The Divergence Theorem. (Sect. 16.8)

- The divergence of a vector field in space.
- The Divergence Theorem in space.
- The meaning of Curls and Divergences.
- Applications in electromagnetism:
  - Gauss' law. (Divergence Theorem.)
  - Faraday's law. (Stokes Theorem.)

The divergence of a vector field in space

Definition
The *divergence* of a vector field $\mathbf{F} = \langle F_x, F_y, F_z \rangle$ is the scalar field

$$\operatorname{div} \mathbf{F} = \partial_x F_x + \partial_y F_y + \partial_z F_z.$$

Remarks:

- It is also used the notation $\operatorname{div} \mathbf{F} = \nabla \cdot \mathbf{F}$.
- The divergence of a vector field measures the expansion (positive divergence) or contraction (negative divergence) of the vector field.
- A heated gas expands, so the divergence of its velocity field is positive.
- A cooled gas contracts, so the divergence of its velocity field is negative.

The divergence of a vector field in space

Example

Find the divergence and the curl of $\mathbf{F} = \langle 2xyz, -xy, -z^2 \rangle$.

Solution: Recall: $\operatorname{div} \mathbf{F} = \partial_x F_x + \partial_y F_y + \partial_z F_z$.

$$\partial_x F_x = 2yz, \quad \partial_y F_y = -x, \quad \partial_z F_z = -2z.$$

Therefore $\nabla \cdot \mathbf{F} = 2yz - x - 2z$, that is $\nabla \cdot \mathbf{F} = 2z(y-1) - x$.

Recall: $\operatorname{curl} \mathbf{F} = \nabla \times \mathbf{F}$.

$$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial_x & \partial_y & \partial_z \\ 2xyz & -xy & -z^2 \end{vmatrix} = \langle (0-0), -(0-2xy), (-y-2xz) \rangle$$

We conclude: $\nabla \times \mathbf{F} = \langle 0, 2xy, -(2xz + y) \rangle$. $\triangleleft$

The divergence of a vector field in space

Example

Find the divergence of $\mathbf{F} = \dfrac{\mathbf{r}}{\rho^3}$, where $\mathbf{r} = \langle x, y, z \rangle$, and $\rho = |\mathbf{r}| = \sqrt{x^2 + y^2 + z^2}$. (Notice: $|\mathbf{F}| = 1/\rho^2$.)

Solution: The field components are $F_x = \dfrac{x}{\rho^3}$, $F_y = \dfrac{y}{\rho^3}$, $F_z = \dfrac{z}{\rho^3}$.

$$\partial_x F_x = \partial_x \Big[x \big(x^2 + y^2 + z^2 \big)^{-3/2} \Big]$$

$$\partial_x F_x = \big(x^2 + y^2 + z^2 \big)^{-3/2} - \frac{3}{2} x \big(x^2 + y^2 + z^2 \big)^{-5/2} (2x)$$

$$\partial_x F_x = \frac{1}{\rho^3} - 3\frac{x^2}{\rho^5} \;\Rightarrow\; \partial_y F_y = \frac{1}{\rho^3} - 3\frac{y^2}{\rho^5}, \quad \partial_z F_z = \frac{1}{\rho^3} - 3\frac{z^2}{\rho^5}.$$

$$\nabla \cdot \mathbf{F} = \frac{3}{\rho^3} - 3\frac{(x^2 + y^2 + z^2)}{\rho^5} = \frac{3}{\rho^3} - 3\frac{\rho^2}{\rho^5} = \frac{3}{\rho^3} - \frac{3}{\rho^3}.$$

We conclude: $\nabla \cdot \mathbf{F} = 0$. $\triangleleft$

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The Divergence Theorem in space

Theorem

The flux of a differentiable vector field $\mathbf{F} : \mathbb{R}^3 \to \mathbb{R}^3$ across a closed oriented surface $S \subset \mathbb{R}^3$ in the direction of the surface outward unit normal vector $\mathbf{n}$ satisfies the equation

$$\iint_S \mathbf{F} \cdot \mathbf{n} \, d\sigma = \iiint_V (\nabla \cdot \mathbf{F}) \, dV,$$

where $V \subset \mathbb{R}^3$ is the region enclosed by the surface S.

Remarks:

- The volume integral of the divergence of a field $\mathbf{F}$ in a volume V in space equals the outward flux (normal flow) of $\mathbf{F}$ across the boundary S of V.
- The expansion part of the field $\mathbf{F}$ in V minus the contraction part of the field $\mathbf{F}$ in V equals the net normal flow of $\mathbf{F}$ across S out of the region V.

The Divergence Theorem in space

Example

Verify the Divergence Theorem for the field $\mathbf{F} = \langle x, y, z\rangle$ over the sphere $x^2 + y^2 + z^2 = R^2$.

Solution: Recall: $\displaystyle\iint_S \mathbf{F}\cdot\mathbf{n}\,d\sigma = \iiint_V (\nabla\cdot\mathbf{F})\,dV$.

We start with the flux integral across S. The surface S is the level surface $f = 0$ of the function $f(x, y, z) = x^2 + y^2 + z^2 - R^2$. Its outward unit normal vector $\mathbf{n}$ is

$$\mathbf{n} = \frac{\nabla f}{|\nabla f|}, \quad \nabla f = \langle 2x, 2y, 2z\rangle, \quad |\nabla f| = 2\sqrt{x^2 + y^2 + z^2} = 2R,$$

We conclude that $\mathbf{n} = \dfrac{1}{R}\langle x, y, z\rangle$, where $z = z(x, y)$.

Since $d\sigma = \dfrac{|\nabla f|}{|\nabla f\cdot\mathbf{k}|}\,dx\,dy$, then $d\sigma = \dfrac{R}{z}\,dx\,dy$, with $z = z(x, y)$.

The Divergence Theorem in space

Example

Verify the Divergence Theorem for the field $\mathbf{F} = \langle x, y, z\rangle$ over the sphere $x^2 + y^2 + z^2 = R^2$.

Solution: Recall: $\mathbf{n} = \dfrac{1}{R}\langle x, y, z\rangle$, $d\sigma = \dfrac{R}{z}\,dx\,dy$, with $z = z(x, y)$.

$$\iint_S \mathbf{F}\cdot\mathbf{n}\,d\sigma = \iint_S \Big(\langle x, y, z\rangle\cdot\frac{1}{R}\langle x, y, z\rangle\Big)\,d\sigma.$$

$$\iint_S \mathbf{F}\cdot\mathbf{n}\,d\sigma = \frac{1}{R}\iint_S (x^2 + y^2 + z^2)\,d\sigma = R\iint_S d\sigma.$$

The integral on the sphere S can be written as the sum of the integral on the upper half plus the integral on the lower half, both integrated on the disk $R = \{x^2 + y^2 \leqslant R^2,\ z = 0\}$, that is,

$$\iint_S \mathbf{F}\cdot\mathbf{n}\,d\sigma = 2R\iint_R \frac{R}{z}\,dx\,dy.$$

The Divergence Theorem in space

Example

Verify the Divergence Theorem for the field $\mathbf{F} = \langle x, y, z \rangle$ over the sphere $x^2 + y^2 + z^2 = R^2$.

Solution: $\displaystyle\iint_S \mathbf{F} \cdot \mathbf{n}\, d\sigma = 2R \iint_R \frac{R}{z}\, dx\, dy$.

Using polar coordinates on $\{z = 0\}$, we get

$$\iint_S \mathbf{F} \cdot \mathbf{n}\, d\sigma = 2 \int_0^{2\pi} \int_0^R \frac{R^2}{\sqrt{R^2 - r^2}}\, r\, dr\, d\theta.$$

The substitution $u = R^2 - r^2$ implies $du = -2r\, dr$, so,

$$\iint_S \mathbf{F} \cdot \mathbf{n}\, d\sigma = 4\pi R^2 \int_{R^2}^0 u^{-1/2} \frac{(-du)}{2} = 2\pi R^2 \int_0^{R^2} u^{-1/2} du$$

$$\iint_S \mathbf{F} \cdot \mathbf{n}\, d\sigma = 2\pi R^2 \Big(2u^{1/2} \Big|_0^{R^2} \Big) \quad \Rightarrow \quad \iint_S \mathbf{F} \cdot \mathbf{n}\, d\sigma = 4\pi R^3.$$

The Divergence Theorem in space

Example

Verify the Divergence Theorem for the field $\mathbf{F} = \langle x, y, z \rangle$ over the sphere $x^2 + y^2 + z^2 = R^2$.

Solution: $\displaystyle\iint_S \mathbf{F} \cdot \mathbf{n}\, d\sigma = 4\pi R^3$.

We now compute the volume integral $\displaystyle\iiint_V \nabla \cdot \mathbf{F}\, dV$. The divergence of $\mathbf{F}$ is $\nabla \cdot \mathbf{F} = 1 + 1 + 1$, that is, $\nabla \cdot \mathbf{F} = 3$. Therefore

$$\iiint_V \nabla \cdot \mathbf{F}\, dV = 3 \iiint_V dV = 3\Big(\frac{4}{3}\pi R^3\Big)$$

We obtain $\displaystyle\iiint_V \nabla \cdot \mathbf{F}\, dV = 4\pi R^3$.

We have verified the Divergence Theorem in this case. ◁

The Divergence Theorem in space

Example

Find the flux of the field $\mathbf{F} = \dfrac{\mathbf{r}}{\rho^3}$ across the boundary of the region between the spheres of radius $R_1 > R_0 > 0$, where $\mathbf{r} = \langle x, y, z \rangle$, and $\rho = |\mathbf{r}| = \sqrt{x^2 + y^2 + z^2}$.

Solution: We use the Divergence Theorem

$$\iint_S \mathbf{F} \cdot \mathbf{n} \, d\sigma = \iiint_V (\nabla \cdot \mathbf{F}) \, dV.$$

Since $\nabla \cdot \mathbf{F} = 0$, then $\displaystyle\iiint_V (\nabla \cdot \mathbf{F}) \, dV = 0.$ Therefore

$$\iint_S \mathbf{F} \cdot \mathbf{n} \, d\sigma = 0.$$

The flux along any surface S vanishes as long as $\mathbf{0}$ is not included in the region surrounded by S. ($\mathbf{F}$ is not differentiable at $\mathbf{0}$.) $\triangleleft$

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The meaning of Curls and Divergences

Remarks: The meaning of the Curl and the Divergence of a vector field **F** is best given through the Stokes and Divergence Theorems.

- $\nabla \times \mathbf{F} = \lim_{S \to \{P\}} \frac{1}{A(S)} \oint_C \mathbf{F} \cdot d\mathbf{r}$,

  where S is a surface containing the point P with boundary given by the loop C and $A(S)$ is the area of that surface.
- $\nabla \cdot \mathbf{F} = \lim_{R \to \{P\}} \frac{1}{V(R)} \iint_S \mathbf{F} \cdot \mathbf{n}\, d\sigma$,

  where R is a region in space containing the point P with boundary given by the closed orientable surface S and $V(R)$ is the volume of that region.

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Applications in electromagnetism: Gauss' Law

Gauss' law: Let $q : \mathbb{R}^3 \to \mathbb{R}$ be the charge density in space, and $\mathbf{E} : \mathbb{R}^3 \to \mathbb{R}^3$ be the electric field generated by that charge. Then

$$\iiint_R q\, dV = k \iint_S \mathbf{E} \cdot \mathbf{n}\, d\sigma,$$

that is, the total charge in a region R in space with closed orientable surface S is proportional to the integral of the electric field $\mathbf{E}$ on this surface S.

The Divergence Theorem relates surface integrals with volume integrals, that is, $\iint_S \mathbf{E} \cdot \mathbf{n}\, d\sigma = \iiint_R (\nabla \cdot \mathbf{E})\, dV$.

Using the Divergence Theorem we obtain the differential form of Gauss' law,

$$\nabla \cdot \mathbf{E} = \frac{1}{k}\, q.$$

Applications in electromagnetism: Faraday's Law

Faraday's law: Let $B : \mathbb{R}^3 \to \mathbb{R}^3$ be the magnetic field across an orientable surface S with boundary given by the loop C, and let $\mathbf{E} : \mathbb{R}^3 \to \mathbb{R}^3$ measured on that loop. Then

$$\frac{d}{dt} \iint_S \mathbf{B} \cdot \mathbf{n}\, d\sigma = -\oint_C \mathbf{E} \cdot d\mathbf{r},$$

that is, the time variation of the magnetic flux across S is the negative of the electromotive force on the loop.

The Stokes Theorem relates line integrals with surface integrals, that is, $\oint_C \mathbf{E} \cdot \mathbf{r} = \iiint_S (\nabla \times \mathbf{E})\, d\sigma$.

Using the Divergence Theorem we obtain the differential form of Gauss' law,

$$\partial_t \mathbf{B} = -\nabla \times \mathbf{E}.$$