Geometric Estimates from spanning surfaces of knots

joint w/ Stephan Burton (MSU)

Michigan State University

Knots in Hellas, Olymbia, Greece, July 18, 2016

Knot complement:

- Given a knot K in S^3 the complement $M = S^3 \setminus K$ is a 3-manifold with ∂M= a torus.
- For a slope σ on ∂M let $M(\sigma)$ denote the 3-manifold obtained by Dehn filling M along σ.
- By the knot complement theorem of Gordon and Luecke, there is a unique slope μ, called the meridian of K, such that $M(\mu)$ is S^3.
- A λ-curve of K is a slope on ∂M that intersects μ exactly once.
- *spanning surface* of K: properly embedded surface in M whose boundary is a λ-curve.
- S is *essential* if it is π_1-injective and cannot be homotoped onto ∂M (i.e. not boundary parallel torus or annulus).
- **Talk Theme.** Restrict to *hyperbolic knots:* Give bounds of slope lengths on the *maximal cusp* and of the *cusp volume* in terms of the the topology of essential surfaces spanned by the knots
- We show that there is an algorithmically checkable criterion to decide whether a hyperbolic knot has meridian length less than a given bound, and we use it to we obtain large families of knots with meridian lengths bounded above by four.

Hyperbolic knots: some terminology

- *hyperbolic knot K*: The interior of $M = S^3 \setminus K$ admits hyperbolic structure of finite volume. Then, M has one end of the form $T^2 \times [1, \infty)$, where T^2 denotes a torus.
- $\mathbb{H}^3$= upper half space hyperbolic model and let $\rho : \mathbb{H}^3 \to M$ be the covering map. $\{(x, y, z) : z > 0\}$
- The end is geometrically realized as the image of some $C = \rho(H)$ of some *horoball* $H \in \mathbb{H}^3$ at ∞. The pre-image $\rho^{-1}(C)$ is a collection of horoballs in $\mathbb{H}^3$. $H = \{(x, y, z) : z > \alpha\}$
- There is an 1-parameter cusp family each obtained by each other by expanding the horoball H (vary α) while keeping the same limiting point on the sphere at infinity. By expanding the cusp until in the pre-image $\rho^{-1}(C)$ each horosphere is tangent to another, we obtain a choice of *maximal cusps*. Since M has a single end then there is a well defined maximal cusp referred to as the *maximal cusp* of M.
- The *cusp* of K, denoted by C, is the maximal cusp of M. The boundary R_H of the horoball H is a plane and the boundary of C, denoted by ∂C, inherits a Euclidean structure from $\rho | R_H : R_H \longrightarrow \partial C$.

Hyperbolic knots: some terminology

- The *cusp area* of K, denoted by $\mathrm{Area}(\partial C)$ is the Euclidean area of ∂C and the *cusp volume* of K, denoted by $\mathrm{Vol}(C)$ is the volume of C. Note that we have $\mathrm{Area}(\partial C) = 2\mathrm{Vol}(C)$.
- **Definition** The *meridian* μ of M can be defined to be the geodesic representative on ∂C of a meridian curve of K; let $\ell(\mu)$ denote the Euclidean length of μ on ∂C.
- A λ-curve on ∂C is one that intersects the meridian exactly once. The length of a geodesic representative of a shortest λ-curve on ∂C will be denoted by $\ell(\lambda)$. Note that there may be multiple shortest λ-curves.
- *the* length of *the* shortest λ-curve on ∂C. The cusp volume is bounded above by $\ell(\mu)\ell(\lambda)$.
- **Definition.** A slope s on the boundary of a hyperbolic knot complementMK is called *exceptional* if the 3-manifold $M(\sigma)$ is not hyperbolic.
  (Possibilities: π_1 may be finite or N may be reducible or toroidal or “small” Seifert fibered space).

Length of exceptional slopes

- Gromov-Thurston "2π-theorem" asserts that if $\ell(\sigma) > 2\pi$ then $M(\sigma)$ admits a Riem. metric of negative curvature. By the Geometrization Theorem $M(\sigma)$ is hyperbolic.
- Agol and Lackenby independently improved 2π to 6. Thus exceptional slopes must have length less or equal to 6.
- There are examples of exceptional slopes with length 6 (Agol, Adams etal)
- If $M = S^3 \setminus K$, then $\ell(\mu) \leq 6$.
- Adams showed that the meridian of a 2-bridge hyperbolic knot has length less than 2.
- Adams, Colestock, Fowler, Gillam, and Katerman showed that $\ell(\mu) < 6$ and that for alternating knots $\ell(\mu) < 3$.
- Examples of knots whose meridian length approaches 4 below are given by Agol. Purcell also gave examples and showed that for " highly twisted" knots $\ell(\mu) \leq 4$.

Length of exceptional slopes

- Gromov-Thurston "2π-theorem" asserts that if $\ell(\sigma) > 2\pi$ then $M(\sigma)$ admits a Riem. metric of negative curvature. By the Geometrization Theorem $M(\sigma)$ is hyperbolic.
- Agol and Lackenby independently improved 2π to 6. Thus exceptional slopes must have length less or equal to 6.
- There are examples of exceptional slopes with length 6 (Agol, Adams etal)
- If $M = S^3 \setminus K$, then $\ell(\mu) \leq 6$.
- Adams showed that the meridian of a 2-bridge hyperbolic knot has length less than 2.
- Adams, Colestock, Fowler, Gillam, and Katerman showed that $\ell(\mu) < 6$ and that for alternating knots $\ell(\mu) < 3$.
- Examples of knots whose meridian length approaches 4 below are given by Agol. Purcell also gave examples and showed that for " highly twisted" knots $\ell(\mu) \leq 4$.
- **Question.** It is true that $\ell(\mu) \leq 4$ for all hyperbolic knots in S^3?

Meridian length bounds

Theorem

Let K be hyperbolic knot with meridian length $\ell(\mu)$. Suppose that K admits essential spanning surfaces S_1 and S_2 such that

$$|\chi(S_1)| + |\chi(S_2)| \leq \frac{b}{6} \cdot i(\partial S_1, \partial S_2), \qquad (1)$$

where b is a positive real number and $i(\partial S_1, \partial S_2)$ the minimal intersection number of the λ-curves $\partial S_1, \partial S_2$ on ∂M.
Then the meridian length satisfies $\ell(\mu) \leq b$. Moreover, given a hyperbolic knot K and $b > 0$, there an algorithm which determines if there are essential surfaces S_1 and S_2 satisfying inequality (1).

- For $b = 4$ we have an algorithmic criterion to check if $\ell(\mu)$.

Meridian length bounds

Theorem

Let K be hyperbolic knot with meridian length $\ell(\mu)$. Suppose that K admits essential spanning surfaces S_1 and S_2 such that

$$|\chi(S_1)| + |\chi(S_2)| \leq \frac{b}{6} \cdot i(\partial S_1, \partial S_2), \tag{1}$$

where b is a positive real number and $i(\partial S_1, \partial S_2)$ the minimal intersection number of the λ-curves $\partial S_1, \partial S_2$ on ∂M.
Then the meridian length satisfies $\ell(\mu) \leq b$. Moreover, given a hyperbolic knot K and $b > 0$, there an algorithm which determines if there are essential surfaces S_1 and S_2 satisfying inequality (1).

- For $b = 4$ we have an algorithmic criterion to check if $\ell(\mu)$.
- **Proof Ingredients:**
- Pleated surface arguments reminiscent to proof of "6-Theorem" by Agol.
- Algorithm relies on normal surface theory algorithms due to Jaco-Rubinstein, Jaco-Tollefson, and Jaco-Sedwick.

Example: Pretzel Knots

Example

Let $K = P(a, -b, -c)$ where $a, b, c > 1$ are odd. The pretzel surface S_P is a minimum genus Seifert surface. K also has a spanning surface S of K is given $s(S_A) = -2b - 2c$ and $s(S_P) = 0$. The difference in slopes of two surfaces is equal to the geometric intersection number, so we obtain that $i(\partial S_A, \partial S_P) = 2b + 2c$. An easy calculation shows that $\chi(S_A) = 1 - b - c$ and $\chi(S_P) = -1$. Using above theorem we then see that $\ell(\mu) < 3$.

- Theorem can be applied to knots that admit alternating projections on closed surfaces so that they define essential checkerboard surfaces.
- General pretzel knot: $K = P(n_1, \ldots, n_s, p_1, \ldots, p_r)$, where $p_i > 0$ and $n_j < 0$. For $r, s > 1$, K projects in alternating fashion on a torus and admits essential checkerboard surfaces that give $\ell(\mu) \leq 3$. (more later)

Example: Alternating Knots

- Let $D = D(K)$ be a prime, reduced, alternating diagram that is not a diagram of a $(2, p)$ torus knot. Then
- Menasco showed that K is hyperbolic.
- Aumann showed that the checkerboard surfaces of D are essential in $S^3 \setminus K$.
- Adams, Colestock, Fowler, Gillam, and Katerman showed the following

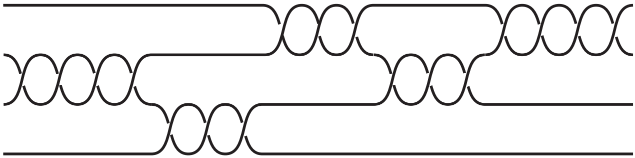
Theorem

Let K be a hyperbolic alternating knot with crossing number $c = c(K)$. Let C denote the maximal cusp of $S^3 \setminus K$ and let $\mathrm{Area}(\partial C)$ denote the cusp area. Finally let $\ell(\mu)$ and $\ell(\lambda)$ denote the length of the meridian and the shortest λ-curve of K. Then we have

1. $\ell(\mu) \leq 3 - \dfrac{6}{c}$
2. $\ell(\lambda) \leq 3c - 6$
3. $\mathrm{Area}(\partial C) \leq 9c \left(1 - \dfrac{2}{c}\right)^2$

"Complicated" positive/negative braids

- Let B_n be the braid group on n strands, with $n \geq 3$, and let $\sigma_1, \ldots, \sigma_{n-1}$ be the elementary braid generators.
- Let $b = \sigma_{i_1}^{r_1} \sigma_{i_2}^{r_2} \cdots \sigma_{i_k}^{r_k}$ be a braid in B_n.
- Futer-K.-Purcell shoed that if either $r_j \geq 3$ for all j, or else $r_j \leq -3$ for all j, then the braid closure D_b of b represents hyperbolic knot.



S

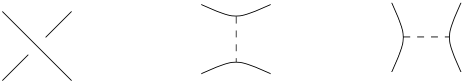
Corollary (Burton-K.)

Suppose that a knot K is represented by a braid closure D_b such that either $r_j \geq 3$ for all j, or else $r_j \leq -3$ for all j. Suppose moreover D_b is prime diagram. Then K is hyperbolic and the meridian length satisfies $\ell(\mu) < 4$.

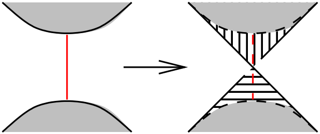
- **Remark.** Similar claims hold for "complicated" plat closures.

I. Prelims:State Graphs

Two choices for each crossing, of knot diagram D: A or B resolution.

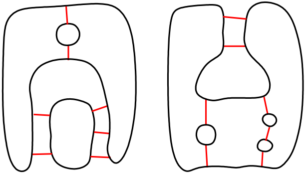


- A Kauffman *state* $\sigma(D)$ is a choice of A or B resolutions for all crossings.
- $\sigma(D)$: *state circles*
- Form a *fat graph* H_σ by adding edges at resolved crossings.
- Get a *state surface* S_σ: Each state circle bounds a disk in S_σ (nested disks drawn on top).
- At each edge (for each crossing) attach twisted band.



States and "adequacy"

- **Definition.** K is called *A–adequate* if has a diagram $D = D(K)$ where the all-A state graph $H_A = H_A(D)$ has no 1-edge loops. The K is called *adequate* if has a diagram $D = D(K)$ such that both D and its mirror image are adequate.

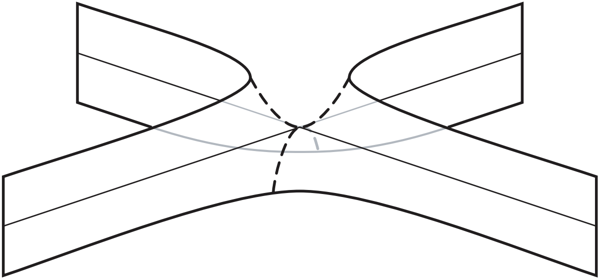


Facts:

- (Lickorish–Thistlethwaite, 80's) Adequate diagrams behave well with respect to the Kaufman state expansion of the Jones and colored Jones polynomials. There is no cancellation between states contributing to the maximum degree and minimum degrees.
- (Futer-K.-Purcell, Ozawa)The all-A and all-B state surfaces of adequate knot diagrams are essential in the corresponding knot complement.

Turaev surfaces:

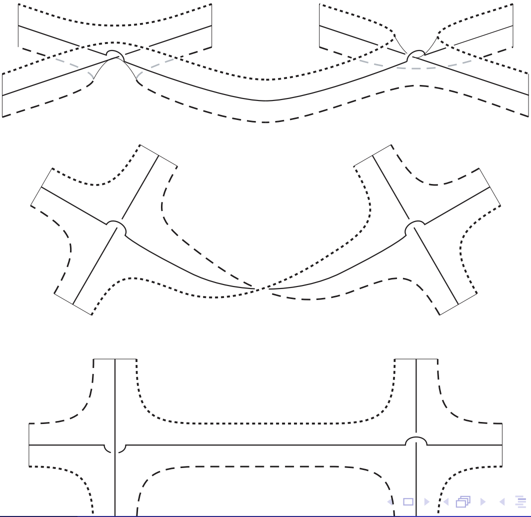
- Given D:=knot diagram we have the *Turaev surface* $F(D)$:
- Let $\Gamma \subset S^2$ be the planar, 4–valent graph of the diagram D.
- Thicken the (compactified) projection plane to $S^2 \times [-1, 1]$, so that Γ lies in $S^2 \times \{0\}$. Outside a neighborhood of the vertices (crossings), $\Gamma \times [-1, 1]$ will be part of $F(D)$.
- In the neighborhood of each vertex, we insert a saddle, so that the boundary circles on $S^2 \times \{1\}$ are the components of the A–resolution and the boundary circles on $S^2 \times \{-1\}$ are the components of the B–resolution.



- Cap off each circle with a disk, obtaining a closed surface $F(D)$.

Properties:

- $F(D)$ is a Heegaard surface of S^3. D is alternating on $F(D)$; in particular D is an alternating diagram if and only if $g_T(F(D)) = 0$. S_A, S_B are the checkerboard surfaces of alternating projection.



Turaev genus:

- The *Turaev genus* of a knot diagram $D = D(K)$ is defined by

$$g_T(D) = (2 - v_A(D) - v_B(D) + c(D))/2 \tag{2}$$

- $v_A(D)$, $v_B(D)$=# of components of all A, all-B resolution
- The Turaev genus of a knot K is defined by

$$g_T(K) = \min\{g_T(D) \mid D = D(K)\} \tag{3}$$

- We will need the following result of T. Abe.

Theorem

Suppose that D is an adequate diagram of a knot K. Then we have

$$g_T(K) = g_T(D) = (2 - v_A(D) - v_B(D) + c(D))/2.$$

- if D is adequate $c(D) = c(K)$=knot invariant (Lickorish).

Diagrammatic bounds of lengths and cusp shapes

Theorem

Let K be an adequate hyperbolic knot in S^3 with crossing number $c = c(K)$ and Turaev genus g_T. Let C denote the maximal cusp of $S^3 \setminus K$ and let $\mathrm{Area}(\partial C)$ denote the cusp area. Finally let $\ell(\mu)$ and $\ell(\lambda)$ denote the length of the meridian and the shortest λ-curve of K. Then we have

1. $\ell(\mu) \leq 3 + \dfrac{6g_T - 6}{c}$
2. $\ell(\lambda) \leq 3c + 6g_T - 6$
3. $\mathrm{Area}(\partial C) \leq 9c\left(1 - \dfrac{2 - 2g_T}{c}\right)^2$

- For $g_T = 0$ we have the estimates of Adams etal. So $\ell(\mu) < 3$.
- For $g_T = 1$, also get $\ell(\mu) \leq 3$.
- Given $g_T > 0$, there can be at most finely many hyperbolic adequate knots of Turaev genus g_T with $\ell(\mu) > 4$. If $g_T \leq 3$ then $\ell(\mu) \leq 4$.
- Let K be a hyperbolic knot with an adequate diagram D with c crossings and t twist regions. If $c \geq 3t$, we have $\ell(\mu) < 4$.

Proof idea

[adequate knot case]

- $M = S^3 \setminus K$= hyperbolic knot complement ; K has adequate diagram D. Let C the maximal cusp of K.
- Consider S to be the disjoint union of the checkerboard surfaces S_A, S_B corresponding to the Turaev surface $F(D)$. and consider $f : S \longrightarrow M = S^3 \setminus K$, where $f(S)$ is the union of S_A, S_B in the complement of K.
- Since S_A, S_B is essential we can *pleat* f. Then we obtain a hyperbolic structure on S by pulling back the metric from M via f. Understand the cusp geometry of $f(S)$; that is $f(S) \cap C$.
- **Key point.** Show that there are disjoint horocusps neighborhoods $H = H_A \cup H_B$, such that $f(H_A), f(H_A) \subset C$, $\ell(\partial H) = \mathrm{Area} H$ and such that $\ell(\partial H)$ is at least as big as the length of $f(\partial H)$ measured on C. Thus we have

$$\ell_C(S) \leq \mathrm{Area}(H),$$

where $\ell_C(S)$ denote the total length of the intersection curves in $f(S) \cap \partial C$.
[The fact that S is boundary incompressible is important.]

Proof idea cont'n

- A result of Böröczky on the density of horocycle packings on the hyperbolic plane gives

$$\mathrm{Area} H_A \leq \frac{6}{2\pi}\mathrm{Area}(S) = \frac{6}{2\pi}(2\pi|\chi(S)|),$$

where the last equation follows by the Gauss-Bonnet theorem.

- We get

$$\ell_C(S) \;\leq\; 6|\chi(S)|, \tag{4}$$

where $\ell_C(S)$ is the total length of the curves $f(S) \cap \partial C$.

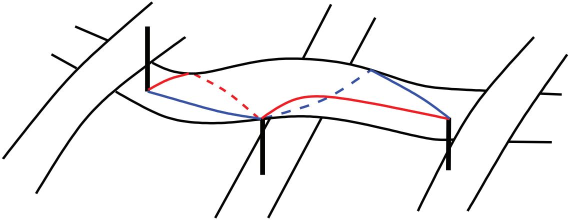
- Calculate $\ell_C(S)$ and show that

$$2c\ell(\mu) \leq \ell_C(S),$$

where $c = i(\partial S_A, \partial S_B)$ is the number of crossings of D.

Meridian length bound

- Consider the curves ∂S_A and ∂S_B near two consecutive crossings of D. If one crossing is an over-crossing and the other crossing is an under-crossing in the diagram D, then the intersection curves look like:



- ∂S_A and ∂S_B form a diamond pattern on ∂C.
- By the definitions of S_A, S_B, $g_T(D)$ and Abe's result

$$|\chi(S_A)| + |\chi(S_B)| = 2c - v_A(D) - v_B(D) = c + 2g_T - 2.$$

- Finally,

$$2c\ell(\mu) \leq 6(c + 2g_T - 2).$$

Shortest λ-curve length bound

- Orient $\partial S_A, \partial S_B$ and μ so that $\partial S_A, \partial S_B$ have opposite algebraic intersection numbers with μ.
- By resolving the crossings of ∂S_A with ∂S_B in a manner not consistent consistent with the orientations of ∂S_A and ∂S_B, one obtains two ℓ-curves in ∂C.
- Thus $2\ell(\lambda) \leq \ell_C(S)$.
- As before we get

$$2\ell(\lambda) < 6|\chi(S_1)| + 6|\chi(S_2)| = 6c + 12g_T - 12,$$

- or

$$\ell(\lambda) < 3c + 6g_T - 6.$$

Dehn surgery

- K hyperbolic adequate knot with $M = S^3 \setminus K$. Let $\delta = \frac{2g_T - 2}{c}$. It is an invariant of K that is calculated from any adequate diagram and let σ be a slope on ∂M. If

$$\Delta(\mu, \sigma) > \frac{18}{3.35}\left(1 + \frac{2g_T - 2}{c}\right) = 5.37(1+\delta),$$

then $\ell(\sigma) > 6$ and thus σ cannot be an exceptional slope.
- If σ is a slope represented by $p/q \in \mathbb{Q}$ in $H_1(\partial C)$ then $\Delta(\mu, \sigma) = |q|$.

Theorem

Let K be a hyperbolic adequate knot and let δ be as above. If $|q| \geq 6(1+\delta)$, then the 3-manifold N obtained by p/q surgery along K is hyperbolic and the volume satisfies the following

$$\mathrm{vol}(S^3 \setminus K) > \mathrm{vol}(N) \geq \left(1 - \frac{36(1+\delta)^2}{q^2}\right)^{3/2} \mathrm{vol}(S^3 \setminus K).$$

- With Futer-Purcell we gave diagrammatic 2-sided bounds of $\mathrm{vol}(S^3 \setminus K)$. Thus the volume of N can be estimated from any adequate diagram of K.