Cosmetic crossing changes on knots

parts joint w/ Cheryl Balm, Stefan Friedl and Mark Powell

2012 Joint Mathematics Meetings in Boston, MA, January 4-7.

The Setting:

Question: Suppose that changing a single crossing C of a knot produces the "same" knot:

- How does C look like?
- Can we characterize such crossing changes?
- Do there exist such crossing changes besides the "obvious" ones?

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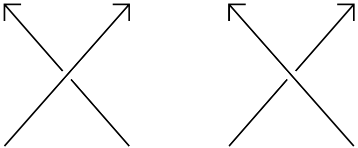

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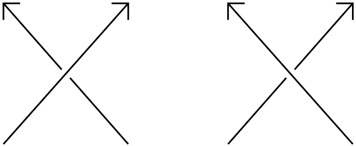

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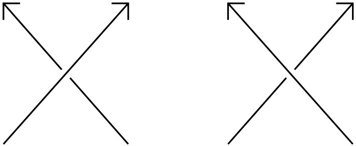

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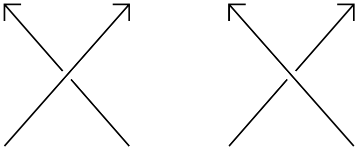

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- **Cosmetic crossing question.** Are there knots that admit cosmetic crossing changes?
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- **Remarks:**
- Question is stated as Problem 1.58 of Kirby's Problem list; attributed to Xiao-Song Lin.
- There are examples of K, K' that differ by a single crossing and there is an **orientation reversing** $f : S^3 \longrightarrow S^3$ with $f(K) = K'$. E. g. $K = P(3, 1, -3)$ and $K' = P(3, -1, -3)$.

Knots without cosmetic crossings:

- Unknot: Scharlemann-Thompson (CMH, '87)— by work of Gabai.
- 2-bridge knots: I.Torisu (TAIA, '97)— Montesinos trick, Cyclic Surgery Theorem.

  Torisu also showed that the question reduces to that for *prime* knots

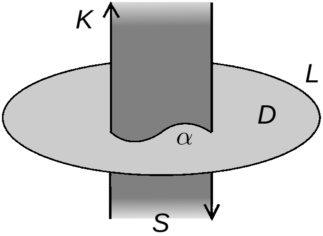
  Important: 2-fold branch covers are Lens spaces.
- Fibered Knots: K.– (Crelle, '11)—Sutured manifold techniques, results on commutator length of powers of Dehn twists in surface mapping class groups.

  Important: Fibrations of knot complements are unique!
- More recently; Genus one knots: w/ Balm, Friedl and Powell—- Abelian invariants.

  Next: More on the genus one case.

Cosmetic crossings and genus

Let K be an oriented knot and $L = \partial D$ a crossing circle supporting a crossing C and K' obtained from K by changing C. Since the linking number of L and K is zero, K bounds a Seifert surface in the complement of L. Let S be a Seifert surface that is of minimal genus in the complement of L. After an isotopy we can arrange so that $S \cap D$ is a single arc α .



The surface S gives rise to Seifert surfaces S and S' of K and K'.

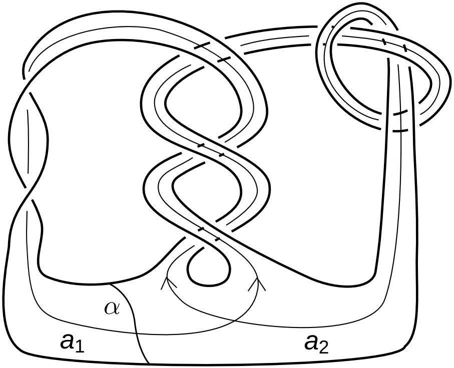
Proposition. Suppose C is cosmetic. Then S and S' are Seifert surfaces of minimal genus for K and K', respectively.

Notation. For a link J in S^3 let $\eta(J)$ denote a regular neighborhood of J in S^3 and $M_J = \overline{S^3 \setminus \eta(J)}$.

Proof of Proposition. Consider the surface S properly embedded in $M_{K\cup L}$ so that it is disjoint from $\partial\eta(L) \subset \partial M$. Since C is *cosmetic* $M_{K\cup L}$ is *irreducible*. Since S has minimum genus, the foliation machinery of Gabai applies. In particular, S is taut in the Thurston norm. The manifolds M_K and $M_{K'}$ are obtained by Dehn filling of $M_{K\cup L}$ along $\partial\eta(L)$. By a deep result of Gabai S can fail to remain taut in the Thurston norm (i.e. genus minimizing) in at most one of M_K and $M_{K'}$ (JDG, '87). But M_K and $M_{K'}$ are homeomorphic (by an orientation-preserving homeomorphism). Thus S remains taut in both of M_K and $M_{K'}$. This implies that S and S' are Seifert surfaces of minimal genus for K and K', respectively. **Q.E.D.**

Conclusion for genus one knots

Cosmetic crossing changes are realized by twisting along an essential arc $\alpha \subset S$ on a genus one surface S; twisting produces S'.



Since the genus of S is one, α is non-separating!

Pick s.c.c. a_1, a_2 on S that give symplectic bases for $H_1(S)$ and $H_1(S')$ and a_1 intersects α exactly once.

Abelian obstructions to CC.

Theorem(Balm-K.-Friedl-Powell)
Given an oriented genus one knot K let $\Delta_K(t)$ denote the Alexander polynomial of K and let Y_K denote the double cover of S^3 branching over K. Suppose that K admits a cosmetic crossing. Then

1. K is algebraically slice.
   In particular, $\Delta_K(t) \doteq f(t)f(t^{-1})$ for some linear polynomial $f(t)$ and $\det(K) = |\Delta_K(-1)| = (f(-1))^2$; a perfect square.

2. The homology group $H_1(Y_K)$ is a finite cyclic group.

3. K admits Seifert matrices

$$V = \begin{pmatrix} a & b \\ c & 0 \end{pmatrix} \quad \text{and} \quad \mathrm{V}' = \begin{pmatrix} a+\epsilon & b \\ c & 0 \end{pmatrix}$$

   where $a, b, c \in \mathbb{Z}$ and $\epsilon = \pm 1$.

Knots with unique surfaces

For knots that admit *unique* (up to isotopy) minimal genus Seifert surfaces we have the following stronger result.

Theorem. Let K be an oriented genus one knot with a unique minimal genus Seifert surface, which admits a cosmetic crossing. Then $\Delta_K(t) \doteq 1$.

Thus genus one knots with unique Seifert surfaces admit NO cosmetic crossing changes. Whitehead doubles have unique genus one surfaces –H. Lyons (Topology '74), W. Whitten ((Topology '73).

Corollary. The twisted Whitehead doubles of any knot admit no cosmetic crossing changes.

Low crossing knots

KnotInfo gives the 23 knots of genus one with at most 12 crossings, with the values of their determinants.

K	$\det(K)$	K	$\det(K)$	K	$\det(K)$
3_1	3	9_2	15	$11a_{362}$	39
4_1	5	9_5	23	$11a_{363}$	35
5_2	7	9_{35}	27	$\mathbf{11n_{139}}$	**9**
$\mathbf{6_1}$	**9**	$\mathbf{9_{46}}$	**9**	$11n_{141}$	21
7_2	11	10_1	17	$12a_{803}$	21
7_4	15	$\mathbf{10_3}$	**25**	$12a_{1287}$	37
8_1	13	$11a_{247}$	19	$12a_{1166}$	33
8_3	17	$11a_{343}$	31	-	-

- For all but $6_1, 9_{46}, 10_3$ and $11n_{139}$ the determinant is NOT a perfect square. The determinant criterion works for all but **four** examples.
- 6_1 and 10_3 are 2-bridge knots— no cosmetic crossings (Torisu)
- $K = 9_{46}$ is isotopic to the pretzel knot $P(3, 3, -3)$. The first homology group of the 2-fold branched cover is not cyclic; we have $H_1(Y_K) \cong \mathbb{Z}_3 \oplus \mathbb{Z}_3$. Thus no cosmetic crossings.

Low crossing knots Cont'

- The knot $K = 11\mathrm{n}_{139}$ is isotopic to the pretzel knot $P(-5, 3, -3)$. There is a genus one surface for which a Seifert matrix is

$$V = \begin{pmatrix} -1 & 2 \\ 1 & 0 \end{pmatrix},$$

  Using this Seifert matrix we calculate $H_1(Y_K) \cong \mathbb{Z}_9$ (cyclic!)— 2-fold cover fails.

- Turn to S-equivalence of matrices: If there are cosmetic crossing changes then V is (integrally) S-equivalent to to

$$\begin{pmatrix} 0 & 2 \\ 1 & 0 \end{pmatrix} \text{ or to } \begin{pmatrix} -2 & 2 \\ 1 & 0 \end{pmatrix}.$$

  Since $|\det(V)| = 2$ is prime, by a result of H. Trotter (Inventiones '73), V will be *congruent* to one of these two matrices. Check Impossible!

Corollary. Let K be a genus one knot that has a diagram with at most 12 crossings. Then K admits no cosmetic crossings.

Some Questions:

If K is an algebraically slice knot of genus one then every genus one surface contains exactly two *metabolizers*: s.c.c. with self linking number zero.

1. Do slicing obstructions of the metabolizer curves (such us Casson-Gordon invariants, ρ-invariants of Cochran-Harvey-Leidy) provide further obstructions to cosmetic crossings of K? Can they be used to show

   If K admits cosmetic crossing changes then $\Delta_K(t) = 1$? (in progress, w/ M. Powell).

2. What about the case $\Delta_K(t) = 1$?
3. More invariants that obstruct to cosmetic crossing changes?