Properties of degrees of colored Jones polynomials

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Talk theme

Knots: Smooth embedding $K : S^1 \to S^3$.
Equivalence: K_1, K_2 are equivalent if $f(K_1) = K_2$, f homeomorphism of S^3.

Relations among the two knot theory perspectives:

3-manifold topology/geometry

- Geometric structures and geometric invariants of the complement $S^3 \setminus K$.
- Essential surfaces in the complement $S^3 \setminus K$.

Physics/ representation theory originated invariants.

- Quantum invariants: Jones polynomial and Colored Jones polynomial.
- Defined/computed from knot diagrams.

Knots and 3-manifolds:

Given K remove an open tube around K to obtain the

Knot complement: $M_K = S^3 \setminus K$

Compact, orientable 3-manifold with torus boundary.

- Map $\pi_1(\partial M_K) \to \pi_1(M_K)$ is injection unless K=*Trivial Knot*.
- S properly embedded surface in M_K with our without boundary.
- **Definition.** S is *essential* if it is π_1-injective and cannon be homotopied onto ∂M_K (i.e. not boundary parallel torus or annulus).

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- *Satellites:* M_K contains essential tori. There is a *canonical* (finite) collection of such tori.
- *Torus knots:* M_K contains no essential torus but contains essential annulus (*cabling annulus*)
- *Hyperbolic Knots:* M_K can be given a complete Riemannian metric of constant negative curvature–(metric is unique Mostow-Prasad Rigidity Theorem)

Boundary Slopes:

- Recall $M_K = S^3 \setminus N_K$ where N_K= tubular neighborhood of K.
- $\langle \mu, \lambda \rangle$ = meridian–*canonical* longitude basis of $H_1(\partial N_K)$.
- **Defin.** $p/q \in \mathbb{Q} \cup \{1/0\}$ is called a *boundary slope* of K if there is an essential surface $(S, \partial S) \subset (M_K, \partial N_K)$, such that ∂S represents $p\mu + q\lambda \in H_1(\partial N_K)$.

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- (Hatcher, 80's) Every knot $K \subset S^3$ has finitely many boundary slopes.
- (Hatcher-Thurston, 80's) Gave algorithm to find all boundary slopes of 2-bridge knots.
- (Hatcher-Oertel) Gave algorithm to find all boundary slopes of Montesinos knots. – Algorithm allows to find all essential surfaces.
- (Jaco-Sedwick, 2003) Reproved Hatcher's finiteness result and generalized it to *normal surfaces*: There are finitely many slopes on ∂N_K that are realized by normal surfaces with respect to any " nice" (= one vertex) triangulation of M_K.
- Normal surface contain the essential ones–not every normal surface is essential.

Colored Jones Polynomials

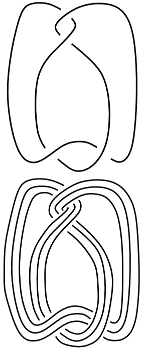
- For a knot K, the colored Jones function $J_K(n)$ is a sequence

$$J_K : \mathbb{Z} \to \mathbb{C}[t^{\pm 1}]$$

of Laurent polynomials in t. Extended to $\mathbb{Z}$ by $J_K(n) = -J_K(-n)$.
- Normalized so that $J_{\text{unknot}}(n) = (t^{2n} - t^{-2n})/(t^2 - t^{-2})$.

  - Encodes information about the Jones polynomial of K and its parallels K^n. The Jones polynomial corresponds to $n = 2$.
  - Technically, $J_K(n)$ is the quantum invariant using the n-dimensional representation of $SU(2)$.
  - Structure of quantum invariants and representation theory of $SU(2)$ (decomposition of tensor products of representations) lead to formulae in terms of "parallel" cables:

$$J_K(1) = 1, \qquad J_K(2)(t) = J_K(t),$$
$$J_K(3)(t) = J_{K^2}(t) - 1, \quad J_K(4)(t) = J_{K^3}(t) - 2J_K(t), \ \ldots$$



Colored Jones Polynomials

- (Garoufalidis- Le, 2005) The colored Jones function "*t-holonomic*": It satisfies satisfies non-trivial linear recurrence relations.
- Given K, there are polynomials $a_j(t^{2n}, t) \in \mathbb{C}[t^{2n}, t)]$, so that

$$a_d(t^{2n}, t)J_K(n+d) + \cdots + a_0(t^{2n}, t)J_K(n) = 0,$$

for all n.

- **Example.** K=right hand side trefoil.
- Colored Jones Function

$$J_K(n) = t^{-6(n^2-1)} \sum_{j=-\frac{n-1}{2}}^{\frac{n-1}{2}} t^{24j^2+12j} \frac{t^{8j+2} - t^{-(8j+2)}}{t^2 - t^{-2}}.$$

- Linear recurrence relation

$$(t^{8n+12} - 1)J_K(n+2) + (t^{-4n-6} - t^{-12n-10} - t^{8n+10} + t^{-2})J_K(n+1)$$
$$-(t^{-4n+4} - t^{-12n-8})J_K(n) = 0.$$

Degree of CJP: Slope conjecture

- $d_+[J_K(n)]$= highest degree of $J_K(n)$ in t
- $d_-[J_K(n)]$=lowest degree.
- **Notation.** $\{x_n\}'$ = set of *cluster points* of the sequence $\{x_n\}$.
- The sets of cluster points

$$js_K := \left\{n^{-2} d_+[J_K(n)]\right\}' \quad \text{and} \quad js_K^* := \left\{n^{-2} d_-[J_K(n)]\right\}'.$$

are finite. This follows by the "t-holonomicity" property of CJP.

- The elements of $js_K \cup js_K^*$ are called *Jones slopes*.
- **Conjecture 1.** (Garoufalidis, '10): For every knot the *Jones slopes* are *boundary slopes*!

The slope conjecture con't

- Warm up examples
- **Example 1.** $K = T_{p,q}$= Torus knot.
- Only two boundary slopes

$$\{0, pq\},$$

realized by a minimum genus Seifert surface and the cabling annulus.

- Jones slopes $js_K \cup js^*_K = \{0, pq\}$.
- **Example 2.** $K = P(-2, 3, 7)$-pretzel knot:

$$\begin{aligned} 4d_+[J_K(n)] &= 37/2n^2 + 34n + e(n), \\ 4d_-[J_K(n)] &= 0n^2 + 18n - 18, \end{aligned}$$

where $e(n) : \mathbb{Z} \to \mathbb{Q}$ is a periodic function of period $p - 3$.

Slope Conjecture Con't:

- Hatcher-Oertel algorithm for finding boundary slopes applies to $K = P(-2, 3, 7)$. Dunfield has implemented the algorithm– (calculation is fast for examples). We get

$$\text{Boundary slopes} = \{37/2, 0, 16, 20\}.$$

- The slope conjecture was confirmed for the following knots:
  - alternating knots and adequate knots
  - knots with up to nine crossings, torus knots,
  - "Most" of (p, q, r)-pretzel knots
  - families of closed 3-braids (2-fusion knots)
  - Iterated cables and connect sums of any of the above.

(Garoufalidis, Garoufalidis-Dunfield-Van der Veen, C. Lee- Van der Veen, Futer-K.-Purcell, K.- A. Tran, Motegi-Takata ...)

- **Remark.** Curtis and Taylor were one of the first authors to study the relation between boundary slopes and the degree of the Jones polynomial.

More on the structure of the degree of CJP

- Garoufalidis observed: Given K there is $N_K > 0$, such that, for $n \geq N_K$,

  $$d_+[J_K(n)] = a_K(n)\, n^2 + b_K(n)n + c_K(n),$$

  where $a_K(n), b_K(n), c_K(n) : \mathbb{Z} \to \mathbb{Q}$ are periodic functions.
- We have $b_{\text{Unknot}}(n) = 1/2$.
- **Conjecture 2** (K-Tran) If $K \neq \text{Unknot}$, we have

  $$b_K(n) \leq 0.$$

  That is $b_K(n)$ detects the unknot.
- Results and numerical evidence suggest that if K is hyperbolic, then

  $$b_K(n) < 0.$$

  In fact, if $b_K(n) = 0$ then the complement of K contains an embedded essential annulus.
- **Question.** Does $b_K(n)$ represent Euler characteristic of surfaces in the complement of K? Does it predict the topology of essential surfaces realizing the Jones slopes of K?

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Some data on the degree of the CJP

K	js_K	$\{2b_K(n)\}'$	$\chi(S)$	$\lvert\partial S\rvert$
8_{19}	$\{12\}$	$\{0\}$	0	2
8_{20}	$\{8/3\}$	$\{-1, -5/3\}$	-3	1
8_{21}	$\{1\}$	$\{-2\}$	-4	2
9_{42}	$\{6\}$	$\{-1\}$	-2	2
9_{43}	$\{32/3\}$	$\{-1, -5/3\}$	-3	1
9_{44}	$\{14/3\}$	$\{-2, -8/3\}$	-6	1
9_{45}	$\{1\}$	$\{-2\}$	-4	2
9_{46}	$\{2\}$	$\{-1\}$	-2	2
9_{48}	$\{11\}$	$\{-3\}$	-6	2

Table: Non-alternating Montesinos knots up to nine crossings.

- s= denominator of Jones slope, $|\partial S|$ = # of boundary components.

$$\frac{\chi(S)}{s|\partial S|} \in \{2b_K(n)\}'.$$

$s|\partial S|$ is called *the number of sheets* of S.

Strong Slope Conjecture

- Recall $d_+[J_K(n)] = a_K(n)\, n^2 + b_K(n)n + c_K(n)$,
- $js_K := \{2a_K(n)\}'$=Jones slopes
- **Conjecture 3: Strong slope conjecture.** (K.-Tran) Let K be a knot and $r/s \in js_K$, with $s > 0$ and $\gcd(r, s) = 1$, a Jones slope of K. Then there is an essential surface $S \subset M_K$ with boundary slope r/s, and such that

$$\frac{\chi(S)}{|\partial S| s} \in \{2b_K(n)\}'.$$

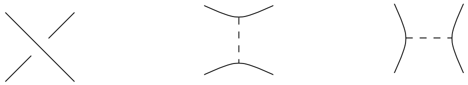
- Conjecture 3 holds for:
- adequate knots (including alternating ones) (Futer-K.-Purcell)
- Knots up to nine crossings
- Iterated torus knots (K.-Tian)
- 3-string Pretzel knots (Lee-Van der veen)
- Conjecture 3 is closed under knot cabling (K.–Tran) and connect sums (Montegi-Takata)

Results:

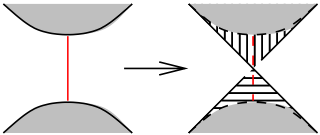
- Next:
- Outline how Strong Slope Conjecture follows for
- I. Adequate knots
- II. Pretzel knots/ 2-fusion knots
- III. Behavior of degree of CJP and boundary slopes under satellite operations.
- V. Further indirect evidence/topological consequences.

I. Prelims:State Graphs

Two choices for each crossing, of knot diagram D: A or B resolution.

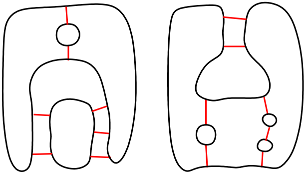


- A Kauffman *state* $\sigma(D)$ is a choice of A or B resolutions for all crossings.
- $\sigma(D)$: *state circles*
- Form a *fat graph* H_σ by adding edges at resolved crossings.
- Get a *state surface* S_σ: Each state circle bounds a disk in S_σ (nested disks drawn on top).
- At each edge (for each crossing) attach twisted band.



States and “adequacy”

- **Definition.** K is called *A–adequate* if has a diagram $D = D(K)$ where the all-A state graph $H_A = H_A(D)$ has no 1-edge loops. The K is called *adequate* if has a diagram $D = D(K)$ such that both D and its mirror image are adequate.



Key facts:

- (Lickorish–Thistlethwaite, 80’s) Adequate diagrams behave well with respect to the Kaufman state expansion of CJP. There is no cancellation between states contributing to the maximum degree and minimum degrees.
- (Futer-K.-Purcell, Ozawa, 2011)The all-A and all-B state surfaces of adequate knot diagrams are essential in the corresponding knot complement.

State surfaces and Slope Conjectures

- Given a knot diagram $D = D(K)$, let
- S_A=all -A state surface for D, S_B=all -B state surface.
- $c_+ := c_+(D)$ = number of positive crossings in D.

Theorem

Let D be an A–adequate diagram of a knot K. Then the surface S_A is essential in the knot complement M_K, and it has boundary slope $-2c_+$. Furthermore, we have

$$4d_+[J_K(n)] = 2c_+n^2 + 2\chi(S_A)n + \text{constant term.}$$

Similarly for B-adequate diagrams
In particular, if K is adequate, then it satisfies the Strong Slope Conjecture.

- Adequate knots have period one/Jones slopes are integers
- Alternating knots are adequate.
- All knots up to 10 crossings are A or B-adequate.
- Montesinos Knots, sums of alternating tangles, Positive knots, all closed 3-braids, are A or B adequate.

Pretzel knots family:

- General pretzel knot $K = P(n_1, \ldots, n_s, p_1, \ldots, p_r)$, where $p_i > 0$ and $n_j < 0$.
- For $r, s > 1$, K is adequate. Otherwise, K is only A or B adequate.
- Key remaining case: $K(n, q, p)$ (Lee-v.d. Veen):
- $4d_+[J_K(n)]$ is calculated using "*fusion*" (trivalent graphs and 6j-sympols).
- The Hatcher-Oertel algorithm is used to produce essential surfaces proving the the Strong Slope Conjecture.
- **Example.** $K = P(-2, 3, p)$, where $p \geq 5$ is an odd integer.

$$d_+[J_K(n)] = a_K(n)n^2 + b_K(n)n + c_K(n).$$

$$4a_K(n) = 2(p^2 - p - 5)/(p - 3) \quad \text{and} \quad 2b_K(n) = -(p - 5)/(p - 3).$$

- K has an essential surface S with slope $2(p^2 - p - 5)/(p - 3)$, $|\partial S| = 2$, and

$$\chi(S) = -(p - 5) = (p - 3)(2b_K(n)).$$

Satellites: Cabling and CJP

- Quantum invariants admit satellite decomposition formulae. Simpler case: "Cabling"
- Suppose K is a knot, and p, q are co-prime integers:
- **Definition.** The (p, q)-cable $K_{p,q}$ of K is the satellite of K with pattern (p, q)-torus knot.
- **Cabling formula:** (Morton, v.d. Veen)
- For $n > 0$ we have

$$J_{K_{p,q}}(n) = t^{-pq(n^2-1)/4} \sum_{k \in \mathcal{S}_n} t^{4pk(qk+1)} J_K(2qk+1)$$

where $\mathcal{S}_n$ be the set of all k such that

$$|k| \leq (n-1)/2 \quad \text{and} \quad k \in \begin{cases} \mathbb{Z} & \text{if } n \text{ is odd,} \\ \mathbb{Z}+1/2 & \text{if } n \text{ is even.} \end{cases}$$

Cabling slopes

- bs_K=set of boundary slopes of K.
- Klaff-Shalen studied boundary slopes of cables from the viewpoint of character varieties.

Theorem

(K.-Tran) For every knot $K \subset S^3$ and (p, q) co-prime integers we have

$$\left(q^2 bs_K \cup \{pq\}\right) \subset bs_{K_{p,q}}.$$

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- Meanwhile.....
- Using the cabling formula, we can see that under certain hypotheses,

$$\left(q^2 js_K \cup \{pq\}\right) \subset js_{K_{p,q}},$$

and the change of linear term of $d_+[J_K(n)]$, mimics that of the topology of essential surfaces for boundary slopes.

Cabling slopes Con't

- Proof allows to record "Euler characteristic behavior" under cabling. Say, we have an integral boundary slope $a \in bs_K$ of K.
- Suppose there is a essential surface S' in the complement of K that realizes the boundary slope $a \in \mathbb{Z}$. Then we obtain an essential surface S in the complement of $K_{p,q}$, that has boundary slope $q^2 a$, and

$$\chi(S) = |q|\chi(S') + |\partial S'|(1 - |q|)|p - aq| \text{ and } |\partial S| = |\partial S'|.$$

- For instance, if K had Jones period one with $d_+[J_K(n)] = an^2 + bn + c$. Then.
- Linear term for $K_{p,q}$ will be

$$b_1 = 2|q|b + (1 - |q|)|p - 4aq|.$$

Corollary

SSS is true for iterated cables of adequate knots. In particular it holds for iterated torus knots

2-fusion knots:

- 2-fusion knots (a family of closed 3-braids)
- Realize the knots as Dehn filling of a 3-component link (hyperbolic complement)
- Find a nice ideal triangulation. Use character variety methods (Culler-Shalen theory) to “enumerate” the normal surfaces realizing the boundary slopes.
- (Dunfeld- Garoufalidis:) Prove a criterion for normal a surface to be essential.
- Study behavior of these surfaces under Dehn filling— get essential surfaces to match the Jones slopes.
- Jones slopes were calculated using “quadratic programing” (v. d. Veen)

Indirect evidence

- If the Strong Slope conjecture is true then we have the following characterzation of alternating knots.
- K is alternating if and only if admits Jones slopes s, s^*, realized by essential spanning surfaces S, S^*, with

$$(s - s^*)/2 + \chi(S) + \chi(S^*) = 2 \text{ and } (s - s^*) = 2c(K), \qquad (1)$$

- J. Howie and J. Greene have recently proved a stronger result that implies above characterization!
- SSS gives similar characterization for adequate knots: Lead to following problem
- **Problem.** K is an adequate knot if and only if it admits Jones slopes s, s^*, realized by essential spanning surfaces S, S^*, with

$$(s - s^*)/2 + \chi(S) + \chi(S^*) = 2 - 2g_T(K) \text{ and } (s - s^*) = 2c(K).$$

where $g_T(K)$=the Turaev genus of K.

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