Geometric structures of 3-manifolds and quantum invariants

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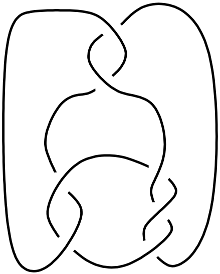
Settings and general theme of talk

3-manifolds: M=compact, orientable, with empty or tori boundary.

Knots: Smooth embedding $K : S^1 \to S^3$. Knots K_1, K_2 are equivalent if $f(K_1) = K_2$, f orientation preserving diffeomorphism of S^3.



Talk: Relations among three perspectives.

Combinatorial presentations

- knot diagrams, triangulations –Cut/paste

3-manifold topology/geometry

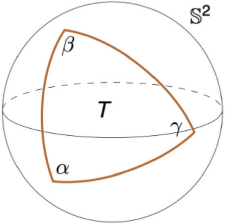
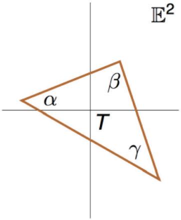
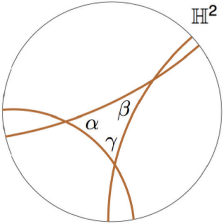
- Geometric structures on M (e.g. $M = S^3 \setminus K$) and geometric invariants

Physics originated invariants

- Quantum invariants of knots/3-manifolds

Warm up: 2-d Model Geometries:

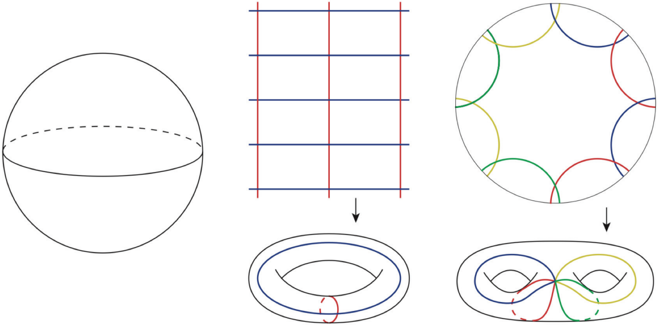
For this talk, an n-dimensional *model geometry* is a simply connected n-manifold with a “homogeneous” Riemannian metric.
In dimension 2, there are exactly three model geometries, up to scaling:

Spherical	Eucledian	Hyperbolic
		
curvature $= +1$	curvature $= 0$	curvature $= -1$
$\text{Area}(T) = (\alpha+\beta+\gamma) - \pi$	$\alpha+\beta+\gamma = \pi$	$\text{Area}(T) = \pi - (\alpha+\beta+\gamma)$

Geometrization (a.k.a. Uniformization) in 2-d:

Every (closed, orientable) surface can be written as $S = X/G$, where X is a model geometry and G is a discrete group of isometries.

$X = \mathbf{S}^2$ $\qquad$ $X = \mathbb{E}^2$ $\qquad$ $X = \mathbb{H}^2$



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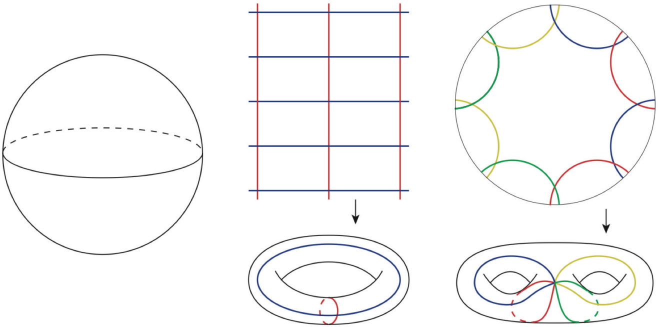


- Geometry relates to topology: $k \cdot Area(S) = 2\pi\chi(S)$, $k = 1, 0, -1$ (*curvature*).

Geometrization in 3-d:

In dimension 3, there are eight model geometries:

$$X = \mathbf{S}^3 \quad \mathbb{E}^3 \quad \mathbb{H}^3, \ \mathbf{S}^2 \times \mathbb{R}, \quad \mathbb{H}^2 \times \mathbb{R}, \quad Sol, \quad Nil, \quad \widetilde{SL_2(\mathbb{R})}$$

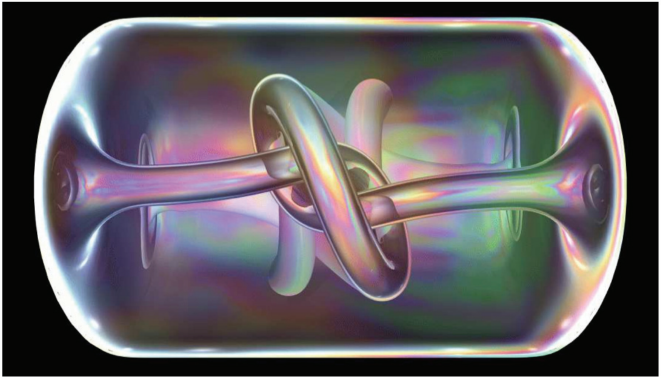
Theorem (Thurston 1980 + Perelman 2003)

For every (compact, oriented) 3-manifold M, there is a canonical way to cut M along spheres and tori into pieces $M_1, \ldots, M_n$, such that each piece is $M_i = X_i/G_i$, where G_i is a discrete group of isometries of the model geometry X_i .

- Canonical : "Unique" collection of spheres and tori.
- The Poincare conjecture is a special case ($\mathbf{S}^3$ is the only compact model).
- Hyperbolic 3-manifolds are a prevalent, rich and very interesting class.
- Because of cutting along tori, manifolds with toroidal boundary will naturally arise. Knot complements fit in this class.

Knots complements; nice 3-manifolds with boundary:

Given K remove an open tube around K to obtain the *Knot complement:*
Notation. $M_K = S^3 \setminus K$.



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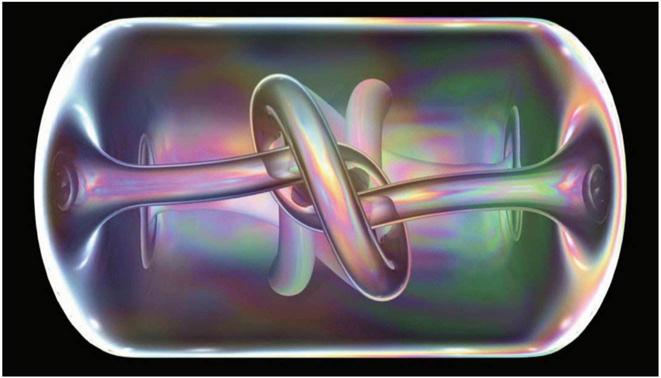


Knot complements can be visualized!

More on the Geometric decomposition

Theorem (Knesser, Milnor 60's, Jaco-Shalen, Johanson 1970, Thurston 1980 + Perelman 2003)

M=oriented, compact, with empty or toroidal boundary.

1. *There is a unique collection of 2-spheres that decompose M*

$$M = M_1 \# M_2 \# \ldots \# M_p \# (\# S^2 \times S^1)^k,$$

   where $M_1, \ldots, M_p$ are compact orientable irreducible 3-manifolds.

2. *For M=irreducible, there is a unique collection of disjointly embedded essential tori $\mathcal{T}$ such that all the connected components of the manifold obtained by cutting M along $\mathcal{T}$, are either Seifert fibered manifolds or hyperbolic.*

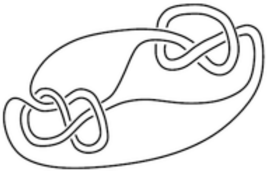
- *Seifert fibered manifolds:* For this talk, think of it as

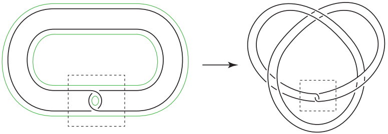
$$S^1 \times \text{surface with boundary} + \text{union of solid tori}.$$

  Complete topological classification [Seifert, 60']
- *Hyperbolic:* Interior admits complete, hyperbolic metric of finite volume.

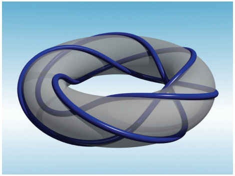
Thee types of knots:

Satellite Knots: Complement contains embedded “essential” tori; There is a *canonical* (finite) collection of such tori.





Torus knots: Knot embeds on standard torus in T in S^3 and is determined by its class in $H_1(T)$. Complement is SFM.



Hyperbolic knots: Rest of them.

Rigidity for hyperbolic 3-manifolds:

Theorem (Mostow, Prasad 1973)

Suppose M is compact, oriented, and ∂M is a possibly empty union of tori. If M is hyperbolic (that is: $M \setminus \partial M = \mathbb{H}^3/G$), then G is unique up to conjugation by hyperbolic isometries. In other words, a hyperbolic metric on M is essentially unique.

M =hyperbolic 3-manifold:

- By rigidity, every geometric measurement of M (volume, areas of surface etc.) is a *topological invariant*
- In practice M is represented by combinatorial data such as, *a triangulation*, a *Heegaard diagram*, a *Dehn surgery diagram* or a *knot diagram* (in case of knot complements).

Challenging Question: How do we see geometry in the combinatorial descriptions of M? Can we calculate/estimate geometric invariants from combinatorial ones? (Highly active research area)

Gromov Norm/Volume highlights:

- Recall M uniquely decomposes along spheres and tori into disjoint unions of Seifert fibered spaces and hyperbolic pieces $M = S \cup H$,
- Gromov, Thurston, 80's:
- *Gromov norm of M:* $||M|| = v_3 \mathrm{Vol}\,(H)$, $\mathrm{Vol}\,(H)$ is the sum of the hyperbolic volumes of components of H and v_3 is the volume of the regular hyperbolic tetrahedron.
- $||M||$ is additive under disjoint union and connected sums of manifolds.
- If M hyperbolic $||M|| = v_3 \mathrm{Vol}\,(M)$.
- If M Seifert fibered then $||M|| = 0$
- If M contains an embedded torus T and M' is obtained from M by cutting along T then

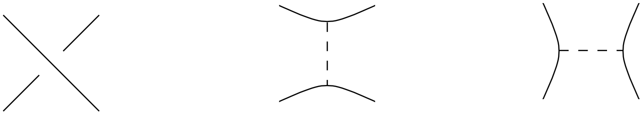
$$||M|| \leqslant ||M'||.$$

  Moreover, the inequality is an equality if T is incompressible in M.

Quantum invariants: Jones Polynomials

1980's: Ideas originated in physics and in representation theory led to vast families invariants of knots and 3-manifolds. (*Quantum invariants*)

- *Jones Polynomials*: Discovered by V. Jones (1980's); using braid group representations coming from the theory of certain operator algebras (sub factors).
- Can be calculated from any link diagram using, for example, Kaufman states:
- Two choices for each crossing, *A* or *B* resolution.



- Choice of *A* or *B* resolutions for all crossings: *state* σ.
- Assign a "*weight*" to every state.
- JP calculated as a certain "*state sum*" over all states of any diagram.

Quantum invariants: Colored Jones Polynomials

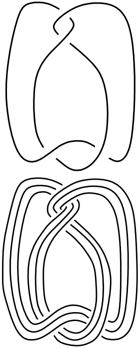
For this talk we discuss:

- The *Colored Jones Polynomials*: Infinite sequence of Laurent polynomials $\{J_{K,n}(t)\}_n$ encoding the *Jones polynomial* of K and these of the links K^s that are the parallels of K.
- Formulae for $J_{K,n}(t)$ come from representation theory of Lie Groups!: representation theory of $SU(2)$ (decomposition of tensor products of representations). For example,
  They look like

$$J_{K,1}(t) = 1, \quad J_{K,2}(t) = J_K(t)\text{- } \textit{Original JP,}$$

$$J_{K,3}(t) = J_{K^2}(t) - 1, \quad J_{K,4}(t) = J_{K^3}(t) - 2J_K(t), \ \ldots$$

- $J_{K,n}(t)$ can be calculated from any knot diagram via processes such as *Skein Theory*, *State sums*, *R-matrices*, *Fusion rules*....



The CJP predicts Volume?

Question: How do the *CJP* relate to geometry/topology of knot complements?

Kashaev+ H. Murakami - J. Murakami (2000) proposed

Volume conjecture. Suppose K is a knot in S^3. Then

$$2\pi \cdot \lim_{n\to\infty} \frac{\log |J_K(e^{2\pi i/n})|}{n} = v_3||S^3 \setminus K||$$

- The conjecture is wide open: Few verifications by brute force calculations.
- Knots up to 7 crossings Ohtsuki),
- Simple families of knots of zero Gromov norm zero (Zheng, Kashaev).

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Some difficulties:

- "State sum" for $J_K(e^{\pi i/2n})$ very oscillating; is often $J_K(e^{\pi i/2n}) = 0$.
- No good behavior of $J_K(e^{\pi i/2n})$ with respect to geometric decompositions.

Coarse relations: Colored Jones polynomial

For a knot K, and $n = 1, 2, \ldots,$ we write its *n-colored Jones polynomial*:

$$J_{K,n}(t) := \alpha_n t^{m_n} + \beta_n t^{m_n - 1} + \cdots + \beta'_n t^{k_n + 1} + \alpha'_n t^{k_n} \in \mathbb{Z}[t, t^{-1}]$$

- (Garoufalidis-Le, 04): Each of $\alpha'_n, \beta'_n \ldots$ satisfies a *linear recursive relation* in n, with integer coefficients .

  (e. g. $\alpha'_{n+1} + (-1)^n \alpha'_n = 0$).

- Given a knot K any diagram $D(K)$, there exist explicitly given functions $M(n, D)$ $m_n \leq M(n, D)$. For nice knots where $m_n = M(n, D)$ we have *stable coefficients*
- (Dasbach-Lin, Armond) If $m_n = M(n, D)$, then

  $$\beta'_K := |\beta'_n| = |\beta'_2|, \quad \text{and} \quad \beta_K := |\beta_n| = |\beta_2|,$$

  for every $n > 1$.
- Stable coefficients control the volume of the link complement.

A Coarse Volume Conjecture

Theorem (Dasbach-Lin, Futer-K.-Purcell, Giambrone, 05-'15')

There universal constants $A, B > 0$ such that for any hyperbolic link that is nice we have

$$A\,(\beta'_K + \beta_K) \;\leq\; Vol\,(S^3 \smallsetminus K) \;<\; B\,(\beta'_K + \beta_K).$$

Question. Does there exist function $B(K)$ of the coefficients of the colored Jones polynomials of a knot K, that is easy to calculate from a "nice" knot diagram such that for hyperbolic knots, $B(K)$ is coarsely related to hyperbolic volume $\mathrm{Vol}\,(S^3 \smallsetminus K)$?
Are there constants $C_1 \geq 1$ and $C_2 \geq 0$ such that

$$C_1^{-1} B(K) - C_2 \;\leq\; \mathrm{Vol}\,(S^3 \smallsetminus K) \;\leq\; C_1 B(K) + C_2,$$

for all hyperbolic K?

- C. Lee, Proved CVC for more classes of knots (2017)

Turaev-Viro invariants and a more general volume conjecture

- Families of real valued invariants $TV_r(M, q)$ of a compact oriented 3-manifold M; indexed by a positive integer r, *the level* and for each r they depend on an $2r$-th root of unity, q. [Turaev-Viro, 1990]
- $TV_r(M, q)$ are combinatorially defined invariants and can be computed from triangulations of M by a *state sum* formula. Sums involve *quantum* $6j$*-sympols*. Terms are highly "oscillating". Combinatorics relay have roots on representations of Lie groups.
- For this talk: $TV_r(M) := TV_r(M, e^{\frac{2\pi i}{r}})$, $r = odd$ and $q = e^{\frac{2\pi i}{r}}$.
- For experts: These correspond to the $SO(3)$ quantum group.
- (Q. Chen- T. Yang, 2015): compelling experimental evidence supporting
- **Conjecture :** For M compact, orientable

$$\lim_{r \to \infty} \frac{2\pi}{r} \log(TV_r(M, e^{\frac{2\pi i}{r}})) = v_3 ||M||,$$

where r runs over odd integers.

Recent results (Detcherrry-K.-Yang, 2016)

- For $M = S^3 \setminus L$, a link complement in S^3, the invariants $TV_r(M)$ can be expressed in terms of the colored Jones polynomial of L.
- Gave first examples of "large r" asymptotics of $TV_r(M, e^{\frac{2\pi i}{r}})$ are calculated and verified the Chen-Yang conjecture for some link complements (Borromean rings, Figure-eight-hyperbolic manifolds).
- Proved Conjecture for Knots of Gromov norm zero.
- Conjecture is compatible with disjoint unions of links and connect sums (Warning: Original volume conjecture is not!).
- We have
$$\liminf_{r\to\infty} \frac{2\pi}{r} \log(TV_r(M)) \geq 0.$$
- We discover "new" exponential growth phenomena of the colored Jones polynomial at values that are not predicted by the Kashaev-Murakami-Murakami conjecture or generalizations.

Recent results, con't

$$LTV(M) = \limsup_{r \to \infty} \frac{2\pi}{r} \log(TV_r(M))$$

where r runs over all odd integers. The main result of this article is the following:

Theorem (Detcherry-K., 2017)

There exists a universal constant $B > 0$ such that for any compact orientable 3-manifold M with empty or toroidal boundary we have

$$B \cdot LTV(M) \leqslant ||M||,$$

where the constant B is about 1.1964×10^{-10}.

Corollary

For any link $K \subset S^3$ with $||S^3 \smallsetminus K|| = 0$, we have

$$LTV(M) = \lim_{r \to \infty} \frac{2\pi}{r} \log(TV_r(M \smallsetminus K)) = v_3 ||S^3 \smallsetminus K|| = 0,$$

Why are TV invariants “better”?

- TV invariants are defined for all compact, oriented 3-manifolds.
- TV invariants are defined on triangulations of 3-manifolds: For hyperbolic 3-manifolds the (hyperbolic) volume can be estimated/calculated from appropriate triangulations.
- TV invariants are part of a Topological Quantum Field Theory (TQFT) and they cab be computed by cutting and gluing 3-manifolds along surfaces. The TQFT behaves when citing along spheres and tori; in particular with respect to prime and JSJ decompositions.
- For experts: The TQFT is the $SO(3)$- Reshetikhin-Turaev and Witten TQFT as constructed by Blanchet, Habegger, Masbaum and Vogel (1995)

Outline of proof of main result:

1. Study the large-r asymptotic behavior of the quantum $6j$-symbols, and using the state sum formulae for the invariants TV_r, to prove give linear upper bound of $LTV(M$ in terms of the number of tetrahedra in any triangulation of M. In particular, $LTV(M) < \infty$.
2. Use step (1) and a theorem of Thurston to show that there is $C > 0$ such that for any hyperbolic 3-manifold M $LTV(M) \leq C||M||$.
3. Use TQFT properties to show that if M is a Seifert fibered manifold, then $LTV(M) = ||M|| = 0$.
4. Show that If M contains an embedded tori T and M' is obtained from M by cutting along T then
$$LTV(M) \leqslant LTV(M').$$
5. Show that $LTV(M)$ is (sub)additive under connected sum and disjoint unions.
6. Use geometric decomposition of 3-manifolds and parallel behavior of $LTV(M)$ and $||M||$ to prove theorem.

New exponential growth results:

- Chen-Tian Conjecture implies that

$$ITV(M) := \liminf_{r\to\infty} \frac{2\pi}{r} \log(TV_r(M)) > 0,$$

  for any complete, hyperbolic 3-manifold of finite volume. This property is very hard to establish using the state sum expressions of the Tureav-Viro invariants.
- Detcherry-K. showed that for M, M' compact orientable with empty or toroidal boundary, and such that M is obtained by Dehn filling from M' we have $ITV(M') > ITV(M)$. Thus exponential growth of the Turaev-Viro invariants for M implies the exponential growth for the invariants of M'.
- We have

Corollary

Let L be a link in S^3 that contains the figure-8 knot or the Borromean rings as a sublink. Then we have

$$ITV(S^3 \smallsetminus L) \geqslant 2v_3.$$