the isometry Φ reshuffles the integrals associated with different Killing vectors,

$$m(\mathscr{S}, \Phi^* g, b, (\Phi_*)^{-1} X) = m(\mathscr{S}, g, b, X) \; , \tag{4.3}$$

according to the action of $\mathrm{Iso}^{\uparrow}(\mathscr{S}, b)$ by push-forward on $\mathscr{K}$, or $\mathscr{K}_{\mathscr{S}^\perp}$, or $\mathscr{K}_{\mathscr{S}\|}$. (We note that since Φ is an b-isometry preserving $\mathscr{S}$, Φ_* preserves the field of b-unit normals to $\mathscr{S}$, hence the space $\mathscr{K}_{\mathscr{S}^\perp}$ of those Killing vector fields which are normal to $\mathscr{S}$. Similarly Φ preserves the space $\mathscr{K}_{\mathscr{S}\|}$ of Killing vector fields tangent to $\mathscr{S}$.) Equations (4.2)-(4.3) show that any invariant of the action of $\mathrm{Iso}^{\uparrow}(\mathscr{S}, b)$ on $\mathscr{K}$, or on $\mathscr{K}_{\mathscr{S}^\perp}$, or on $\mathscr{K}_{\mathscr{S}\|}$, gives a geometric invariant which can be associated to $\mathscr{S}$, independently of the choice the coordinate systems in $\mathscr{C}$.

When b is the $(n+1)$-dimensional anti-de Sitter metric, the relevant invariants based on Killing vector fields in $\mathscr{K}_{\mathscr{S}^\perp}$ have already been discussed in detail in the introduction, Section 1. Consider the remaining Killing vector fields $L_{(\mu)(\nu)} \in \mathscr{K}_{\mathscr{S}\|}$, as given by Equation (B.6). Equation (B.9) shows that under the action of $\mathrm{Iso}^{\uparrow}(\mathscr{S}, b) = O^{\uparrow}(1, n)$, the orthochronous $(n+1)$-dimensional Lorentz group, the integrals

$$Q_{(\mu)(\nu)} \equiv m(\mathscr{S}, g, b, L_{(\mu)(\nu)})$$

transform as the components of a two-covariant antisymmetric tensor. One then obtains a geometric invariant of $\mathscr{S}$ by calculating

$$Q \equiv Q_{(\mu)(\nu)} Q_{(\alpha)(\beta)} \eta^{(\mu)(\alpha)} \eta^{(\nu)(\beta)} \tag{4.4}$$

(for conventions see Appendix B.1). In dimension $3+1$ another independent geometric invariant is obtained from

$$Q^* \equiv Q_{(\mu)(\nu)} Q_{(\alpha)(\beta)} \epsilon^{(\mu)(\alpha)(\nu)(\beta)} . \tag{4.5}$$

In higher dimensions further invariants are obtained by calculating $\mathrm{tr}(P^{2k})$, $2 \le 2k \le (n+1)$, where $P^{(\alpha)}{}_{(\beta)} = \eta^{(\alpha)(\mu)} Q_{(\mu)(\beta)}$. (In this notation Q given by Equation (4.4) equals $\mathrm{tr}(P^2)$.) When $n+1$ is even one also has obvious generalisations of (4.5).

Consider, next, a (compact) strictly negatively curved (M,h), as considered in Appendix (B.2). In that case all Killing vector fields are in $\mathscr{K}_{\mathscr{S}^\perp}$, the action of $\mathrm{Iso}^\uparrow(\mathscr{S},b)$ on $\mathscr{K}_{\mathscr{S}^\perp}$ is trivial, and all the geometric invariants of $\mathscr{S}$ given by (1.3) are provided by the mass integrals considered in the Introduction.

Let, finally, (M,h) be a flat $(n-1)$-dimensional torus $\mathbb{T}^n$, $n \ge 2$; as discussed in Appendix (B.2), all conformal isometries of (M,h) are isometries and the action of $\mathrm{Iso}^\uparrow(\mathscr{S},b)$ on $\mathscr{K}$ is trivial. It follows that in addition to the mass we have $n-1$ independent invariants

$$m_A(\mathscr{S},g) \equiv m(\mathscr{S},g,b,\partial_A)$$

associated with the Killing vectors ∂_A of (M,h); here the ∂_A's have been chosen so that they are tangent to the S^1 factors of $\mathbb{T}^N = S^1 \times \cdots \times S^1$, and normalized to have unit length; such vector fields can loosely be thought of as generating "rotations" of the torus $\mathbb{T}^n$ into itself, giving the m_A's an angular-momentum type character.

A The phase space and the Hamiltonians

In [10] the starting point of the analysis is the Hilbert Lagrangian for vacuum Einstein gravity,

$$\mathcal{L} = \sqrt{-\det g_{\mu\nu}}\ \frac{g^{\alpha\beta} R_{\alpha\beta}}{16\pi} .$$

With our signature $(-+\cdots+)$ one needs to repeat the analysis in [10] with $\mathcal{L}$ replaced by

$$\frac{\sqrt{-\det g_{\mu\nu}}}{16\pi} \left(g^{\alpha\beta} R_{\alpha\beta} - 2\Lambda \right) ,$$

and without making the assumption $n+1=4$ done there; we follow the presentation in [15]: Consider the Ricci tensor,

$$R_{\mu\nu} = \partial_\alpha \left[\Gamma^\alpha_{\mu\nu} - \delta^\alpha_{(\mu} \Gamma^\kappa_{\nu)\kappa} \right] - \left[\Gamma^\alpha_{\sigma\mu} \Gamma^\sigma_{\alpha\nu} - \Gamma^\alpha_{\mu\nu} \Gamma^\sigma_{\alpha\sigma} \right] , \tag{A.1}$$

where the Γ's are the Christoffel symbols of g. Contracting $R_{\mu\nu}$ with the contravariant density of metric,

$$\mathfrak{g}^{\mu\nu} := \frac{1}{16\pi} \sqrt{-\det g}\ g^{\mu\nu} , \tag{A.2}$$

one obtains the following expression for the Hilbert Lagrangian density:

$$\begin{aligned}\tilde{L} &= \frac{1}{16\pi}\sqrt{-\det g}R = \mathfrak{g}^{\mu\nu}R_{\mu\nu} \\ &= \partial_\alpha\left[\mathfrak{g}^{\mu\nu}\left(\Gamma^\alpha_{\mu\nu} - \delta^\alpha_{(\mu}\Gamma^\kappa_{\ \nu)\kappa}\right)\right] + \mathfrak{g}^{\mu\nu}\left[\Gamma^\alpha_{\sigma\mu}\Gamma^\sigma_{\alpha\nu} - \Gamma^\alpha_{\mu\nu}\Gamma^\sigma_{\alpha\sigma}\right] . \end{aligned} \tag{A.3}$$

Here we have used the metricity condition of Γ, which is equivalent to the following identity:

$$\mathfrak{g}^{\mu\nu}{}_{,\alpha} := \partial_\alpha \mathfrak{g}^{\mu\nu} = \mathfrak{g}^{\mu\nu}\Gamma^\sigma_{\alpha\sigma} - \mathfrak{g}^{\mu\sigma}\Gamma^\nu_{\sigma\alpha} - \mathfrak{g}^{\nu\sigma}\Gamma^\mu_{\sigma\alpha} . \tag{A.4}$$

Suppose now, that $B^\alpha_{\sigma\mu}$ is another symmetric connection in M, which will be used as a background (or reference) connection. Denote by $\mathring{R}_{\mu\nu}$ its Ricci tensor. From the metricity condition (A.4) we similarly obtain

$$\begin{aligned}\mathfrak{g}^{\mu\nu}\mathring{R}_{\mu\nu} &= \partial_\alpha\left[\mathfrak{g}^{\mu\nu}\left(B^\alpha_{\mu\nu} - \delta^\alpha_{(\mu}B^\kappa_{\ \nu)\kappa}\right)\right] - \mathfrak{g}^{\mu\nu}\left[B^\alpha_{\sigma\mu}B^\sigma_{\alpha\nu} - B^\alpha_{\mu\nu}B^\sigma_{\alpha\sigma}\right] \\ &\quad + \mathfrak{g}^{\mu\nu}\left[\Gamma^\alpha_{\sigma\mu}B^\sigma_{\alpha\nu} + B^\alpha_{\sigma\mu}\Gamma^\sigma_{\alpha\nu} - \Gamma^\alpha_{\mu\nu}B^\sigma_{\alpha\sigma} - B^\alpha_{\mu\nu}\Gamma^\sigma_{\alpha\sigma}\right] . \end{aligned} \tag{A.5}$$

It is useful to introduce the tensor field

$$p^\alpha_{\mu\nu} := \left(B^\alpha_{\mu\nu} - \delta^\alpha_{(\mu}B^\kappa_{\ \nu)\kappa}\right) - \left(\Gamma^\alpha_{\mu\nu} - \delta^\alpha_{(\mu}\Gamma^\kappa_{\ \nu)\kappa}\right) . \tag{A.6}$$

Once the reference connection $B^\alpha_{\mu\nu}$ is given, the tensor $p^\alpha_{\mu\nu}$ encodes the entire information about the connection $\Gamma^\alpha_{\mu\nu}$:

$$\Gamma^\alpha_{\mu\nu} = B^\alpha_{\mu\nu} - p^\alpha_{\mu\nu} + \frac{2}{n}\delta^\alpha_{(\mu}p^\kappa_{\nu)\kappa}$$

(recall that the space-time dimension is $n+1$). Subtracting Equation (A.5) from (A.3), and using the definition of $p^\alpha_{\mu\nu}$, we arrive at the equation

$$\mathfrak{g}^{\mu\nu}R_{\mu\nu} - \frac{\sqrt{-\det g_{\mu\nu}}}{8\pi}\Lambda = -\partial_\alpha\left(\mathfrak{g}^{\mu\nu}p^\alpha_{\mu\nu}\right) + L ,$$

where

$$\begin{aligned}L &:= \mathfrak{g}^{\mu\nu}\left[\left(\Gamma^\alpha_{\sigma\mu} - B^\alpha_{\sigma\mu}\right)\left(\Gamma^\sigma_{\alpha\nu} - B^\sigma_{\alpha\nu}\right) - \left(\Gamma^\alpha_{\mu\nu} - B^\alpha_{\mu\nu}\right)\left(\Gamma^\sigma_{\alpha\sigma} - B^\sigma_{\alpha\sigma}\right) + \mathring{R}_{\mu\nu}\right] \\ &\quad - \frac{\sqrt{-\det g_{\mu\nu}}}{8\pi}\Lambda . \end{aligned}$$

This result may be used as follows: the quantity L differs by a total divergence from the gravitational Lagrangian, and hence the associated variational principle leads to the same equations of motion. Further, the metricity condition (A.4) enables us to rewrite L in terms of the first derivatives of $\mathfrak{g}^{\mu\nu}$: indeed, replacing in (A.4) the partial derivatives $\mathfrak{g}^{\mu\nu}{}_{,\alpha}$ by the covariant derivatives $\mathfrak{g}^{\mu\nu}{}_{;\alpha}$, calculated with respect to the background connection B,

$$\mathfrak{g}^{\mu\nu}{}_{;\alpha} := \mathfrak{g}^{\mu\nu}\left(\Gamma^\sigma_{\alpha\sigma} - B^\sigma_{\alpha\sigma}\right) - \mathfrak{g}^{\mu\sigma}\left(\Gamma^\nu_{\sigma\alpha} - B^\nu_{\sigma\alpha}\right) - \mathfrak{g}^{\nu\sigma}\left(\Gamma^\mu_{\sigma\alpha} - B^\mu_{\sigma\alpha}\right) , \tag{A.7}$$

we may calculate $p^{\alpha}_{\mu\nu}$ in terms of the latter derivatives. The final result is:

$$\begin{aligned} p^{\lambda}_{\mu\nu} &= \frac{1}{2}\mathfrak{g}_{\mu\alpha}\mathfrak{g}^{\lambda\alpha}{}_{;\nu} + \frac{1}{2}\mathfrak{g}_{\nu\alpha}\mathfrak{g}^{\lambda\alpha}{}_{;\mu} - \frac{1}{2}\mathfrak{g}^{\lambda\alpha}\mathfrak{g}_{\sigma\mu}\mathfrak{g}_{\rho\nu}\mathfrak{g}^{\sigma\rho}{}_{;\alpha} \\ &\quad + \frac{1}{2(n-1)}\mathfrak{g}^{\lambda\alpha}\mathfrak{g}_{\mu\nu}\mathfrak{g}_{\sigma\rho}\mathfrak{g}^{\sigma\rho}{}_{;\alpha} , \end{aligned} \tag{A.8}$$

where by $\mathfrak{g}_{\mu\nu}$ we denote the matrix inverse to $\mathfrak{g}^{\mu\nu}$; we assume that $n \geq 2$. Further,

$$\Gamma^{\alpha}_{\mu\nu} - B^{\alpha}_{\mu\nu} = -p^{\alpha}_{\mu\nu} + \frac{1}{n-1}\mathfrak{g}_{\sigma\rho}\mathfrak{g}^{\sigma\rho}{}_{;(\mu}\delta^{\alpha}_{\nu)} .$$

We have

$$\frac{\partial L}{\partial \mathfrak{g}^{\mu\nu}{}_{,\lambda}} = \frac{\partial L}{\partial \mathfrak{g}^{\mu\nu}{}_{;\lambda}} = \frac{\partial L}{\partial \Gamma^{\alpha}_{\beta\gamma}}\frac{\partial \Gamma^{\alpha}_{\beta\gamma}}{\partial \mathfrak{g}^{\mu\nu}{}_{;\lambda}} = p^{\lambda}_{\mu\nu} , \tag{A.9}$$

with the last equality being obtained by tedious but otherwise straightforward algebra. It follows that the tensor $p^{\lambda}_{\mu\nu}$ is the momentum canonically conjugate to the contravariant tensor density $\mathfrak{g}^{\mu\nu}$; prescribing this last object is of course equivalent to prescribing the metric. Alternatively, one can calculate

$$\begin{aligned} L &= \frac{1}{2}\mathfrak{g}_{\mu\alpha}\mathfrak{g}^{\mu\nu}{}_{;\lambda}\mathfrak{g}^{\lambda\alpha}{}_{;\nu} - \frac{1}{4}\mathfrak{g}^{\lambda\alpha}\mathfrak{g}_{\sigma\mu}\mathfrak{g}_{\rho\nu}\mathfrak{g}^{\mu\nu}{}_{;\lambda}\mathfrak{g}^{\sigma\rho}{}_{;\alpha} \\ &\quad + \frac{1}{8}\mathfrak{g}^{\lambda\alpha}\mathfrak{g}_{\mu\nu}\mathfrak{g}^{\mu\nu}{}_{;\lambda}\mathfrak{g}_{\sigma\rho}\mathfrak{g}^{\sigma\rho}{}_{;\alpha} + \mathfrak{g}^{\mu\nu}\mathring{R}_{\mu\nu} - \frac{\sqrt{-\det g_{\mu\nu}}}{8\pi}\Lambda . \end{aligned} \tag{A.10}$$

and check directly that

$$\frac{\partial L}{\partial \mathfrak{g}^{\mu\nu}{}_{;\lambda}} = p^{\lambda}_{\mu\nu} . \tag{A.11}$$

Given a symmetric background connection B on M, we take L given by Equation (A.10) as the Lagrangian for the theory. The canonical momentum $p^{\lambda}_{\mu\nu}$ is defined by Equation (A.8) or, equivalently, by Equation (A.11). If $\mathscr{S}$ is any piecewise smooth hypersurface in M, we define *the space-time phase bundle over* $\mathscr{S}$ as the collection of the $(p^{\lambda}_{\mu\nu}, \mathfrak{g}^{\alpha\beta})$'s over $\mathscr{S}$. If $(\delta_a p^{\lambda}_{\mu\nu}, \delta_a \mathfrak{g}^{\alpha\beta})$, $a = 1, 2$, are two sections over $\mathscr{S}$ of the bundle of vertical vectors tangent to the space-time phase bundle, following [27] we set

$$\Omega_{\mathscr{S}}((\delta_1 p^{\lambda}_{\mu\nu}, \delta_1 \mathfrak{g}^{\alpha\beta}), (\delta_2 p^{\lambda}_{\mu\nu}, \delta_2 \mathfrak{g}^{\alpha\beta})) = \int_{\mathscr{S}} (\delta_1 p^{\mu}_{\alpha\beta}\delta_2 \mathfrak{g}^{\alpha\beta} - \delta_2 p^{\mu}_{\alpha\beta}\delta_1 \mathfrak{g}^{\alpha\beta})\, dS_{\mu} , \tag{A.12}$$

with the fields $(\delta_1 p^{\lambda}_{\mu\nu}, \delta_1 \mathfrak{g}^{\alpha\beta})$ and $(\delta_2 p^{\lambda}_{\mu\nu}, \delta_2 \mathfrak{g}^{\alpha\beta})$ such that the integrals converge. Here dS_{μ} is defined as

$$\frac{\partial}{\partial x^{\mu}} \rfloor dx^0 \wedge \cdots \wedge dx^n , \tag{A.13}$$

where $\rfloor$ denotes contraction. This can be loosely thought of as being the "symplectic" form on the gravitational phase space; however we will avoid this terminology since the definition of a symplectic form involves non-degeneracy conditions, which are quite subtle in an infinite dimensional context, and which we do not want to address.

To be more specific, let $\mathscr{S}$ be a hypersurface which is the union of a compact set with an asymptotic region $\mathscr{S}_{\rm ext} \approx [R_0,\infty)\times M$ parameterized by (r,v^A) as in the body of this paper. Consider a background metric b of the form (1.1) defined on $\mathscr{S}_{\rm ext}$, with its associated tetrad e_a; we define the phase space $\mathscr{P}_b$ as the space of those smooth[11] sections $(p^\lambda_{\mu\nu},\mathfrak{g}^{\alpha\beta})$ along $\mathscr{S}$ of the space-time phase bundle which satisfy the following conditions:

$\mathscr{C}1$. First, we only allow those sections of the space-time phase bundle which arise from solutions of the vacuum Einstein equations with cosmological constant Λ — in particular the general relativistic constraint equations with cosmological constant Λ have to be satisfied by the fields $(p^\lambda_{\mu\nu},\mathfrak{g}^{\alpha\beta})$.

$\mathscr{C}2$. Next, the e_a-tetrad components of g are required to be bounded on $\mathscr{S}_{\rm ext}$. Moreover, we impose the integral condition

$$\int_{\mathscr{S}_{\rm ext}} r\left(\sum_{a,b,c}|\mathring{\nabla}_a e^{bc}|^2+\sum_{d,e}|e^{de}|^2\right)\,d\mu_b<\infty\;, \tag{A.14}$$

where e^{ab} are the e_a-tetrad components of $g-b$. In (A.14) $d\mu_b$ is, as before, the measure arising from the metric induced on $\mathscr{S}$ by the background metric b; in local coordinates such that $\mathscr{S}_{\rm ext}=\{t=0\}$ we have $d\mu_b=\sqrt{\det b_{ij}}\,dr d^{n-1}v$, with the indices i,j running from 1 to n.

$\mathscr{C}3$. Further, the fall-off conditions

$$e^{ab}=o(r^{-n/2})\;,\quad e_c(e^{ab})=o(r^{-n/2})\;. \tag{A.15}$$

are assumed to hold on $\mathscr{S}_{\rm ext}$.

$\mathscr{C}4$. Finally, we shall assume that the following "volume normalization condition" is satisfied:

$$\int_{\mathscr{S}_{\rm ext}} r|b_{cd}e^{cd}|\,d\mu_b<\infty\;. \tag{A.16}$$

(Recall that when M is not parallelizable, then conditions (A.14), (A.15), *etc.*, should be understood as the requirement of existence of a covering of M by a finite number of open sets $\mathscr{O}_i$ together with frames defined on $[R_0,\infty)\times\mathscr{O}_i$ satisfying the relevant conditions.)

Whenever we consider variations $(\delta p^\lambda_{\mu\nu},\delta\mathfrak{g}^{\alpha\beta})$ of the fields in $\mathscr{P}_b$, we will require that those variations satisfy the same decay conditions as the fields in $\mathscr{P}_b$.

From now on we shall assume that $B^\alpha_{\beta\gamma}$ is the Levi-Civita connection of the background metric b. A condition equivalent to (A.14), and slightly more convenient to work with, is

$$\int_{\mathscr{S}_{\rm ext}} r\left(\sum_{a,b,c}|e_a(e^{bc})|^2+\sum_{d,e}|e^{de}|^2\right)\,d\mu_b<\infty\;. \tag{A.17}$$

[11] The condition of smoothness of the relevant fields is certainly not needed, and should be relaxed if an attempt is made to obtain a full symplectic description of the situation at hand.

This follows immediately from Equations (3.4) and (3.6), which show that the $\mathring{\nabla}$-connection coefficients are bounded in the frame e_a. It follows that the fall-off conditions (2.9)-(2.10) will ensure that $\mathscr{C}2$–$\mathscr{C}4$ hold.

Let us show that Equation (A.14) guarantees that the integral defining $\Omega_{\mathscr{S}}$ converges. In order to see that, consider the identity:

$$\begin{aligned}\mathring{\nabla}_\alpha e^{\mu\nu} &= \mathring{\nabla}_\alpha g^{\mu\nu} = (\mathring{\nabla}_\alpha - \nabla_\alpha) g^{\mu\nu} \\ &= -g^{\mu\sigma}(\Gamma^\nu_{\sigma\alpha} - B^\nu_{\sigma\alpha}) - g^{\nu\sigma}(\Gamma^\mu_{\sigma\alpha} - B^\mu_{\sigma\alpha}) \; .\end{aligned}$$

The usual cyclic permutations calculation allows one to express $\Gamma^\alpha_{\beta\gamma} - B^\alpha_{\beta\gamma}$ as a linear combination of the $\mathring{\nabla}_\alpha e^{\mu\nu}$'s. It then follows from Equation (A.6) that the tetrad coefficients p^a_{bc} of $p^\alpha_{\beta\gamma}$ are, on $\mathscr{S}_{\rm ext}$, linear combinations with bounded coefficients of the $\mathring{\nabla}_a e^{bc}$'s. In local coordinates on $\mathscr{S}_{\rm ext}$ we have

$$\sqrt{|\det b_{\mu\nu}|} \sim r\sqrt{\det b_{ij}} \; ,$$

hence

$$\int_{\mathscr{S}_{\rm ext}} |\delta p^t_{ab}| \, |\delta \mathfrak{g}^{ab}| dr d^{n-1}v \leq C \sum_{a,b,c,d,e} \int_{\mathscr{S}_{\rm ext}} r |\mathring{\nabla}_c \delta e^{de}| \, |\delta e^{ab}| d\mu_b < \infty \; .$$

Here the coordinate $x^0 \equiv t$ has been chosen so that $\mathscr{S}_{\rm ext} = \{t = 0\}$. Thus, $\Omega_{\mathscr{S}}$ is well defined on $\mathscr{P}_b$, as desired.

Recall, now, that $\Omega_{\mathscr{S}}$ coincides up to boundary terms with the more familiar "ADM symplectic form" [25, 26]: one sets

$$P^{kl} \;:=\; \sqrt{\det g_{mn}} \, ({}^3g^{ij} K_{ij} \, {}^3g^{kl} - K^{kl}) \; , \tag{A.18}$$

where K_{kl} is the extrinsic curvature of $\mathscr{S}$,

$$K_{kl} \;:=\; -\frac{1}{\sqrt{|g^{tt}|}} \Gamma^t_{kl} \; , \tag{A.19}$$

with ${}^3g^{kl}$ — the three-dimensional inverse of the induced metric g_{kl} on $\mathscr{S}$; the indices on K^{kl} have been raised using ${}^3g^{kl}$. If we further choose the coordinate x^3 in such a way that $\partial\mathscr{S}_{\rm ext} = \{t = 0, x^3 = 1\}$, then the "symplectic" form (A.12) can be rewritten as [25, 26]

$$\begin{aligned}\Omega_{\mathscr{S}}((\delta_1 p^\lambda_{\mu\nu}, \delta_1 \mathfrak{g}^{\alpha\beta}), (\delta_2 p^\lambda_{\mu\nu}, \delta_2 \mathfrak{g}^{\alpha\beta})) &= \frac{1}{16\pi} \int_{\mathscr{S}} \left(\delta_1 g_{kl} \delta_2 P^{kl} - \delta_2 g_{kl} \delta_1 P^{kl} \right) d^n x \\ &+ \frac{1}{16\pi} \int_{\partial\mathscr{S}} \left(\delta_1 N^3 \delta_2 \frac{\sqrt{\det g_{kl}}}{N} - \delta_2 N^3 \delta_1 \frac{\sqrt{\det g_{kl}}}{N} \right) d^{n-1}v\end{aligned} \tag{A.20}$$

where

$$N = 1/\sqrt{-g^{tt}} \; , \quad N_k = g_{tk} \; , \quad N^3 = (g^{3k} - \frac{g^{t3} g^{tk}}{g^{tt}}) N_k \; .$$

Let us show that $\Omega_{\mathscr{S}}$ actually coincides with the ADM "symplectic form" on $\mathscr{P}_b$. It clearly follows from (A.14) (with e^{ab} replaced by δe^{ab}) that the volume integral there converges as before; it remains to show that the boundary integral vanishes. We have

$$\begin{aligned}
\delta\left(\frac{\sqrt{\det g_{ij}}}{N}\right) &= \delta\left(\sqrt{|\det g_{\mu\nu}|}\right) \\
&= \delta\left(\sqrt{\frac{\det g_{\mu\nu}}{\det b_{\mu\nu}}}\right)\sqrt{|\det b_{\mu\nu}|} \\
&= o(r^{-n/2})O(r^{n-1}) = o(r^{n/2-1}) \; .
\end{aligned}$$

One easily checks the identity

$$N^3 = \frac{f_0(r)}{\sqrt{|g^{tt}|}} \; ,$$

where f_0 is the future directed g-unit-normal to $\mathscr{S}$. We have

$$\begin{aligned}
g^{tt} \equiv g(dt,dt) &= (\eta^{ab}+e^{ab})e_a(t)e_b(t) \\
&= (\eta^{00}+e^{00})\left(e_0(t)\right)^2 \\
&= (\eta^{00}+e^{00})|b^{tt}| \; ,
\end{aligned}$$

which gives

$$\frac{1}{\sqrt{|g^{tt}|}} = O(r) \; ,$$

$$\delta\left(\frac{1}{\sqrt{|g^{tt}|}}\right) = \delta\left(\sqrt{\frac{b^{tt}}{g^{tt}}}\right)\frac{1}{\sqrt{|b^{tt}|}} = o(r^{-n/2+1}) \; .$$

Further,

$$f_0(r) = {f_0}^b e_b(r) = {f_0}^1 e_1(r) = o(r^{-n/2+1}) \; ,$$

$$\delta f_0(r) = \delta {f_0}^1 e_1(r) = o(r^{-n/2+1}) \; ,$$

where e_a is a b-orthonormal frame as in Equation (2.7), and the vanishing of the boundary term in (A.20) readily follows.

According to [28] (see also [15, 27]) the Hamiltonian associated with a one parameter family of maps of the phase space into itself which arise from the flow of a vector field X on the space-time equals

$$H(X,\mathscr{S}) = \int_{\mathscr{S}} (p^{\mu}_{\alpha\beta}\pounds_X \mathfrak{g}^{\alpha\beta} - X^{\mu}L)\, dS_{\mu} \tag{A.21}$$

provided that all the integrals involved are well defined, and that the boundary integral in the variational formula

$$\begin{aligned}
-\delta H &= \int_{\mathscr{S}} \left(\pounds_X {p^{\lambda}}_{\mu\nu}\delta\mathfrak{g}^{\mu\nu} - \pounds_X \mathfrak{g}^{\mu\nu}\delta {p^{\lambda}}_{\mu\nu}\right) dS_{\lambda} \\
&\quad + \int_{\partial\mathscr{S}} X^{[\mu}{p^{\nu]}}_{\alpha\beta}\delta\mathfrak{g}^{\alpha\beta}\, dS_{\mu\nu} \; ,
\end{aligned} \tag{A.22}$$

vanishes. In the case when B is the metric connection of a given background metric $b_{\mu\nu}$, and when X is a Killing vector field of $b_{\mu\nu}$, the identification

$$m(\mathscr{S}, g, b, X) \quad = \quad H(X, \mathscr{S}) \ , \tag{A.23}$$

together with the calculations in [10] leads to Equations (1.3)-(1.5). More precisely, let $E^{\nu\lambda}$ be given by the formula [10]

$$\begin{aligned} E^{\nu\lambda} \quad &= \quad \frac{2|\det b_{\mu\nu}|}{16\pi\sqrt{|\det g_{\rho\sigma}|}} g_{\beta\gamma}(e^2 g^{\gamma[\nu} g^{\lambda]\kappa})_{;\kappa} X^\beta \\ &\quad + \frac{1}{8\pi}\sqrt{|\det g_{\rho\sigma}|}\ g^{\alpha[\nu}\delta^{\lambda]}_\beta X^\beta{}_{;\alpha} \ . \end{aligned} \tag{A.24}$$

$$e \quad = \quad \sqrt{|\det g_{\rho\sigma}|}/\sqrt{|\det b_{\mu\nu}|} \ . \tag{A.25}$$

It can be checked that all the formulae of [10, Appendix B] are dimension independent, and lead to the identity

$$E^\lambda := p^\lambda_{\alpha\beta} \pounds_X \mathfrak{g}^{\alpha\beta} - X^\lambda L = E^{\nu\lambda}{}_{;\nu} + T^\lambda{}_\kappa X^\kappa \ , \tag{A.26}$$

where the matter energy-momentum tensor has been defined in (2.2). Now, when b is the anti de Sitter metric, the integral of $E^\lambda dS_\lambda$ over large "balls" $B_R := \{r \le R\}$ within $\mathscr{S}$ would diverge if we tried to pass with the radius of those balls to infinity because we have

$$E^\lambda\Big|_{g=b} = -(\mathring{R} - 2\Lambda) X^\lambda / 16\pi \ ,$$

with $\mathring{R}$ — the Ricci scalar of the background metric b, and $\mathring{R} - 2\Lambda = 4\Lambda/(n-1)$ in an $(n+1)$-dimensional space-time. We therefore add to E^λ a g-independent term which will cancel this divergence: indeed, such terms can be freely added to the Hamiltonian because they do not affect the variational formula that defines a Hamiltonian. From an energy point of view such an addition corresponds to a choice of the zero point of the energy. We thus set

$$\mathbb{U}^{\alpha\beta} := E^{\alpha\beta} - E^{\alpha\beta}\Big|_{g=b} \ .$$

From the definition of E^λ and from Equation (A.26) one easily finds

$$\begin{aligned} 16\pi \mathring{\nabla}_\beta \mathbb{U}^{\alpha\beta} \quad &= \quad \left(\sqrt{|\det g|} g^{ab} - \sqrt{|\det b|} b^{ab}\right) \mathring{R}_{ab} X^\beta \\ &\quad + 2\Lambda \left(\sqrt{|\det b|} - \sqrt{|\det g|}\right) X^\beta \\ &\quad + \left(\mathring{T}^\lambda{}_\kappa - T^\lambda{}_\kappa\right) X^\kappa \\ &\quad + \sqrt{|\det b|}\ \left(Q^\alpha{}_\beta X^\beta + Q^\alpha{}_{\beta\gamma} \mathring{\nabla}^\beta X^\gamma\right) \ , \end{aligned} \tag{A.27}$$

where $Q^\alpha{}_\beta$ is a quadratic form in $e_a(e^{bc})$, and $Q^\alpha{}_{\beta\gamma}$ is bilinear in $e_a(e^{bc})$ and e^{ab}, both with bounded coefficients. Further, $\mathring{T}^\lambda{}_\kappa$ is defined as in Equation (2.2) with g replaced by b.

From now on we assume that both g and b are Einstein, and we only consider vector fields X which are b-Killing vector fields and satisfy

$$|X| + |\mathring{\nabla} X| \le Cr \tag{A.28}$$

for some constant C; this holds for all the backgrounds considered in Appendix B, in particular for the generalized Kottler metrics (1.10). Theorem 2.1 then shows that the integral defining H converges for fields in $\mathscr{P}_b$.

Suppose, further, that the b-Killing vector field X has the property that *the associated variations of the fields are compatible with the boundary conditions* imposed on fields in $\mathscr{P}_b$. This means in particular that we must have

$$\int_{\mathscr{S}} r \sum_{a,b,c} |\pounds_X \left(\mathring{\nabla}_a e^{bc}\right)|^2 \, d\mu_b < \infty \, . \tag{A.29}$$

Clearly the volume integral in the variational formula (A.22) converges under (A.29) together with the remaining conditions set forth above. Further, the boundary integral there vanishes under (A.15), so that Equation (A.21) does indeed provide the required Hamiltonian on $\mathscr{P}_b$.

For Killing vectors satisfying (A.28) Equation (A.29) will hold if

$$\int_{\mathscr{S}} r^3 \sum_{a,b,c,d} |\left(\mathring{\nabla}_a \mathring{\nabla}_d e^{bc}\right)|^2 \, d\mu_b < \infty \, , \tag{A.30}$$

but we emphasize that the weaker condition (A.29) suffices.

B Isometries and Killing vectors of the background

B.1 $(n+1)$-dimensional anti-de Sitter metrics

For $n \ge 2$ consider the $(n+1)$-dimensional anti-de Sitter space-time $(\mathscr{M}, b)$, thus b is given by (1.8) with h — the unit round metric on the $(n-1)$-dimensional sphere ${}^{(n-1)}S$. As elsewhere we set $\mathscr{S} = \{t = 0\}$. When M is the two-dimensional sphere, the Killing vectors of b are given in [21]. For higher dimensional spheres the b-Killing vector fields are easily found by thinking of b as the metric induced on the covering space of the hyperboloid

$$\eta_{(a)(b)} y^{(a)} y^{(b)} = -\ell^2 \tag{B.1}$$

in the $(n+2)$-dimensional manifold $\mathscr{Y}$ with the metric[12]

$$\eta_{(a)(b)} dy^{(a)} dy^{(b)} = -(dy^{(0)})^2 + \sum_{(i)=(1)}^{(n)} (dy^{(i)})^2 - (dy^{(n+1)})^2 \, . \tag{B.2}$$

[12] $\mathscr{Y}$ can be identified with the universal covering of the space obtained by removing the set $y^{(0)} = y^{(n+1)} = 0$ from $\mathbb{R}^{n+2}$; $\mathscr{Y}$ then inherits the local coordinates $y^{(a)}$ used in Equation (B.2). However, in order to understand the geometry of $\mathscr{M}$ in a neighbourhood of $\mathscr{S}$ it is sufficient — and most convenient — to think of $\mathscr{Y}$ as of $\mathbb{R}^{n+2}$.

Throughout this section the indices $(a), (b)$, *etc.*, run from (0) to $(n+1)$. The hyperboloid can be locally parameterized by coordinates t, x^i implicitly defined by the equations[13]

$$y^{(0)} = \ell \cos(t/\ell)\sqrt{1+r^2/\ell^2}\ , \tag{B.3}$$

$$y^{(n+1)} = \ell \sin(t/\ell)\sqrt{1+r^2/\ell^2}\ , \tag{B.4}$$

$$y^{(i)} = x^i\ , \tag{B.5}$$

with $r^2 = \sum_{i=1}^n (x^i)^2$, where $x^i = rn^i$, and $n^i \in {}^{(n-1)}S$ can eventually be expressed in terms of coordinates on the sphere ${}^{(n-1)}S$. For example, for $n = 3$ we can use $x^1 = r\sin(\theta)\cos(\varphi)$, $x^2 = r\sin(\theta)\sin(\varphi)$, $x^3 = r\cos(\theta)$, with θ, φ — the usual spherical coordinates. It is also convenient to represent the hypersurface $\mathscr{S} \subset \mathscr{M}$ given by $\mathscr{S} = \{t = 0\}$ as $\{\eta_{(a)(b)} y^{(a)} y^{(b)} = -\ell^2\} \cap \{y^{(n+1)} = 0, y^{(0)} > 0\} \subset \mathscr{Y}$. We set

$$L_{(a)(b)} = y_{(a)}\frac{\partial}{\partial y^{(b)}} - y_{(b)}\frac{\partial}{\partial y^{(a)}}\ , \tag{B.6}$$

where $y_{(a)} = \eta_{(a)(b)} y^{(b)}$. The $L_{(a)(b)}$'s are Killing vector fields of $(\mathscr{Y}, \eta_{(a)(b)})$. Further they are tangent to the hyperboloid $\{\eta_{(a)(b)} y^{(a)} y^{(b)} = -\ell^2\}$ and hence define, by restriction, Killing vector fields of the hyperboloid with the induced metric. In fact they span the space of all the Killing vectors of b because there is the right maximal number of them. From the coordinate transformation (B.3)-(B.5) one can compute the corresponding Killing vectors of anti de Sitter space-time in the coordinates $\{t, x^i\}$, obtaining

$$L_{(n+1)(0)} = \ell\frac{\partial}{\partial t}\ ,$$

$$L_{(i)(n+1)} = \frac{x^i}{\sqrt{1+r^2/\ell^2}}\cos(t/\ell)\frac{\partial}{\partial t} + \ell\sqrt{1+r^2/\ell^2}\sin(t/\ell)\frac{\partial}{\partial x^i}\ ,$$

$$L_{(i)(0)} = -\frac{x^i}{\sqrt{1+r^2/\ell^2}}\sin(t/\ell)\frac{\partial}{\partial t} + \ell\sqrt{1+r^2/\ell^2}\cos(t/\ell)\frac{\partial}{\partial x^i}\ ,$$

$$L_{(i)(j)} = x^i\frac{\partial}{\partial x^j} - x^j\frac{\partial}{\partial x^i}\ .$$

Let $\mathscr{K}_{\mathscr{S}^\perp}$ be the set of Killing vector fields of b which are orthogonal to $\mathscr{S}$; from the expressions above it is not difficult to check that a vector basis for $\mathscr{K}_{\mathscr{S}^\perp}$ is given by $\mathfrak{g}_{(\mu)} = L_{(n+1)(\mu)}|_{t=0}$, where (μ) runs from (0) to (n).

Proposition B.1 *Let $\Phi : \mathscr{M} \to \mathscr{M}$ be an isometry of b such that $\Phi(\mathscr{S}) = \mathscr{S}$. Then there exists a Lorentz transformation matrix $\Lambda^{(\nu)}{}_{(\mu)}$ such that the basis vectors of $\mathscr{K}_{\mathscr{S}^\perp}$ satisfy*

$$\Phi_* \mathfrak{g}_{(\mu)} = \Lambda^{(\nu)}{}_{(\mu)} \mathfrak{g}_{(\nu)}\ .$$

[13] The spherical coordinates associated to the "cartesian" coordinates x^i give the form (1.8) of the metric b.

Remark: We note that the property $\Phi_*(\mathscr{K}_{\mathscr{S}^\perp}) = \mathscr{K}_{\mathscr{S}^\perp}$ follows from the fact that Φ preserves $\mathscr{S}$, which implies that Φ maps the field of unit normals to $\mathscr{S}$ into itself.

Proof: As is well known, for every isometry $\Phi : \mathscr{M} \to \mathscr{M}$ of b there exists a diffeomorphism $\hat{\Phi} : \mathscr{Y} \to \mathscr{Y}$, isometry of $\eta_{(a)(b)}$, such that Φ is the restriction of $\hat{\Phi}$ to the hyperboloid (B.1). In coordinates we have

$$\hat{\Phi}^{(a)}(y) = \Lambda^{(a)}{}_{(b)} y^{(b)} , \tag{B.7}$$

where $\Lambda^{(a)}{}_{(b)}$ is a matrix satisfying

$$\Lambda^{(c)}{}_{(a)} \Lambda^{(d)}{}_{(b)} \eta_{(c)(d)} = \eta_{(a)(b)} . \tag{B.8}$$

The hypersurface $\mathscr{S} \subset \mathscr{Y}$ is given by $\eta_{(a)(b)} y^{(a)} y^{(b)} = -1$ and $y^{(n+1)} = 0$ together with the condition $y^{(0)} > 0$, so that the condition $\hat{\Phi}(\mathscr{S}) = \mathscr{S}$ implies

$$\Lambda^{(a)}{}_{(b)} = \left[\begin{array}{cc} \Lambda^{(\mu)}{}_{(\nu)} & 0 \\ 0 & \pm 1 \end{array} \right] ,$$

where we split the indices as $(a) = (\mu), (n+1)$. Equation (B.8) shows that $\Lambda^{(\mu)}{}_{(\nu)}$ is a $n+1$-dimensional Lorentz transformation, $(\Lambda^{(\mu)}{}_{(\nu)}) \in O(1,n)$. Equations (B.7) and (B.6) imply that under push-forward by $\hat{\Phi}$ the Killing vectors of $\eta_{(a)(b)}$ transform as

$$\hat{\Phi}_* L_{(a)(b)} = \Lambda^{(c)}{}_{(a)} \Lambda^{(d)}{}_{(b)} L_{(c)(d)}$$

in particular, the basis vectors of $\mathscr{K}_{\mathscr{S}^\perp}$ transform as

$$\begin{aligned} \hat{\Phi}_* \mathfrak{g}_{(\mu)} &= \hat{\Phi}_* L_{(n+1)(\mu)} \Big|_{\hat{\Phi}(y^{(n+1)})=0} \\ &= \Lambda^{(c)}{}_{(n+1)} \Lambda^{(d)}{}_{(\mu)} L_{(c)(d)} \Big|_{y^{(n+1)}=0} \\ &= \pm \Lambda^{(\nu)}{}_{(\mu)} \mathfrak{g}_{(\nu)} . \end{aligned}$$

Replacing $\Lambda^{(\nu)}{}_{(\mu)}$ by $-\Lambda^{(\nu)}{}_{(\mu)}$ if necessary, the result follows. □

Let $\mathscr{K}_{\mathscr{S}\|}$ be the space of b- Killing vectors spanned by the $L_{(\mu)(\nu)}$'s, thus $\mathscr{K}_{\mathscr{S}\|}$ contains all the $L_{(a)(b)}$'s which are not in $\mathscr{K}_{\mathscr{S}^\perp}$. An identical calculation as in the proof above shows that under isometries of b preserving $\mathscr{S}$ we have

$$\hat{\Phi}_* L_{(\mu)(\nu)} = \Lambda^{(\sigma)}{}_{\mu} \Lambda^{(\rho)}{}_{(\nu)} L_{(\sigma)(\rho)} . \tag{B.9}$$

It follows that the resulting representation of the Lorentz group on $\mathscr{K}_{\mathscr{S}\|}$ is equivalent to a representation on two-contravariant anti-symmetric tensors.

B.2 h's with a non-positive Ricci tensor

We consider metrics (2.6), as in Section 2. In what follows we shall only consider (M,h)'s with a non-positive Ricci curvature, with $n \geq 3$, the case $n = 2$ being covered by the previous section. We shall further assume that the scalar curvature R_h of h (the Ricci scalar) is a constant. We note that the vector fields

$$X = X^0 n = \frac{\lambda}{a} n = \lambda \partial_t \;, \qquad \lambda \in \mathbb{R} \;, \tag{B.10}$$

where $n = e_0$ is the field of future pointing unit normals to the hypersurfaces $\{t = const\}$, are Killing vector fields for the metric b whatever $a = a(r)$. The non-vanishing connection coefficients,

$$\mathring{\omega}^a{}_{bc} \equiv \theta^a(\mathring{\nabla}_{e_c} e_b) \;,$$

with respect to this frame are

$$\mathring{\omega}^A{}_{1B} = \frac{1}{ra(r)} \delta^A_B = -\mathring{\omega}^1{}_{AB} \;, \quad \mathring{\omega}^A{}_{BC} = \frac{1}{r} \beta^A{}_{BC} \;, \quad \mathring{\omega}_{100} = -\mathring{\omega}_{010} = -\frac{a'(r)}{a^2(r)} \;, \tag{B.11}$$

where the $\beta^A{}_{BC}$ are the Levi-Civita connection coefficients of h with respect to the frame α^A. The AB components of the Killing equations read

$$\mathscr{D}_A X_B + \mathscr{D}_B X_A + \frac{1}{a(r)} h_{AB} X^1 = 0 \;, \tag{B.12}$$

where $\mathscr{D}$ is the covariant derivative operator associated with the metric h, which shows that $X^B \partial_B$ is a conformal Killing vector field on M. Uniqueness of solutions of the volume-normalized Yamabe equation in the case under consideration implies that conformal Killing vector fields of (M,h) are necessarily Killing vectors, hence

$$X^1 \equiv 0 \;.$$

The 00, 01 and $0A$ components of the Killing equations read

$$e_0(X_0) = 0 \;, \tag{B.13}$$

$$e_1(X_0) + \frac{a'}{a^2} X_0 = 0 \;, \tag{B.14}$$

$$e_A(X_0) + e_0(X_A) = 0 \;. \tag{B.15}$$

Equation (B.13) shows that X_0 is t-independent.

Suppose, first, that the Ricci tensor of h is strictly negative. It is well known[14] that in this case (M,h) has no non-trivial Killing vector fields so that $X^A \equiv 0$, and Equation (B.15) shows that X_0 is v^A-independent. Integrating (B.14) yields then the one parameter family of Killing vector fields (B.10), which shows that the algebra of all Killing vector fields of b is one-dimensional.

[14] The Killing equations imply $X_B \mathscr{D}_C \mathscr{D}^C X^B = -R_{AB} X^A X^B$, where R_{AB} is the Ricci tensor of h; integration of this equation over M shows that X^A is covariantly constant when R_{AB} is non-positive, and vanishes when R_{AB} is strictly negative.

Suppose, next, that (M,h) is an $(n-1)$-dimensional flat torus $(\mathbb{T}^{n-1},\delta)$. Then the X^A's are covariantly constant[14] vector fields on $\mathbb{T}^{n-1}$, which shows that $X^A = X^A(t,r)$ in coordinates v^A in which the metric δ has constant entries. Integrating (B.15) over $\mathbb{T}^{n-1}$ gives

$$0 = \int_{\mathbb{T}^{n-1}} e_A(X_0) = -e_0(X_A)\,\mathrm{Vol}(\mathbb{T}^{n-1})\ ,$$

hence $X_A = X_A(r)$. Equation (B.15) implies then that X_0 is v^A-independent, so that $X_0 = X_0(r)$, from (B.14) we recover (B.10), and $\mathscr{K}_{\mathscr{S}^\perp}$ is again one-dimensional, as claimed. We note that the $1A$ component of the Killing equation implies that the X^A's are in fact r-independent, which gives a complete description of the set of Killing vector fields occurring in this case.

The above arguments extend to all manifolds with constant scalar curvature and non-positive Ricci curvature, as follows: suppose that (M,h) has non-trivial Killing vector fields. The $1A$ component of the Killing equations gives

$$e_1(X_A) - \frac{1}{ra}X_A = 0 \quad \Longrightarrow \quad X^A = X^A(t,v^A)\ .$$

Integration of Equation (B.14) gives

$$X^0 = \frac{\lambda(v^A)}{a(r)}\ , \tag{B.16}$$

for some function λ on M. Equation (B.10) inserted into (B.15) gives

$$e_A(\lambda) = -a^2(r)\partial_t X^A\ ,$$

which is compatible with Equation (B.10) only if $\partial_t X^A = 0$, hence $e_A(\lambda) = 0$. Summarizing, we have proved

Proposition B.2 *If (M,h) has non-positive Ricci curvature and constant scalar curvature, then all Killing vector fields of the metric b given by Equation (2.6) are of the form*

$$X = \frac{\lambda}{a}n + X^A(v^B)\partial_A\ , \qquad \lambda\in\mathbb{R}\ ,$$

where $X^A(v^B)\partial_A$ is a Killing vector field of the metric h.

C Equality of the Hamiltonian mass with the Abbott-Deser one

In this appendix we consider a subset $\mathbb{R}\times\Sigma_{\mathrm{ext}}$ of a four dimensional space-time $(\mathscr{M},g)$ defined by a coordinate system $\{x^\alpha\}$; we identify Σ_{ext} with the set $\{x^\alpha : x^0 = 0\}$. The space coordinates (x^i) on Σ_{ext} will be written as (r,v^A), with the range of r being $[R_0,\infty)$, and with the v^A being local coordinates on some compact two dimensional manifold. Assume that there exists a frame $\{e_a\}_{a=0}^3$ defined on Σ_{ext}, which defines a background metric $b^{\alpha\beta} = \eta^{ab}e_a^\alpha e_b^\beta$, where $\eta^{ab} = \mathrm{diag}(-1, 1, 1, 1)$. In other words, the tetrad $\{e_a\}$ is orthonormal with respect to $b_{\alpha\beta}$. Assume that in this frame the space-time metric g has the form

$$g_{ab} = \eta_{ab} + e_{ab} \ ,$$

where

$$e_{ab} = o(1/r^\alpha) \ , \qquad e_a(e_{bc}) = o(1/r^\alpha) \ ,$$

for some $\alpha > 0$. The Abbott-Deser mass M_{AB} associated with X is defined as [1]

$$M_{AB} = \frac{1}{2} \lim_{R\to\infty} \int_{\partial\Sigma_R} V^{\alpha\beta} dS_{\alpha\beta} \ , \tag{C.1}$$

where

$$V^{\alpha\beta}(h) = \frac{1}{8\pi} b \left(K^{\alpha\beta\sigma\kappa}{}_{;\kappa} X_\sigma - K^{\alpha\kappa\sigma\beta} X_{\sigma;\kappa} \right) \ , \tag{C.2}$$

with

$$K^{\alpha\beta\sigma\kappa} = b^{\alpha[\kappa} H^{\sigma]\beta} + H^{\alpha[\kappa} b^{\sigma]\beta} \ , \quad H^{\alpha\beta} = e^{\alpha\beta} - \frac{1}{2} e_\gamma{}^\gamma b^{\alpha\beta} \ ,$$

$$b = \sqrt{|\det b_{\mu\nu}|} \ .$$

Note that $K^{\alpha\beta\sigma\kappa}$ has the same symmetries as the Riemann tensor. Let $\mathbb{U}^{\alpha\beta}$ be the "Hamiltonian superpotential" defined by (1.4); assume that

$$(|X| + |\nabla X|) b = O(r^\beta) \ , \tag{C.3}$$

we then claim that

$$\mathbb{U}^{\alpha\beta} = \tilde{V}^{\alpha\beta} + o(r^{\beta-2\alpha}) \ .$$

In order to establish this, recall that

$$\begin{aligned} \det(g_{\alpha\beta}) &= \det\left(b_{\alpha\gamma}\left[\delta^\gamma{}_\beta + b^{\gamma\sigma} e_{\sigma\beta} + o(r^{-2\alpha})\right]\right) \\ &= \det(b_{\alpha\gamma}) \det(\delta^\gamma{}_\beta + e^\gamma{}_\beta) \ , \end{aligned}$$

where $e^\gamma{}_\beta = b^{\gamma\sigma} e_{\sigma\beta} + o(r^{-2\alpha})$. A well known identity gives

$$\det(g_{\alpha\beta}) = \det(b_{\alpha\beta}) \left(1 + e_\gamma{}^\gamma + o(r^{-2\alpha})\right) \ .$$

Let us write $\mathbb{U}^{\alpha\beta} = \mathbb{U}^{\alpha\beta}{}_\gamma X^\gamma + \widehat{\mathbb{U}}^{\alpha\beta}$, where

$$\mathbb{U}^{\alpha\beta}{}_\gamma := \frac{1}{8\pi} \frac{b}{e} \left(e^2 g^{\sigma[\alpha} g^{\beta]\kappa} \right)_{;\kappa} g_{\gamma\sigma} \ ,$$

$$\widehat{\mathbb{U}}^{\alpha\beta} := \frac{1}{8\pi} \left(\sqrt{|\det g_{\rho\sigma}|}\, g^{\kappa[\alpha} \delta^{\beta]}{}_\gamma - b\, b^{\kappa[\alpha} \delta^{\beta]}{}_\gamma \right) X^\gamma{}_{;\kappa} \ .$$

We have

$$e^2 = 1 + e_\alpha{}^\alpha + o(r^{-2\alpha}) \ ,$$

so that the first term above can be written as

$$\begin{aligned}
\mathbb{U}^{\alpha\beta}{}_{\gamma} &= \frac{1}{8\pi} b \left[(1 + e_\alpha{}^\alpha + o(r^{-2\alpha})) \left(b^{\sigma[\alpha} b^{\beta]\kappa} - b^{\sigma[\alpha} e^{\beta]\kappa} - e^{\sigma[\alpha} b^{\beta]\kappa} + o(r^{-2\alpha}) \right) \right]_{;\kappa} b_{\gamma\sigma} \\
&= \frac{b}{8\pi}\, b_{\gamma\sigma} \left(e_\rho{}^\rho\, b^{\sigma[\alpha} b^{\beta]\kappa} - b^{\sigma[\alpha} e^{\beta]\kappa} - e^{\sigma[\alpha} b^{\beta]\kappa} \right)_{;\kappa} + o(r^{\beta-2\alpha}) \\
&= -\frac{b}{8\pi}\, b_{\gamma\sigma} \left(b^{\sigma[\alpha} H^{\beta]\kappa} + H^{\sigma[\alpha} b^{\beta]\kappa} \right)_{;\kappa} + o(r^{\beta-2\alpha})
\end{aligned}$$

so that

$$\mathbb{U}^{\alpha\beta}{}_{\gamma} = \frac{b}{8\pi}\, b_{\gamma\sigma} K^{\alpha\beta\sigma\kappa}{}_{;\kappa} + o(r^{\beta-2\alpha})\ .$$

Similarly,

$$\begin{aligned}
\widehat{\mathbb{U}}^{\alpha\beta} &:= \frac{1}{8\pi} \left(\sqrt{|\det g_{\rho\sigma}|}\, g^{\kappa[\alpha} \delta^{\beta]}{}_{\gamma} - b\, b^{\kappa[\alpha} \delta^{\beta]}{}_{\gamma} \right) X^{\gamma}{}_{;\kappa} \\
&= \frac{1}{8\pi} \left(\sqrt{|\det g_{\rho\sigma}|}\, g^{\kappa[\alpha} b^{\beta]\gamma} - b\, b^{\kappa[\alpha} b^{\beta]\gamma} \right) X_{\gamma;\kappa} \\
&= \frac{b}{8\pi} \left[\left(1 + \frac{1}{2} e + o(r^{-2\alpha}) \right) \left(b^{\kappa[\alpha} b^{\beta]\gamma} - e^{\kappa[\alpha} b^{\beta]\gamma} + o(r^{-2\alpha}) \right) - b^{\kappa[\alpha} b^{\beta]\gamma} \right] X_{\gamma;\kappa} \\
&= -\frac{b}{8\pi} H^{\kappa[\alpha} b^{\beta]\gamma} X_{\gamma;\kappa} + o(r^{\beta-2\alpha})\ .
\end{aligned}$$

Now, X_γ is a Killing vector of $b_{\mu\nu}$, therefore

$$\begin{aligned}
H^{\kappa[\beta} b^{\alpha]\gamma} X_{\gamma;\kappa} &= H^{\kappa[\beta} b^{\alpha]\gamma} X_{[\gamma;\kappa]} \\
&= \frac{1}{2} \left(H^{\kappa[\beta} b^{\alpha]\gamma} - H^{\gamma[\beta} b^{\alpha]\kappa} \right) X_{\gamma;\kappa} \\
&= \frac{1}{2} K^{\kappa\gamma\alpha\beta} X_{\gamma;\kappa}\ ,
\end{aligned}$$

so, we have obtained

$$\widehat{\mathbb{U}}^{\alpha\beta} = \frac{b}{16\pi} K^{\kappa\gamma\alpha\beta} X_{\gamma;\kappa} + o(r^{\beta-2\alpha})\ .$$

The two terms together give

$$\begin{aligned}
\mathbb{U}^{\alpha\beta} &= \frac{b}{8\pi} \left(K^{\alpha\beta\sigma\kappa}{}_{;\kappa} X_\sigma + \frac{1}{2} K^{\kappa\gamma\alpha\beta} X_{\gamma;\kappa} \right) + o(r^{\beta-2\alpha}) \\
&= \frac{b}{8\pi} \left(K^{\alpha\beta\sigma\kappa}{}_{;\kappa} X_\sigma - \frac{1}{2} K^{\alpha\beta\gamma\kappa} X_{\gamma;\kappa} \right) + o(r^{\beta-2\alpha})\ .
\end{aligned}$$

As $K^{\alpha\beta\gamma\kappa}$ has the same symmetries as the Riemann tensor, we have $K^{\alpha[\beta\gamma\kappa]} = 0$, which implies that $\frac{1}{2} K^{\alpha\beta\gamma\kappa} = K^{\alpha[\kappa\gamma]\beta}$, and

$$\begin{aligned}
\mathbb{U}^{\alpha\beta} &= \frac{b}{8\pi} \left(K^{\alpha\beta\sigma\kappa}{}_{;\kappa} X_\sigma - K^{\alpha[\kappa\gamma]\beta} X_{\gamma;\kappa} \right) + o(r^{\beta-2\alpha}) \\
&= \tilde{V}^{\alpha\beta} + o(r^{\beta-2\alpha})\ .
\end{aligned}$$

If

$$\beta - 2\alpha \leq 0 \ , \tag{C.4}$$

we obtain equality of the Abbott-Deser mass with the Hamiltonian one; recall that $\beta = n$ for the anti-de Sitter type metrics considered in the body of the paper, and Equation (C.4) reproduces the condition $\alpha \geq n/2$, identical to that which arises in the proof of coordinate-invariance of the mass integral.

Summarizing, we have proved:

Proposition C.1 *Suppose that Equations (C.2), (C.3) and (C.4) hold. Then the Hamiltonian mass coincides with the Abbott-Deser one; in particular, either they are both undefined, or both diverge, or both converge to the same values.*

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