Partial derivatives and differentiability (Sect. 14.3)

- Partial derivatives and continuity.
- Differentiable functions $f : D \subset \mathbb{R}^2 \to \mathbb{R}$.
- Differentiability and continuity.
- A primer on differential equations.

Partial derivatives and continuity

Recall: The following result holds for single variable functions.

Theorem

If the function $f : \mathbb{R} \to \mathbb{R}$ is differentiable, then f is continuous.

Proof:

$$\lim_{h\to 0}[f(x+h) - f(x)] = \lim_{h\to 0}\Big[\frac{f(x+h) - f(x)}{h}\Big]\, h,$$

$$\lim_{h\to 0}[f(x+h) - f(x)] = f'(x)\, \lim_{h\to 0} h = 0.$$

That is, $\lim_{h\to 0} f(x+h) = f(x)$, so f is continuous. □

Remark: However, the claim "If $f_x(x,y)$ and $f_y(x,y)$ exist, then $f(x,y)$ is continuous" is false.

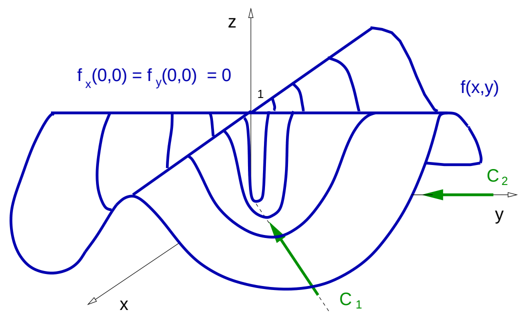
Partial derivatives and continuity

Theorem

If the function $f : \mathbb{R} \to \mathbb{R}$ is differentiable, then f is continuous.

Remark:

- This Theorem is not true for the partial derivatives of a function $f : \mathbb{R}^2 \to \mathbb{R}$.
- There exist functions $f : \mathbb{R}^2 \to \mathbb{R}$ such that $f_x(x_0, y_0)$ and $f_y(x_0, y_0)$ exist but f is not continuous at (x_0, y_0).



Remark: This is a bad property for a differentiable function.

Partial derivatives and continuity

Remark: Here is another discontinuous function at $(0,0)$ having partial derivatives at $(0,0)$.

Example

(a) Show that f is not continuous at $(0,0)$;

(b) Find $f_x(0,0)$ and $f_y(0,0)$, where

$$f(x,y) = \begin{cases} \dfrac{2xy}{x^2+y^2} & (x,y) \neq (0,0), \\ 0 & (x,y) = (0,0). \end{cases}$$

Solution: (a) Along $x = 0$, $f(0,y) = 0$, so $\lim_{y \to 0} f(0,y) = 0$.

Along the path $x = y$, $f(x,x) = \dfrac{2x^2}{2x^2} = 1$, so $\lim_{x \to 0} f(x,x) = 1$.

The Two-Path Theorem implies that $\lim_{(x,y) \to (0,0)} f(x,y)$ DNE.

Partial derivatives and continuity

Example

(a) Show that f is not continuous at $(0,0)$;

(b) Find $f_x(0,0)$ and $f_y(0,0)$, where

$$f(x,y) = \begin{cases} \dfrac{2xy}{x^2+y^2} & (x,y) \neq (0,0), \\ 0 & (x,y) = (0,0). \end{cases}$$

Solution: Recall: $\lim_{(x,y)\to(0,0)} f(x,y)$ DNE.

(b) The partial derivatives are defined at $(0,0)$.

$$f_x(0,0) = \lim_{h\to 0} \frac{1}{h}\big[f(0+h,0) - f(0,0)\big] = \lim_{h\to 0} \frac{1}{h}\,[0-0] = 0.$$

$$f_y(0,0) = \lim_{h\to 0} \frac{1}{h}\big[f(0,0+h) - f(0,0)\big] = \lim_{h\to 0} \frac{1}{h}\,[0-0] = 0.$$

Therefore, $f_x(0,0) = f_y(0,0) = 0$. $\triangleleft$

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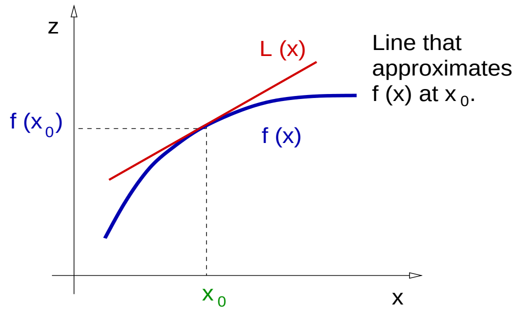
- **Differentiable functions $f : D \subset \mathbb{R}^2 \to \mathbb{R}$.**
- Differentiability and continuity.
- A primer on differential equations.

Differentiable functions $f : D \subset \mathbb{R}^2 \to \mathbb{R}$

Recall: A differentiable function $f : \mathbb{R} \to \mathbb{R}$ at x_0 must be approximated by a line $L(x)$ by $(x_0, f(x_0))$ with slope $f'(x_0)$.



The equation of the tangent line is

$$L(x) = f'(x_0)\,(x - x_0) + f(x_0).$$

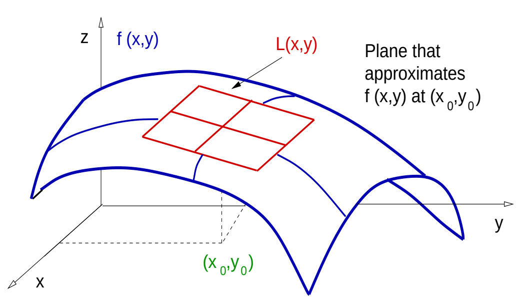
The function f is approximated by the line L near x_0 means

$$f(x) = L(x) + \epsilon_1\,(x - x_0) \quad \text{with } \epsilon_1(x) \to 0 \text{ as } x \to x_0.$$

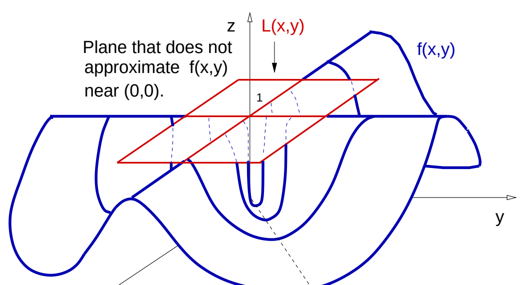
Remark: The graph of a differentiable function $f : D \subset \mathbb{R} \to \mathbb{R}$ is approximated by a line at every point in D.

Differentiable functions $f : D \subset \mathbb{R}^2 \to \mathbb{R}$

Remark: The idea to define differentiable functions:
The graph of a differentiable function $f : D \subset \mathbb{R}^2 \to \mathbb{R}$ is approximated by a plane at every point in D.



Function f is differentiable at (x_0, y_0). (And in its whole domain.)



Function f is not differentiable ONLY at (0,0).

We will show next week that the equation of the plane L is

$$L(x, y) = f_x(x_0, y_0)\,(x - x_0) + f_y(x_0, y_0)\,(y - y_0) + f(x_0, y_0).$$

Differentiable functions $f : D \subset \mathbb{R}^2 \to \mathbb{R}$

Definition

Given a function $f : D \subset \mathbb{R}^2 \to \mathbb{R}$ and an interior point (x_0, y_0) in D, let L be the linear function

$$L(x, y) = f_x(x_0, y_0)\,(x - x_0) + f_y(x_0, y_0)\,(y - y_0) + f(x_0, y_0).$$

The function f is called *differentiable at* (x_0, y_0) iff the function f is approximated by the linear function L near (x_0, y_0), that is,

$$f(x, y) = L(x, y) + \epsilon_1\,(x - x_0) + \epsilon_2\,(y - y_0)$$

where the functions ϵ_1 and $\epsilon_2 \to 0$ as $(x, y) \to (x_0, y_0)$.
The function f is *differentiable* iff f is differentiable at every interior point of D.

Differentiable functions $f : D \subset \mathbb{R}^2 \to \mathbb{R}$

Remark: Recalling the linear function L given above,

$$L(x, y) = f_x(x_0, y_0)\,(x - x_0) + f_y(x_0, y_0)\,(y - y_0) + f(x_0, y_0),$$

an equivalent expression for f being differentiable,

$$f(x, y) = L(x, y) + \epsilon_1\,(x - x_0) + \epsilon_2\,(y - y_0),$$

is the following: Denote $z = f(x, y)$ and $z_0 = f(x_0, y_0)$, and introduce the increments

$$\Delta z = (z - z_0), \quad \Delta y = (y - y_0), \quad \Delta x = (x - x_0);$$

then, the equation above is

$$\Delta z = f_x(x_0, y_0)\,\Delta x + f_y(x_0, y_0)\,\Delta y + \epsilon_1\,\Delta x + \epsilon_2\,\Delta y.$$

(Equation used in the textbook to define a differentiable function.)

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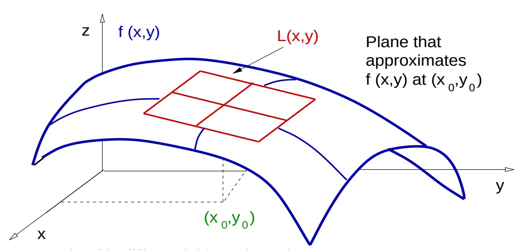
Differentiability and continuity

Remark: We will show in Sect. 14.6 that the graph of a differentiable function $f : D \subset \mathbb{R}^2 \to \mathbb{R}$ is approximated by a plane at every point in D.



Function f is differentiable at (x 0,y 0). (And in its whole domain.)

Theorem
If a function $f : D \subset \mathbb{R}^2 \to \mathbb{R}$ is differentiable, then f is continuous.

Remark: A simple sufficient condition on a function $f : D \subset \mathbb{R}^2 \to \mathbb{R}$ guarantees that f is differentiable.

Theorem
If the partial derivatives f_x and f_y of a function $f : D \subset \mathbb{R}^2 \to \mathbb{R}$ are continuous in an open region $R \subset D$, then f is differentiable in R.

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A primer on differential equations

Remark: A differential equation is an equation where the unknown is a function and the function together with its derivatives appear in the equation.

Example

Given a constant $k \in \mathbb{R}$, find all solutions $f : \mathbb{R} \to \mathbb{R}$ to the differential equation

$$f'(x) = k\, f(x).$$

Solution: Multiply by e^{-kx} the equation above $f'(x) - kf(x) = 0$. The result is $f'(x)\, e^{-kx} - f(x)\, ke^{-kx} = 0$.

The left-hand side is a total derivative, $\big[f(x)\, e^{-kx}\big]' = 0$.

The solution of the equation above is $f(x)e^{-kx} = c$, with $c \in \mathbb{R}$.

Therefore, $f(x) = c\, e^{kx}$. ◁

Remark: Often in physical applications appear three differential equations for functions $f : D \subset \mathbb{R}^n \to \mathbb{R}$, with $n = 2, 3, 4$.

- The Laplace equation: (Gravitation, electrostatics.)

$$f_{xx} + f_{yy} + f_{zz} = 0.$$

- The Heat equation: (Heat propagation, diffusion.)

$$f_t = k\left(f_{xx} + f_{yy} + f_{zz}\right).$$

- The Wave equation: (Light, sound, gravitation.)

$$f_{tt} = v\left(f_{xx} + f_{yy} + f_{zz}\right).$$

Example

Verify that the function $T(t, x) = e^{-4t}\sin(2x)$ satisfies the one-space dimensional heat equation $T_t = T_{xx}$.

Solution: We first compute T_t,

$$T_t = -4e^{-t}\sin(2x).$$

Now compute T_{xx},

$$T_x = 2e^{-t}\cos(2x) \quad \Rightarrow \quad T_{xx} = -4e^{-t}\sin(2x)$$

We conclude that $T_t = T_{xx}$. $\triangleleft$

A primer on differential equations

Example

Verify that $f(x,y,z) = \dfrac{1}{\sqrt{x^2+y^2+z^2}}$ satisfies the Laplace equation : $f_{xx} + f_{yy} + f_{zz} = 0$.

Solution: Recall: $f_x = -\dfrac{x}{(x^2+y^2+z^2)^{3/2}}$. Then,

$$f_{xx} = -\frac{1}{(x^2+y^2+z^2)^{3/2}} + \frac{3}{2}\,\frac{2x^2}{(x^2+y^2+z^2)^{5/2}}.$$

Denote $r = \sqrt{x^2+y^2+z^2}$, then $f_{xx} = -\dfrac{1}{r^3} + \dfrac{3x^2}{r^5}$.

Analogously, $f_{yy} = -\dfrac{1}{r^3} + \dfrac{3y^2}{r^5}$, and $f_{zz} = -\dfrac{1}{r^3} + \dfrac{3z^2}{r^5}$. Then,

$$f_{xx} + f_{yy} + f_{zz} = -\frac{3}{r^3} + \frac{3(x^2+y^2+z^2)}{r^5} = -\frac{3}{r^3} + \frac{3r^2}{r^5} = 0.$$

We conclude that $f_{xx} + f_{yy} + f_{zz} = 0$. $\triangleleft$

A primer on differential equations

Example

Verify that the function $f(t,x) = (vt - x)^3$, with $v \in \mathbb{R}$, satisfies the one-space dimensional wave equation $f_{tt} = v^2 f_{xx}$.

Solution: We first compute f_{tt},

$$f_t = 3v(vt-x)^2 \quad \Rightarrow \quad f_{tt} = 6v^2(vt-x).$$

Now compute f_{xx},

$$f_x = -3(vt-x)^2 \quad \Rightarrow \quad f_{xx} = 6(vt-x).$$

Since $v^2 f_{xx} = 6v^2\,(vt-x)$, then $f_{tt} = v^2 f_{xx}$. $\triangleleft$

A primer on differential equations

Example

Given any $v \in \mathbb{R}$ and any twice continuously differentiable function $u : \mathbb{R} \to \mathbb{R}$, verify that $f(t,x) = u(vt - x)$, satisfies the one-space dimensional wave equation $f_{tt} = v^2 f_{xx}$.

Solution: We first compute f_{tt},

$$f_t = v\, u'(vt - x) \quad \Rightarrow \quad f_{tt} = v^2\, u''(vt - x).$$

Now compute f_{xx},

$$f_x = -u'(vt - x)^2 \quad \Rightarrow \quad f_{xx} = u''(vt - x).$$

Since $v^2 f_{xx} = v^2 u''(vt - x)$, then $f_{tt} = v^2 f_{xx}$. $\lhd$