• Recently, progress in this direction was made my Chris Cornwell; he shows that in every Lens space, and with an appropriate choice of the "trivial links", the power series of Theorem 2 are polynomials. So we have a HOMFLY polynomial for links in Lens spaces. He uses techniques employed in works of Baker-Grigsby- Hedden and Baker-Grigsby to study Legendrian links in Lens spaces and ideas underlying the combinatorial structure of the Ozsvath-Szabo Homology link invariants.

• Cornwell also shows that there are applications to contact topology in Lens spaces: He finds inequalities generalizing those discovered by Franks-Williams, Morton, Fuchs-Tabachnikov for links in S^3.

Inequalities for Legendrian links in Lens spaces: $M = L(p,\ q)$. There is a "tight" contact structure ξ_M on M that pulls back to the standard contact structure on S^3; ξ_M is "unique" up to a certain co-orientation ambiguity. The TB invariant of Legendrian links and the self-linking invariant of transversal links in $(M,\ \xi_M)$ are rational numbers!

- L=topological type in $M = L(p,\ q)$;
- $P_M(L) \in \mathbf{C}[v^{\pm 1},\ z^{\pm 1}]$ the HOMFLY polynomial of L.
- $e_w(L)$=minimum degree in w, $v := w^p$.

Theorem. (Cornwell) (a) For every Legendrian representative L_l of L in $(M,\ \xi_M)$ we have

$$TB(L_l) \leq \frac{e_w(L)-1}{p}$$

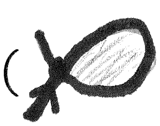
(b) For every transversal representative L_t of L we have $sl(L_t) \leq \frac{e_w(L)-1}{p}$

Constructing the power series: M-rational homology sphere as before. From a link invariant $F : \mathbb{L} \longrightarrow \mathbf{C}$ we derive a singular link invariant $f : \mathbb{L}^1 \longrightarrow \mathbf{C}$ by

$$f(L_\times) = F(L_+) - F(L_-) \qquad (1)$$

When can the process be reversed?

Key step: An integration theorem. A singular link invariant $f : \mathbb{L}^1 \longrightarrow \mathbf{C}$ is derived from a link invariant F via (1) if and only if f satisfies

$f($  $) = 0 \qquad (I_1)$

$$f(L_{\times +}) - f(L_{\times -}) = f(L_{+\times}) - f(L_{-\times}) \qquad (I_2)$$

Note: In (I_2) we start with *any* singular link $L_{\times\times} \in \mathbb{L}^2$. The four singular links in $\mathbb{L}^1$ are obtained by resolving one double point of $L_{\times\times}$ at a time.

Applying the integration theorem. To construct a Jones (formal) power-series

$J_M(L) = \sum_{m=0}^{\infty} v_n^m(L)x^m$ such that for $t :=$ $e^x := 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \cdots$ we have

$$t^{-1}J_M(L_+) - tJ_M(L_-) = (\sqrt{t} - \tfrac{1}{\sqrt{t}})J_M(L_0).$$

Step 1. Define J_M on the "trivial links".

Step 2. Define v_0 on all links in M.

Step 3. Assume $J_M^n(L) = \sum_{m=0}^{n} v_m(L)x^m$ has been defined so that

$$t^{-1}J_M(L_+) - tJ_M(L_-) \equiv (\sqrt{t} - \tfrac{1}{\sqrt{t}})J_M(L_0) \bmod n.$$

Guided by the skein relation we define

$$J_M^{n+1}(L_\times) := J_M^n(L_+) - J_M^n(L_-) \; mod(n+1)$$

which is the mod $(n+1)$ part of

$$(t^2-1)J_M^n(L_-) + t(\sqrt{t} - \frac{1}{\sqrt{t}})J_M^n(L_0).$$

This is a polynomial of degree $n+1$. The coefficients of terms of degree $\leq n$ are singular link invariants derived by $v_0, \ldots, v_n$. However, the coefficient of x^{n+1} is a "new" invariant of singular links . Call it $V_n(L_\times)$.

Step 4. Apply the Integration theorem to V_n: Check $(I_1) - (I_2)$ are satisfied (easy); conclude it integrates to a link invariant $v_n(L)$.

Step 5. Check that $J_M^{n+1}(L) := \sum_{m=0}^{n+1} v_m(L)x^m$ satisfies the crossing change formula mod(n+1).

Main ingredients of the proof Integration theorem:

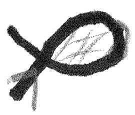
• Toroidal Decompositions of Haken 3-manifolds; Characteristic sub-manifolds (Jaco-Shalen, Johannson) [JSJ-decompositions]

• Results of Johannson and Scott on the classification of essential tori in Seifert fibered spaces up to homotopy.

• Torus Theorem (Gabai, Casson-Jungreis)

Summary: Let M be a rational homology sphere without sub-manifolds that fiber over non-orientable surfaces. Suppose $\Phi : T := S^1 \times S^1 \longrightarrow M$ is an essential map. Then, there is an essential Seifert fibered sum-manifold $S \subseteq M$ such that Φ is homotopic to a map $\Phi_1 : T \longrightarrow S$ that is vertical *w.r.t.* the fibration of S. [i.e. $p^{-1}(p(\Phi_1(T))) = \Phi_1(T)$]

Outline of the proof. The hard direction is to show that a singular link invariant that satisfies

$f($  $) = 0$ $\qquad\qquad (I_1)$

$$f(L_{\times +}) - f(L_{\times -}) = f(L_{+\times}) - f(L_{-\times}) \qquad (I_2)$$

integrates to a link invariant via

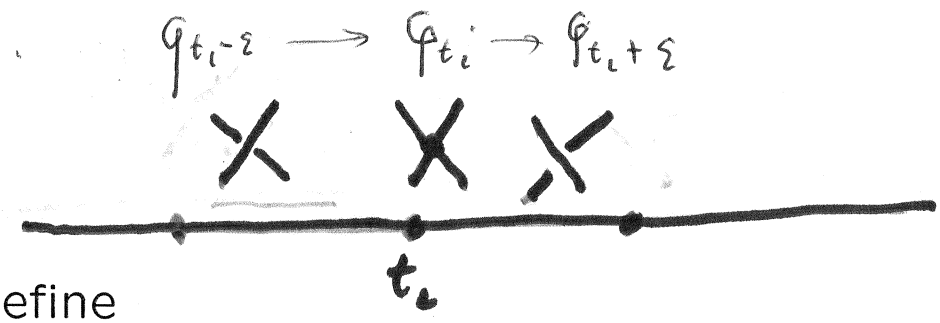
$$f(L_{\times}) = F(L_{+}) - F(L_{-}) \qquad\qquad (1).$$

Define F on the trivial links (integration constants). Let $P := \coprod S^1$.

For a (fixed) "trivial link" $\lambda : P \longrightarrow M$ define $\mathbb{M}_{\lambda} := \{L : P \longrightarrow M;\ p.l.\ \text{map}\}$. Given $L \in \mathbb{M}_{\lambda}$, choose a homotopy $\varphi_t : P \times [0,1] \longrightarrow M$ from L to λ. After a small perturbation, we can assume

– For only finitely many points $0 < t_1 < t_2 < \cdots < t_n < 1$, φ_t is not an embedding.

– φ_{t_i} is a singular link with one double point and the links on "right: and "left" side of φ_{t_i} are the two resolutions of the double point.



Define

$$F(L) = F(\lambda) + \sum_{i=1}^{n} \epsilon_i f(\varphi_{t_i}),$$

where $\epsilon_i = \pm 1$ as determined by (1).

Now F is well defined iff , modulo $F(\lambda)$, $F(L)$ is independent of the choice of the homotopy. Enough to show that for *any* closed homotopy $\Phi : P \times S^1 \longrightarrow M$, from λ to itself,

$$X_\Phi := \sum_{i=1}^{n} \epsilon_i f(\varphi_{t_i}) = 0.$$

where $\epsilon_i = \pm 1$ is determined by the same rule as above.

Step 1. Let $\pi_\lambda := \pi_1(\mathbb{M}_\lambda)$. Show that the assignment $\Phi \to X_\Phi$ gives a group homomorphism $\psi : \pi_\lambda \longrightarrow \mathbf{C}$.

That is $(I_1) - (I_2)$ imply the following:

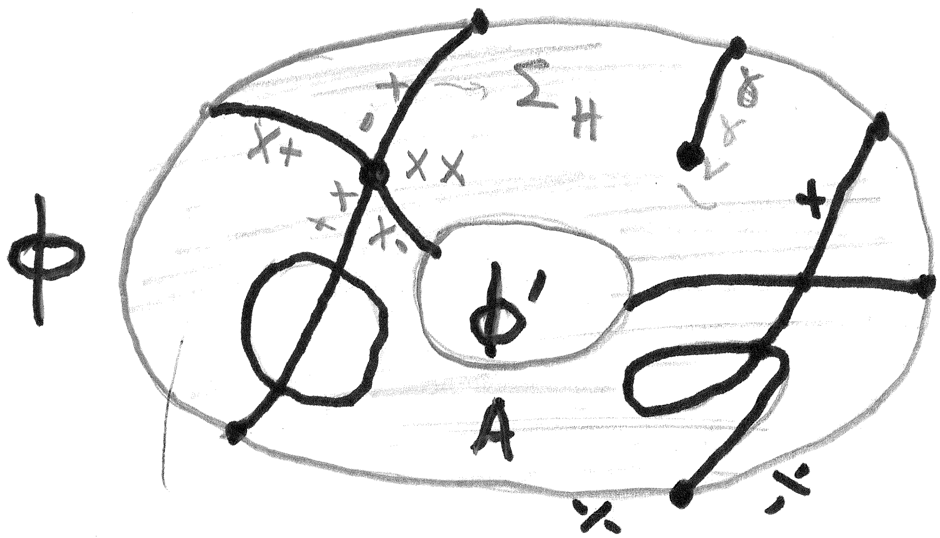
If Φ, Φ' are freely homotopic loops in $\mathbb{M}_\lambda$ then $X_\Phi = X_{\Phi'}$.

- π_λ is a subgroup of $\pi_1(M)$; the centralizer of the class of λ.

Example. If $< \lambda >= 1$ in $\pi_1(M)$ then $\pi_\lambda = \pi_1(M)$. Thus we have homomorphism $\psi : \pi_1(M) \longrightarrow \mathbf{C}$. Since $\mathbf{C}$ is abelian ψ factors to homomorphism $H_1(M) \longrightarrow \mathbf{C}$ which must be zero since $H_1(M)$ is finite! Thus, in this case, $X_\Phi = 0$ holds.

Note. This proves the result in S^3.

"Cerf type" argument: Put a free homotopy $H : A := P \times [0,\ 1] \longrightarrow \mathbb{M}_\lambda$ from Φ to Φ' into a "nice" generic position:



The singular set Σ_H of a generic H

- components of $A \setminus \Sigma_H \longrightarrow$ isotopy classes of links

- points on $\Sigma_H \cap \partial A \longleftrightarrow$ s-links along $\Phi \cup \Phi'$

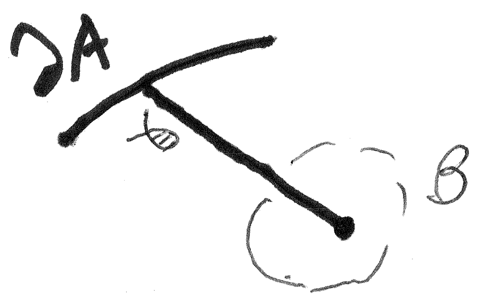
- edges of $\Sigma_H \longrightarrow$ isotopy classes of s-links

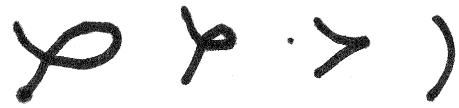
- vertices of valence 1 $\longrightarrow$ condition (I_1)

- vertices of valence 4 $\longrightarrow$ singular links with two double points (condition (I_2))

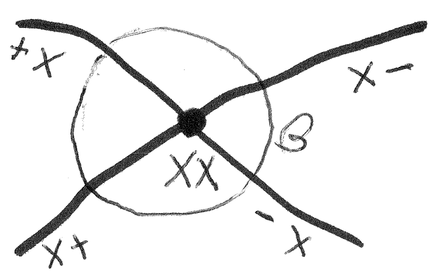
Now $X_\Phi - X_{\Phi'} = \sum_\beta (\pm X_\beta)$,

β small loops each encircling once an interior vertex of Σ_H. Now $(I_1) - (I_2)$, imply $X_\beta = 0$. Hence $X_\Phi = X_{\Phi'}$.





1-valent vertex: $X_\beta = f($  $)$



4-valent: $X_\beta = f(L_{\times +}) - f(L_{\times -}) - f(L_{+ \times}) + f(L_{- \times})$

Handling Tori: Loops $\Phi : S^1 \longrightarrow \mathbb{M}_\lambda$ are viewed as maps from the torus $\Phi : T := S^1 \times S^1 \longrightarrow M$. Since X_Φ is unchanged under homotopy, we to apply the JSJ-type results to homotope Φ into a "nice" position.

- If Φ is *inessential* then we may extend $\Phi : S^1 \times D^2 \longrightarrow M$. This gives the starting point for apply a "Cerf type" argument.

- If Φ is *essential* then homotope to Φ' that is vertical with respect to a fibration of the characteristic sub-manifold. Then in some cases one can deduce $X_{\Phi'} = 0$, directly. In the remaining cases we extend $\Phi' : S^1 \times F \longrightarrow M$ where F is a "nice" surface and apply the "Cerf type" argument again.