Two-by-Two Phase Portraits (Sect. 5.4)

- Review: 2×2 Diagonalizable Systems
- Phase Portraits
  - Real Distinct Eigenvalues
  - Complex Eigenvalues
  - Repeated Eigenvalue
- 2×2 Nondiagonalizable Systems

Review: 2×2 Diagonalizable Systems

Theorem

If the 2×2 constant matrix A is diagonalizable with eigenvalues $\lambda_{\pm}$ and corresponding eigenvectors $\mathbf{v}^{\pm}$, then the general solution to the linear system $\mathbf{x}' = A\,\mathbf{x}$ is

$$\mathbf{x}_{\rm gen}(t) = c_{+}\mathbf{v}^{+}e^{\lambda_{+}t} + c_{-}\mathbf{v}^{-}e^{\lambda_{-}t}.$$

Remark: We classify these systems by their matrix eigenvalues:

(A) The eigenvalues λ_{+}, λ_{-} are real and distinct.

(B) The eigenvalues $\lambda_{\pm} = \alpha \pm \beta i$ are distinct and complex.

(C) The eigenvalues $\lambda_{+} = \lambda_{-} = \lambda_{0}$ is repeated and real.

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Phase Portraits

Remark:

- Two types of graphs of solutions to differential systems:
  (i) The graphs of the vector components;
  (ii) The phase portrait.
- Case (i): Express the solution in vector components
  $\mathbf{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$, and graph x_1 and x_2 as functions of t.
  (Recall the solution in the Example from the previous lecture:
  $x_1(t) = 3\,e^{4t} - e^{-2t}$ and $x_2(t) = 3\,e^{4t} + e^{-2t}$.)
- Case (ii): Express the solution as a vector-valued function,
  $$\mathbf{x}(t) = c_1\,\mathbf{v}_1\,e^{\lambda_1 t} + c_2\,\mathbf{v}_2\,e^{\lambda_2 t},$$
  and plot the vector $\mathbf{x}(t)$ for different values of t.
- Case (ii) is called a *phase portrait*.

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Real Distinct Eigenvalues

Example

Plot the phase portrait of several linear combinations of the fundamental solutions

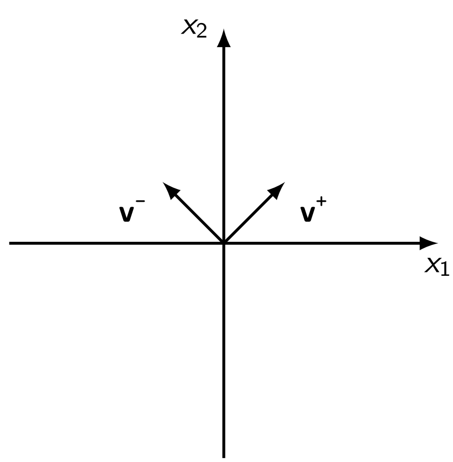
$$\mathbf{x}^{(+)} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{4t}, \quad \mathbf{x}^{(-)} = \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{-2t}, \quad \mathbf{x} = c_- \, \mathbf{x}^{(+)} + c_+ \, \mathbf{x}^{(-)}.$$

Solution:

We start plotting the vectors

$$\mathbf{v}^{+} = \begin{bmatrix} 1 \\ 1 \end{bmatrix},$$

$$\mathbf{v}^{-} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}.$$



Real Distinct Eigenvalues

Example

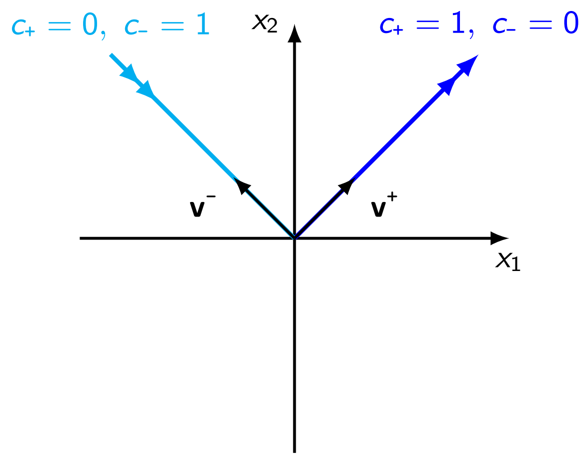
Plot the phase portrait of several linear combinations of the fundamental solutions

$$\mathbf{x}^{(+)} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{4t}, \quad \mathbf{x}^{(-)} = \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{-2t}, \quad \mathbf{x} = c_+ \mathbf{x}^{(+)} + c_- \mathbf{x}^{(-)}.$$

Solution:
We now plot the functions

$$\mathbf{x}^{(+)} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{4t},$$

$$\mathbf{x}^{(-)} = \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{-2t}.$$



Real Distinct Eigenvalues

Example

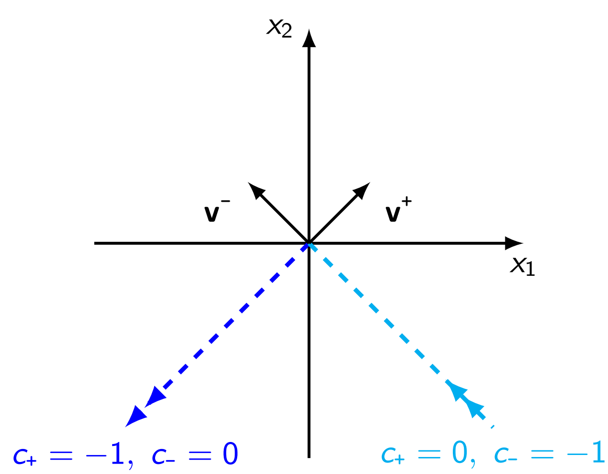
Plot the phase portrait of several linear combinations of the fundamental solutions

$$\mathbf{x}^{(+)} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{4t}, \quad \mathbf{x}^{(-)} = \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{-2t}, \quad \mathbf{x} = c_+ \mathbf{x}^{(+)} + c_- \mathbf{x}^{(-)}.$$

Solution:
We now plot the functions

$$-\mathbf{x}^{(+)} = -\begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{4t},$$

$$-\mathbf{x}^{(-)} = -\begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{-2t}.$$



Real Distinct Eigenvalues

Example

Plot the phase portrait of several linear combinations of the fundamental solutions found above,

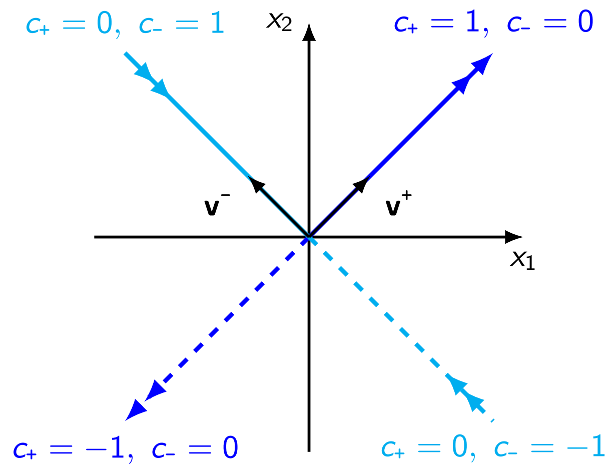
$$\mathbf{x}^{(+)} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{4t}, \quad \mathbf{x}^{(-)} = \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{-2t}, \quad \mathbf{x} = c_+ \mathbf{x}^{(+)} + c_- \mathbf{x}^{(-)}.$$

Solution:

We now plot the four functions

$$\mathbf{x}^{(+)}, \quad -\mathbf{x}^{(+)},$$

$$\mathbf{x}^{(-)}, \quad -\mathbf{x}^{(-)}.$$



Real Distinct Eigenvalues

Example

Plot the phase portrait of several linear combinations of the fundamental solutions found above,

$$\mathbf{x}^{(+)} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{4t}, \quad \mathbf{x}^{(-)} = \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{-2t}, \quad \mathbf{x} = c_+ \mathbf{x}^{(+)} + c_- \mathbf{x}^{(-)}.$$

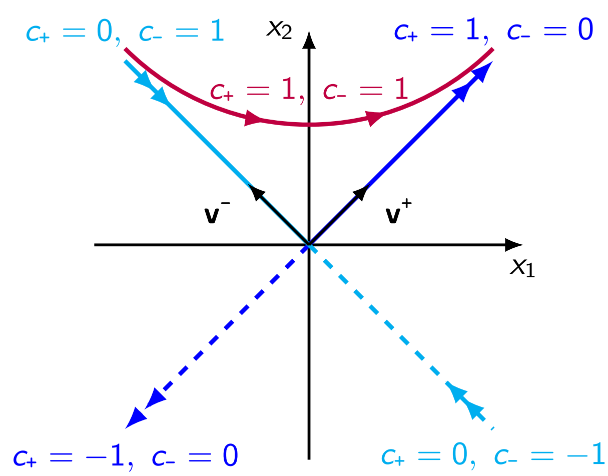
Solution:

We now plot the four functions

$$\mathbf{x}^{(+)}, \; -\mathbf{x}^{(+)}, \; \mathbf{x}^{(-)}, \; -\mathbf{x}^{(-)},$$

and $\mathbf{x}^{(+)} + \mathbf{x}^{(-)}$,

$$\begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{4t} + \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{-2t}.$$



Real Distinct Eigenvalues

Example

Plot the phase portrait of several linear combinations of the fundamental solutions found above,

$$\mathbf{x}^{(+)} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{4t}, \quad \mathbf{x}^{(-)} = \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{-2t}, \quad \mathbf{x} = c_+ \mathbf{x}^{(+)} + c_- \mathbf{x}^{(-)}.$$

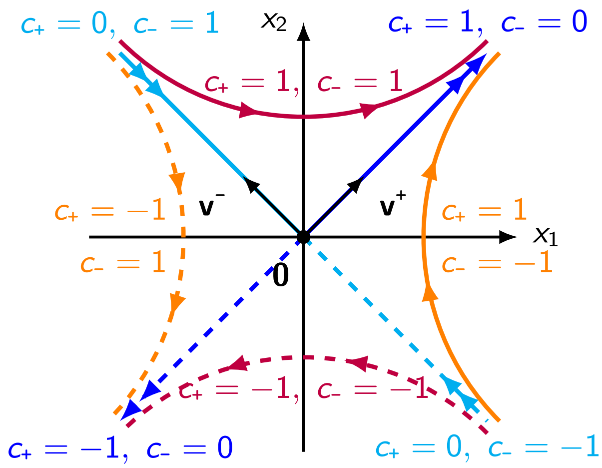
Solution:

We now plot the eight functions

$\mathbf{x}^{(+)}, \ -\mathbf{x}^{(+)}, \ \mathbf{x}^{(-)}, \ -\mathbf{x}^{(-)},$

$\mathbf{x}^{(+)} + \mathbf{x}^{(-)}, \quad -\mathbf{x}^{(+)} + \mathbf{x}^{(-)},$

$\mathbf{x}^{(+)} - \mathbf{x}^{(-)}, \quad -\mathbf{x}^{(+)} - \mathbf{x}^{(-)}.$



Real Distinct Eigenvalues

Problem:

Consider a 2×2 matrix A having two different, real eigenvalues $\lambda_+ \neq \lambda_-$, so A has two non-proportional eigenvectors $\mathbf{v}^+$, $\mathbf{v}^-$ (eigen-directions).

Given a solution $\mathbf{x}(t) = c_+ \mathbf{v}^+ e^{\lambda_+ t} + c_- \mathbf{v}^- e^{\lambda_- t}$, to $\mathbf{x}'(t) = A\,\mathbf{x}(t)$, plot different solution vectors $\mathbf{x}(t)$ on the plane as function of t for different choices of the constants c_+ and c_-.

The plots are different depending on the eigenvalues signs.
We have the following three sub-cases:

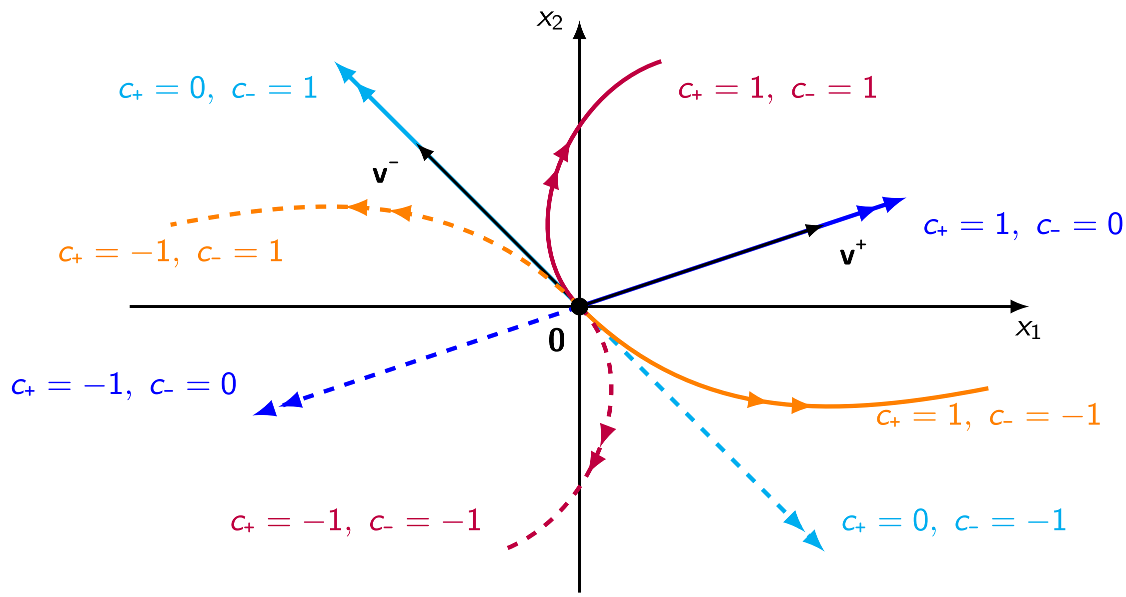
(i) $0 < \lambda_- < \lambda_+$, both positive;

(ii) $\lambda_- < 0 < \lambda_+$, one positive the other negative;

(iii) $\lambda_- < \lambda_+ < 0$, both negative.

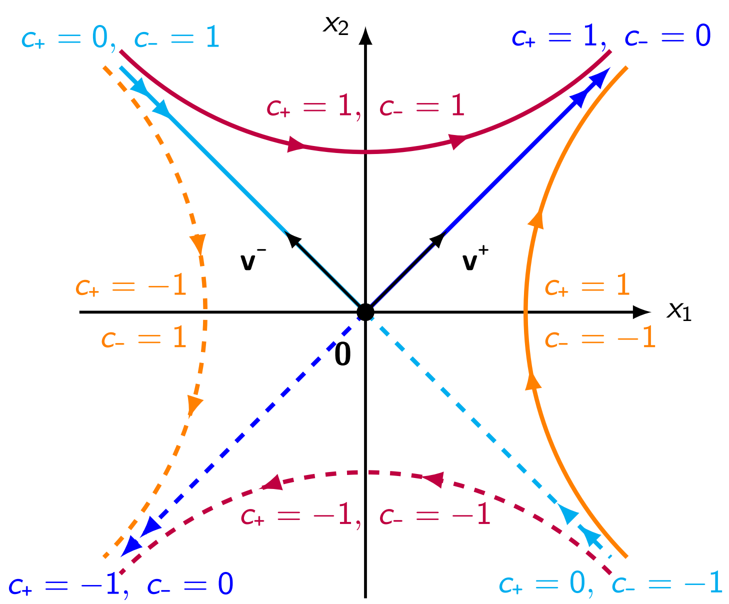
Real Distinct Eigenvalues (Unstable)

Phase portrait: Case (i), two different, real eigenvalues $\lambda_- \neq \lambda_+$, sub-case $0 < \lambda_- < \lambda_+$, both eigenvalue positive.



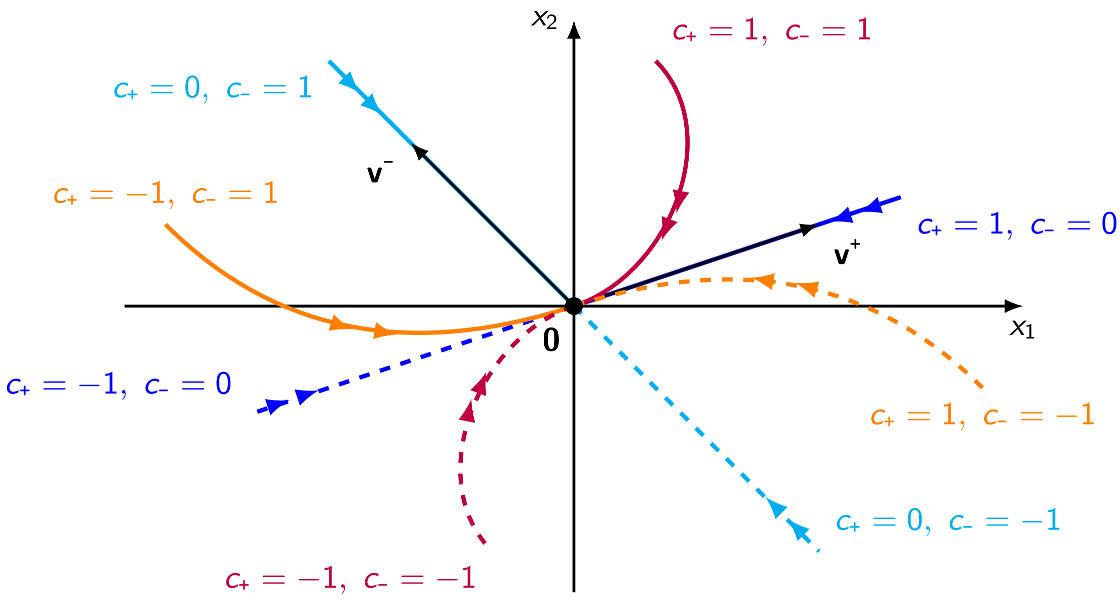
Real Distinct Eigenvalues (Saddle)

Phase portrait: Case (ii), two different, real eigenvalues $\lambda_- \neq \lambda_+$, sub-case $\lambda_- < 0 < \lambda_+$, one eigenvalue positive the other negative.



Real Distinct Eigenvalues (Stable)

Phase Portrait: Case (iii), two different, real eigenvalues $\lambda_- \neq \lambda_+$, sub-case $\lambda_- < \lambda_+ < 0$, both eigenvalues negative.



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Complex Eigenvalues

Example

Sketch a phase portrait for solutions of $\mathbf{x}' = A\mathbf{x}$, $A = \begin{bmatrix} 2 & 3 \\ -3 & 2 \end{bmatrix}$,

with $\lambda_{\pm} = 2 \pm 3i$ and $\mathbf{v}^{\pm} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \pm \begin{bmatrix} -1 \\ 0 \end{bmatrix} i$.

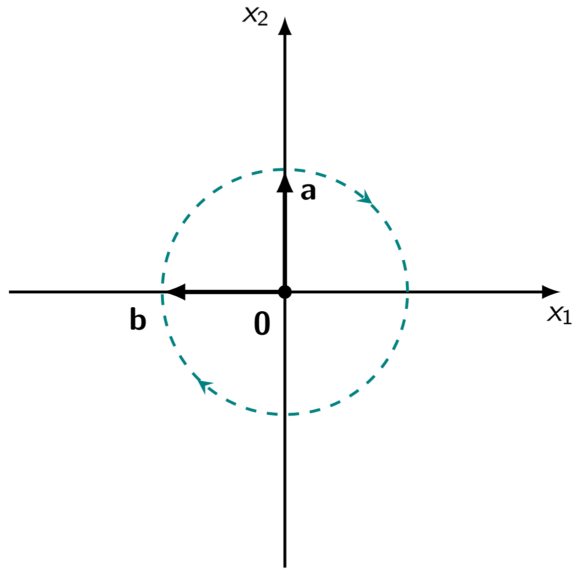
Solution:

The phase portrait of the vectors

$$\tilde{\mathbf{x}}^{(1)} = \begin{bmatrix} \sin(3t) \\ \cos(3t) \end{bmatrix},$$

$$\tilde{\mathbf{x}}^{(2)} = \begin{bmatrix} -\cos(3t) \\ \sin(3t) \end{bmatrix},$$

is a radius one circle.



Complex Eigenvalues (Unstable)

Example

Sketch a phase portrait for solutions of $\mathbf{x}' = A\mathbf{x}$, $A = \begin{bmatrix} 2 & 3 \\ -3 & 2 \end{bmatrix}$,

with $\lambda_{\pm} = 2 \pm 3i$ and $\mathbf{v}^{\pm} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \pm \begin{bmatrix} -1 \\ 0 \end{bmatrix} i$.

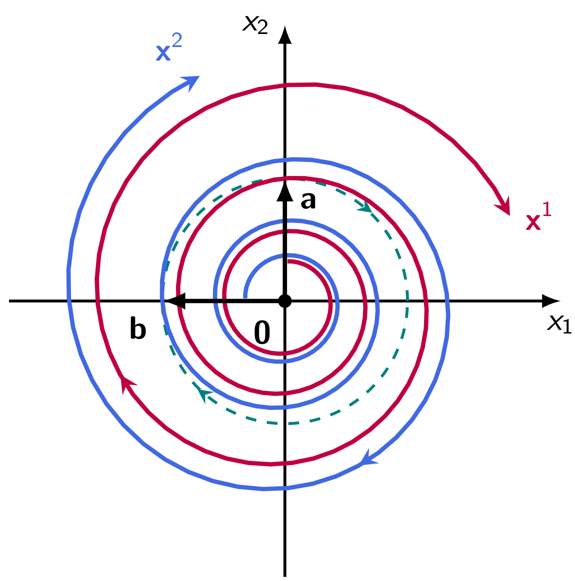
Solution:

The phase portrait of the solutions

$$\mathbf{x}^{(1)} = \begin{bmatrix} \sin(3t) \\ \cos(3t) \end{bmatrix} e^{2t},$$

$$\mathbf{x}^{(2)} = \begin{bmatrix} -\cos(3t) \\ \sin(3t) \end{bmatrix} e^{2t},$$

are outgoing spirals.



Complex Eigenvalues (Unstable) (Center) (Stable)

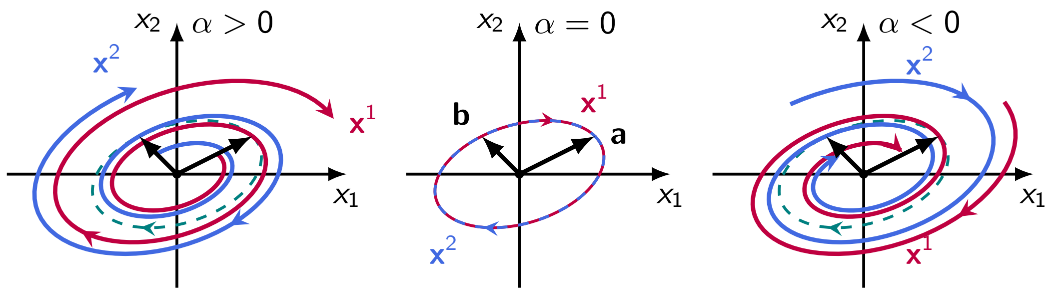
Example

Given any vectors **a** and **b**, sketch qualitative phase portraits of

$$\mathbf{x}^{(1)} = \big[\mathbf{a}\cos(\beta t) - \mathbf{b}\sin(\beta t)\big]\,e^{\alpha t},\ \mathbf{x}^{(2)} = \big[\mathbf{a}\sin(\beta t) + \mathbf{b}\cos(\beta t)\big]\,e^{\alpha t}.$$

for the cases $\alpha > 0$, $\alpha = 0$, and $\alpha < 0$, where $\beta > 0$.

Solution:



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- 2×2 **Nondiagonalizable Systems**

2×2 Nondiagonalizable Systems

Example

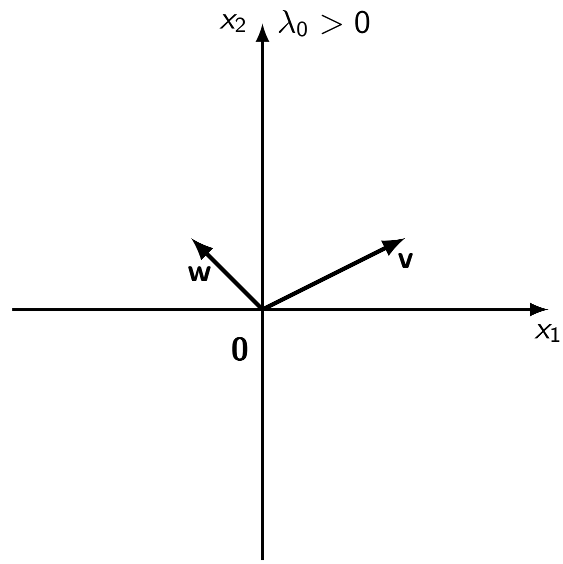
Sketch a phase portrait of solutions to $\mathbf{x}' = A\mathbf{x}$, with A nondiagonalizable and solutions $\mathbf{x}^{(1)} = \mathbf{v}\, e^{\lambda_0 t}$, $\mathbf{x}^{(2)} = (t\mathbf{v} + \mathbf{w})\, e^{\lambda_0 t}$.

Solution:

We assume $\lambda_0 > 0$.

We start plotting the vectors $\mathbf{v}$ and $\mathbf{w}$.

Suppose they are as in the Figure.



2×2 Nondiagonalizable Systems

Example

Sketch a phase portrait of solutions to $\mathbf{x}' = A\mathbf{x}$, with A nondiagonalizable and solutions $\mathbf{x}^{(1)} = \mathbf{v}\, e^{\lambda_0 t}$, $\mathbf{x}^{(2)} = (t\mathbf{v} + \mathbf{w})\, e^{\lambda_0 t}$.

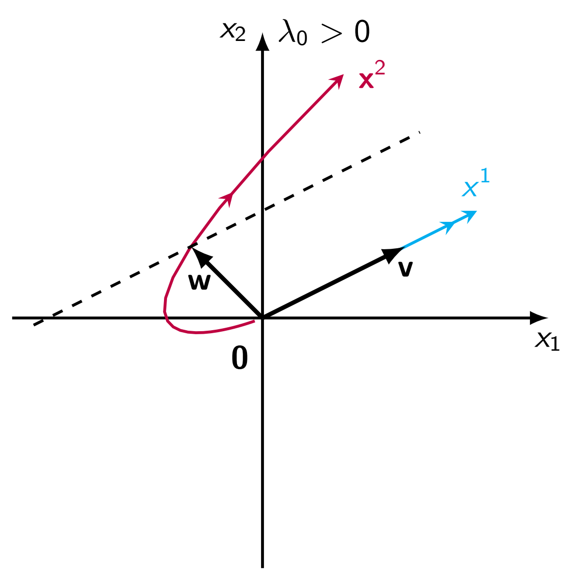
Solution:

We assume $\lambda_0 > 0$.

We plot the functions

$$\mathbf{x}^{(1)} = \mathbf{v}\, e^{\lambda_0 t},$$

$$\mathbf{x}^{(2)} = (t\mathbf{v} + \mathbf{w})\, e^{\lambda_0 t}.$$



2×2 Nondiagonalizable Systems (Unstable)

Example

Sketch a phase portrait of solutions to $\mathbf{x}' = A\mathbf{x}$, with A nondiagonalizable and solutions $\mathbf{x}^{(1)} = \mathbf{v}\, e^{\lambda_0 t}$, $\mathbf{x}^{(2)} = (t\mathbf{v} + \mathbf{w})\, e^{\lambda_0 t}$.

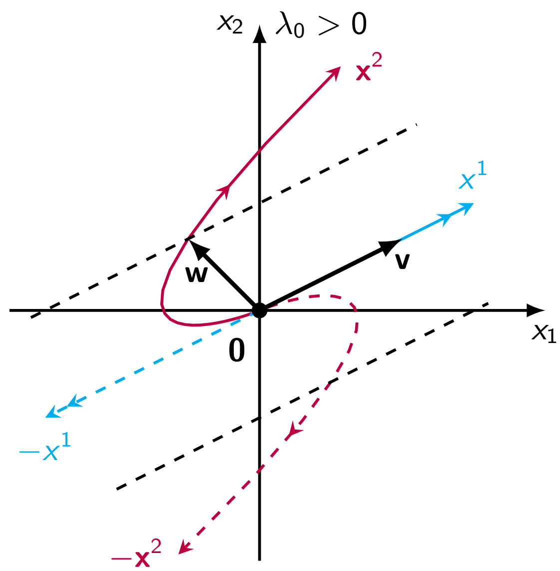
Solution:

We assume $\lambda_0 > 0$.

We plot the functions

$$\mathbf{x}^{(1)}, \quad -\mathbf{x}^{((1)},$$

$$\mathbf{x}^{(2)}, \quad -\mathbf{x}^{(2)}.$$



2×2 Nondiagonalizable Systems (Stable)

Example

Sketch a phase portrait of solutions to $\mathbf{x}' = A\mathbf{x}$, with A nondiagonalizable and solutions $\mathbf{x}^{(1)} = \mathbf{v}\, e^{\lambda_0 t}$, $\mathbf{x}^{(2)} = (t\mathbf{v} + \mathbf{w})\, e^{\lambda_0 t}$.

Solution:

We now assume $\lambda_0 < 0$.

We plot the functions

$$\mathbf{x}^{(1)}, \quad -\mathbf{x}^{(1)},$$

$$\mathbf{x}^{(2)}, \quad -\mathbf{x}^{(2)}.$$