CHAPTER 5

Integrability on R

5.1 THE RIEMANN INTEGRAL

In this chapter we shall study integration of real functions. We begin our discussion by introducing the following terminology.

5.1 Definition.

Let $a, b \in \mathbf{R}$ with $a < b$.

i) A *partition* of the interval $[a, b]$ is a set of points $P = \{x_0, x_1, \dots, x_n\}$ such that

$$a = x_0 < x_1 < \cdots < x_n = b.$$

ii) The *norm* of a partition $P = \{x_0, x_1, \dots, x_n\}$ is the number

$$\|P\| = \max_{1 \le j \le n} |x_j - x_{j-1}|.$$

iii) A *refinement* of a partition $P = \{x_0, x_1, \dots, x_n\}$ is a partition Q of $[a, b]$ which satisfies $Q \supseteq P$. In this case we say that Q is *finer* than P.

5.2 *EXAMPLE.* [THE DYADIC PARTITION].

Prove that for each $n \in \mathbf{N}$, $P_n = \{j/2^n : j = 0, 1, \dots, 2^n\}$ is a partition of the interval $[0, 1]$, and P_m is finer than P_n when $m > n$.

Proof. Fix $n \in \mathbf{N}$. If $x_j = j/2^n$, then $0 = x_0 < x_1 < \cdots < x_{2^n} = 1$. Thus, P_n is a partition of $[0, 1]$. Let $m > n$ and set $p = m - n$. If $0 \le j \le 2^n$, then $j/2^n = j2^p/2^m$ and $0 \le j2^p \le 2^m$. Thus P_m is finer than P_n. ∎

It is clear that by definition, if P and Q are partitions of $[a, b]$, then $P \cup Q$ is finer than both P and Q. (Note that *finer* does not rule out the possibility that $P \cup Q = Q$, which would be the case if Q were a refinement of P.) And if Q is a refinement of P, then $\|Q\| \le \|P\|$. We shall use these observations often.

Let f be nonnegative on an interval $[a, b]$. You may recall that the integral of f over $[a, b]$ (when this integral exists) is the area of the region bounded by the curves $y = f(x)$, $y = 0$, $x = a$, and $x = b$. This area, A, can be approximated by rectangles whose bases lie in $[a, b]$ and whose heights approximate f (see Figure 5.1). If the tops of these rectangles lie above the curve $y = f(x)$, the resulting approximation is larger than A. If the tops of these rectangles lie below

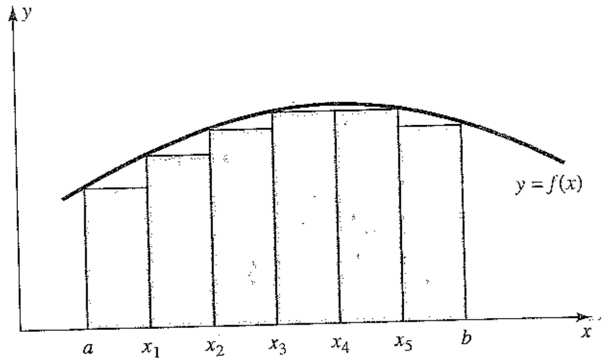

FIGURE 5.1

the curve $y = f(x)$, the resulting approximation is smaller than A. Hence, we make the following definition.

5.3 Definition.

Let $a, b \in \mathbf{R}$ with $a < b$, let $P = \{x_0, x_1, \dots, x_n\}$ be a partition of the interval $[a, b]$, set $\Delta x_j := x_j - x_{j-1}$ for $j = 1, 2, \dots, n$, and suppose that $f : [a, b] \to \mathbf{R}$ is bounded.

i) The *upper Riemann sum* of f over P is the number

$$U(f, P) := \sum_{j=1}^{n} M_j(f)\, \Delta x_j,$$

where

$$M_j(f) := \sup f([x_{j-1}, x_j]) := \sup_{t \in [x_{j-1}, x_j]} f(t).$$

ii) The *lower Riemann sum* of f over P is the number

$$L(f, P) := \sum_{j=1}^{n} m_j(f)\, \Delta x_j,$$

where

$$m_j(f) := \inf f([x_{j-1}, x_j]) := \inf_{t \in [x_{j-1}, x_j]} f(t).$$

(**Note**: Since we assumed that f is bounded, the numbers $M_j(f)$ and $m_j(f)$ exist and are finite.)

Some specific upper and lower Riemann sums can be evaluated with the help of the following elementary observation.

5.4 Remark. *If $g : \mathbf{N} \to \mathbf{R}$, then*

$$\sum_{k=m}^{n}(g(k+1)-g(k)) = g(n+1)-g(m)$$

for all $n \geq m$ in **N**.

Proof. The proof is by induction on n. The formula holds for $n = m$. If it holds for some $n-1 \geq m$, then

$$\sum_{k=m}^{n}(g(k+1)-g(k)) = (g(n)-g(m)) + (g(n+1)-g(n)) = g(n+1)-g(m).$$

∎

We shall refer to this algebraic identity by saying the sum *telescopes* to $g(n+1)-g(m)$. In particular, if $P = \{x_0, x_1, \dots, x_n\}$ is a partition of $[a, b]$, the sum $\sum_{j=1}^{n} \Delta x_j$ telescopes to $x_n - x_0 = b - a$.

Before we define what it means for a function to be integrable, we make several elementary observations concerning upper and lower sums.

5.5 Remark. *If $f(x) = \alpha$ is constant on $[a, b]$, then*

$$U(f, P) = L(f, P) = \alpha(b-a)$$

for all partitions P of $[a, b]$.

Proof. Since $M_j(f) = m_j(f) = \alpha$ for all j, the sums $U(f, P)$ and $L(f, P)$ telescope to $\alpha(b-a)$. ∎

5.6 Remark. *$L(f, P) \leq U(f, P)$ for all partitions P and all bounded functions f.*

Proof. By definition, $m_j(f) \leq M_j(f)$ for all j. ∎

The next result shows that as the partitions get finer, the upper and lower Riemann sums get nearer each other.

5.7 Remark. *If P is any partition of $[a, b]$ and Q is a refinement of P, then*

$$L(f, P) \leq L(f, Q) \leq U(f, Q) \leq U(f, P).$$

Proof. Let $P = \{x_0, x_1, \dots, x_n\}$ be a partition of $[a, b]$. Since Q is finer than P, Q can be obtained from P in a finite number of steps by adding one point at a time. Hence it suffices to prove the inequalities above for the special case $Q = \{c\} \bigcup P$ for some $c \in (a, b)$. Moreover, by symmetry and Remark 5.6, we need only show $U(f, Q) \leq U(f, P)$.

We may suppose that $c \notin P$. Hence, there is a unique index j_0 such that $x_{j_0-1} < c < x_{j_0}$. By definition, it is clear that

$$U(f,Q) - U(f,P) = M^{(\ell)}(c - x_{j_0-1}) + M^{(r)}(x_{j_0} - c) - M\,\Delta x_{j_0},$$

where

$$M^{(\ell)} = \sup f([x_{j_0-1}, c]), \quad M^{(r)} = \sup f([c, x_{j_0}]), \quad \text{and} \quad M = \sup f([x_{j_0-1}, x_{j_0}]).$$

By the Monotone Property of Suprema, $M^{(\ell)}$ and $M^{(r)}$ are both less than or equal to M. Therefore,

$$U(f,Q) - U(f,P) \le M(c - x_{j_0-1}) + M(x_{j_0} - c) - M\,\Delta x_{j_0} = 0. \quad \blacksquare$$

5.8 Remark. *If P and Q are any partitions of $[a,b]$, then*

$$L(f,P) \le U(f,Q).$$

Proof. Since $P \cup Q$ is a refinement of P and Q, it follows from Remark 5.7 that

$$L(f,P) \le L(f,P \cup Q) \le U(f,P \cup Q) \le U(f,Q)$$

for any pair of partitions P, Q, whether Q is a refinement of P or not. $\blacksquare$

We now use the connection between area and integration to motivate the definition of *integrable*. Suppose that $f(x)$ is nonnegative on $[a,b]$ and that the region bounded by the curves $y = f(x)$, $y = 0$, $x = a$, and $x = b$ has a well-defined area A. By Definition 5.3, every upper Riemann sum is an overestimate of A, and every lower Riemann sum is an underestimate of A (see Figure 5.1). Since the estimates $U(f,P)$ and $L(f,P)$ should get nearer to A as P gets finer, the differences $U(f,P) - L(f,P)$ should get smaller. [The shaded area in Figure 5.2 represents the difference $U(f,P) - L(f,P)$ for a particular P.] This leads us to the following definition (see also Exercise 5.1.9).

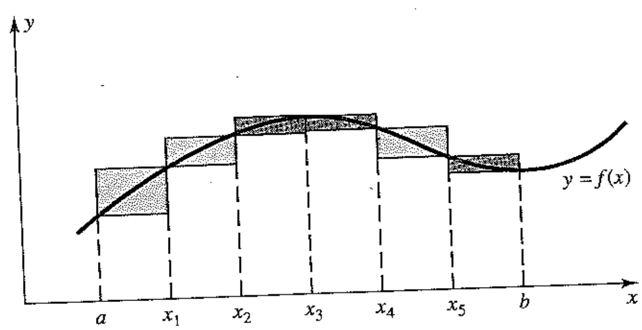

FIGURE 5.2

5.9 Definition.

Let $a, b \in \mathbf{R}$ with $a < b$. A function $f : [a, b] \to \mathbf{R}$ is said to be (*Riemann*) *integrable* on $[a, b]$ if and only if f is bounded on $[a, b]$, and for every $\varepsilon > 0$ there is a partition P of $[a, b]$ such that $U(f, P) - L(f, P) < \varepsilon$.

Notice that this definition makes sense whether or not f is nonnegative. The connection between nonnegative functions and area was only a convenient vehicle to motivate Definition 5.9. Also notice that, by Remark 5.6, $U(f, P) - L(f, P) = |U(f, P) - L(f, P)|$ for all partitions P. Hence, $U(f, P) - L(f, P) < \varepsilon$ is equivalent to $|U(f, P) - L(f, P)| < \varepsilon$.

This section provides a good illustration of how mathematics works. The connection between area and integration leads directly to Definition 5.9. This definition, however, is not easy to apply in concrete situations. Thus, we search for conditions which imply integrability *and* are easy to apply. In view of Figure 5.2, it seems reasonable that a function is integrable if its graph does not jump around too much (so that it can be covered by thinner and thinner rectangles). Since the graph of a continuous function does not jump at all, we are led to the following simple criterion that is sufficient (but not necessary) for integrability.

5.10 Theorem. *Suppose that $a, b \in \mathbf{R}$ with $a < b$. If f is continuous on the interval $[a, b]$, then f is integrable on $[a, b]$.*

Proof. Let $\varepsilon > 0$. Since f is uniformly continuous on $[a, b]$, choose $\delta > 0$ such that

$$|x - y| < \delta \quad \text{implies} \quad |f(x) - f(y)| < \frac{\varepsilon}{b - a}. \tag{1}$$

Let $P = \{x_0, x_1, \ldots, x_n\}$ be any partition of $[a, b]$ which satisfies $\|P\| < \delta$. Fix an index j and notice, by the Extreme Value Theorem, that there are points x_m and x_M in $[x_{j-1}, x_j]$ such that

$$f(x_m) = m_j(f) \quad \text{and} \quad f(x_M) = M_j(f).$$

Since $\|P\| < \delta$, we also have $|x_M - x_m| < \delta$. Hence by (1), $M_j(f) - m_j(f) < \varepsilon/(b - a)$. In particular,

$$U(f, P) - L(f, P) = \sum_{j=1}^{n} (M_j(f) - m_j(f))\, \Delta x_j < \frac{\varepsilon}{b - a} \sum_{j=1}^{n} \Delta x_j = \varepsilon.$$

(The last step comes from telescoping.) ∎

Although the converse of Theorem 5.10 is false (see Example 5.12 and Exercises 5.1.3, 5.1.6, and 5.1.8), there is a close connection between integrability and continuity. Indeed, we shall see (Theorem 9.49) that a function is integrable if and only if it has relatively few discontinuities. This principle is illustrated by the following examples. The nonintegrable function in Example 5.11 is nowhere continuous (hence has many discontinuities) but the integrable function in Example 5.12 has only one discontinuity (hence has few discontinuities).

5.11 ***EXAMPLE.***

The Dirichlet function

$$f(x) = \begin{cases} 1 & x \in \mathbf{Q} \\ 0 & x \notin \mathbf{Q} \end{cases}$$

is not Riemann integrable on $[0, 1]$.

Proof. Clearly, f is bounded on $[0, 1]$. By Theorem 1.18 and Exercise 1.3.3 (Density of Rationals and Irrationals), the supremum of f over any nondegenerate interval is 1, and the infimum of f over any nondegenerate interval is 0. Therefore, $U(f, P) - L(f, P) = 1 - 0 = 1$ for any partition P of the interval $[0, 1]$; that is, f is not integrable on $[0, 1]$. ■

5.12 ***EXAMPLE.***

The function

$$f(x) = \begin{cases} 0 & 0 \le x < 1/2 \\ 1 & 1/2 \le x \le 1 \end{cases}$$

is integrable on $[0, 1]$.

Proof. Let $\varepsilon > 0$. Choose $0 < x_1 < 0.5 < x_2 < 1$ such that $x_2 - x_1 < \varepsilon$. The set

$$P := \{0, x_1, x_2, 1\}$$

is a partition of $[0, 1]$. Since $m_1(f) = 0 = M_1(f)$, $m_2(f) = 0 < 1 = M_2(f)$, and $m_3(f) = 1 = M_3(f)$, it is easy to see that $U(f, P) - L(f, P) = x_2 - x_1 < \varepsilon$. Therefore, f is integrable on $[0, 1]$. ■

We have defined integrability, but not the value of the integral. We remedy this situation by using the Riemann sums $U(f, P)$ and $L(f, P)$ to define upper and lower integrals.

5.13 Definition.

Let $a, b \in \mathbf{R}$ with $a < b$, and $f : [a, b] \to \mathbf{R}$ be bounded.

i) The *upper integral* of f on $[a, b]$ is the number

$$(U) \int_a^b f(x)\, dx := \inf\{U(f, P) : P \text{ is a partition of } [a, b]\}.$$

5.13 Definition. (Continued)

ii) The *lower integral* of f on $[a, b]$ is the number

$$(L)\int_a^b f(x)\,dx := \sup\{L(f, P) : P \text{ is a partition of } [a, b]\}.$$

iii) If the upper and lower integrals of f on $[a, b]$ are equal, we define the *integral* of f on $[a, b]$ to be the common value

$$\int_a^b f(x)\,dx := (U)\int_a^b f(x)\,dx = (L)\int_a^b f(x)\,dx.$$

This defines integration over nondegenerate intervals. Motivated by the interpretation of integration as area, we define the integral of any bounded function f on $[a, a]$ to be zero; that is,

$$\int_a^a f(x)\,dx := 0.$$

Although a bounded function might not be integrable (see Example 5.11 above), the following result shows that the upper and lower integrals of a bounded function always exist.

5.14 Remark. *If $f : [a, b] \to \mathbf{R}$ is bounded, then its upper and lower integrals exist and are finite, and satisfy*

$$(L)\int_a^b f(x)\,dx \leq (U)\int_a^b f(x)\,dx.$$

Proof. By Remark 5.8, $L(f, P) \leq U(f, Q)$ for all partitions P and Q of $[a, b]$. Taking the supremum of this inequality over all partitions P of $[a, b]$, we have

$$(L)\int_a^b f(x)\,dx \leq U(f, Q);$$

that is, the lower integral exists and is finite. Taking the infimum of this last inequality over all partitions Q of $[a, b]$, we conclude that the upper integral is also finite and greater than or equal to the lower integral. ■

Suppose that f is bounded and nonnegative on $[a, b]$. Since the upper and lower sums of f approximate the "area" of the region bounded by the curves $y = f(x)$, $y = 0$, $x = a$, and $x = b$, we guess that f is integrable if and only if the upper and lower integrals of f are equal. The following result shows that this guess is true whether or not f is nonnegative.

5.15 Theorem. *Let $a, b \in \mathbf{R}$ with $a < b$, and $f : [a, b] \to \mathbf{R}$ be bounded. Then f is integrable on $[a, b]$ if and only if*

$$(L) \int_a^b f(x)\,dx = (U) \int_a^b f(x)\,dx. \tag{2}$$

Proof. Suppose that f is integrable. Let $\varepsilon > 0$ and choose a partition P of $[a, b]$ such that

$$U(f, P) - L(f, P) < \varepsilon. \tag{3}$$

By definition, $(U) \int_a^b f(x)dx \leq U(f, P)$ and the opposite inequality holds for the lower integral and the lower sum $L(f, P)$. Therefore, it follows from Remark 5.14 and (3) that

$$\begin{aligned} \left| (U) \int_a^b f(x)\,dx - (L) \int_a^b f(x)\,dx \right| &= (U) \int_a^b f(x)\,dx - (L) \int_a^b f(x)\,dx \\ &\leq U(f, P) - L(f, P) < \varepsilon. \end{aligned}$$

Since this is valid for all $\varepsilon > 0$, (2) holds as promised.

Conversely, suppose that (2) holds. Let $\varepsilon > 0$ and choose, by the Approximation Property, partitions P_1 and P_2 of $[a, b]$ such that

$$U(f, P_1) < (U) \int_a^b f(x)\,dx + \frac{\varepsilon}{2}$$

and

$$L(f, P_2) > (L) \int_a^b f(x)\,dx - \frac{\varepsilon}{2}.$$

Set $P = P_1 \cup P_2$. Since P is a refinement of both P_1 and P_2, it follows from Remark 5.7, the choices of P_1 and P_2, and (2) that

$$\begin{aligned} U(f, P) - L(f, P) &\leq U(f, P_1) - L(f, P_2) \\ &\leq (U) \int_a^b f(x)\,dx + \frac{\varepsilon}{2} - (L) \int_a^b f(x)\,dx + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$ ∎

Since the integral has been defined only on intervals $[a, b]$, we have tacitly assumed that $a \leq b$. We shall use the convention

$$\int_b^a f(x)\,dx = - \int_a^b f(x)\,dx$$

to extend the integral to the case $a > b$. In particular, if $f(x)$ is integrable and nonpositive on $[a, b]$, then the area of the region bounded by the curves $y = f(x)$, $y = 0$, $x = a$, and $x = b$ is given by $\int_b^a f(x)dx$.

In the next section we shall use the machinery of upper and lower sums to prove several familiar theorems about the Riemann integral. We close this section with one more result which reinforces the connection between integration and area.

5.16 Theorem. *If $f(x) = \alpha$ is constant on $[a, b]$, then*

$$\int_a^b f(x)\,dx = \alpha(b-a).$$

Proof. By Theorem 5.10, f is integrable on $[a, b]$. Hence, it follows from Theorem 5.15 and Remark 5.5 that

$$\int_a^b f(x)\,dx = (U)\int_a^b f(x)\,dx = \inf_P U(f, P) = \alpha(b-a).$$

■

EXERCISES

5.1.0. Suppose that $a < b < c$. Decide which of the following statements are true and which are false. Prove the true ones and give counterexamples for the false ones.

a) If f is Riemann integrable on $[a, b]$, then f is continuous on $[a, b]$.
b) If $|f|$ is Riemann integrable on $[a, b]$, then f is Riemann integrable on $[a, b]$.
c) For all bounded functions $f : [a, b] \to \mathbf{R}$,

$$(L)\int_a^b f(x)\,dx \le \int_a^b f(x)\,dx \le (U)\int_a^b f(x)\,dx.$$

d) If f is continuous on $[a, b)$ and on $[b, c]$, then f is Riemann integrable on $[a, c]$.

5.1.1. For each of the following, compute $U(f, P)$, $L(f, P)$, and $\int_0^2 f(x)dx$, where

$$P = \left\{0, \frac{1}{2}, 1, 2\right\}.$$

Find out whether the lower sum or the upper sum is a better approximation to the integral. Graph f and explain why this is so.

a) $f(x) = x^3$
b) $f(x) = 3 - x^2$
c) $f(x) = \sin(x/5)$

5.1.2. a) Prove that for each $n \in \mathbf{N}$,

$$P_n := \left\{ \frac{j}{n} : j = 0, 1, \dots, n \right\}$$

is a partition of $[0, 1]$.

b) Prove that a bounded function f is integrable on $[0, 1]$ if

$$(*) \qquad I_0 := \lim_{n \to \infty} L(f, P_n) = \lim_{n \to \infty} U(f, P_n),$$

in which case $\int_0^1 f(x)dx$ equals I_0.

c) For each of the following functions, use Exercise 1.4.4 to find formulas for the upper and lower sums of f on P_n, and use them to compute the value of $\int_0^1 f(x)dx$.

α) $$f(x) = x$$

β) $$f(x) = x^2$$

γ) $$f(x) = \begin{cases} 0 & 0 \le x < 1/2 \\ 1 & 1/2 \le x \le 1 \end{cases}$$

5.1.3. Let $E = \{1/n : n \in \mathbf{N}\}$. Prove that the function

$$f(x) = \begin{cases} 1 & x \in E \\ 0 & \text{otherwise} \end{cases}$$

is integrable on $[0, 1]$. What is the value of $\int_0^1 f(x)dx$?

5.1.4. **This exercise is used in Section *14.2.** Suppose that $a < b$ and that $f : [a, b] \to \mathbf{R}$ is bounded.

a) Prove that if f is continuous at $x_0 \in [a, b]$ and $f(x_0) \neq 0$, then

$$(L) \int_a^b |f(x)|\, dx > 0.$$

b) Show that if f is continuous on $[a, b]$, then $\int_a^b |f(x)|dx = 0$ if and only if $f(x) = 0$ for all $x \in [a, b]$.

c) Does part b) hold if the absolute values are removed? If it does, prove it. If it does not, provide a counterexample.

5.1.5. Suppose that $a < b$ and that $f : [a, b] \to \mathbf{R}$ is continuous. Show that

$$\int_a^c f(x)\, dx = 0$$

for all $c \in [a, b]$ if and only if $f(x) = 0$ for all $x \in [a, b]$. (Compare with Exercise 5.1.4, and notice that f need not be nonnegative here.)

5.1.6. Let f be integrable on $[a, b]$ and E be a finite subset of $[a, b]$. Show that if g is a bounded function which satisfies $g(x) = f(x)$ for all $x \in [a, b] \setminus E$, then g is integrable on $[a, b]$ and

$$\int_a^b g(x)\,dx = \int_a^b f(x)\,dx.$$

5.1.7. **This exercise is used in Section 12.3**. Let f, g be bounded on $[a, b]$.

a) Prove that

$$(U)\int_a^b (f(x) + g(x))\,dx \le (U)\int_a^b f(x)\,dx + (U)\int_a^b g(x)\,dx$$

and

$$(L)\int_a^b (f(x) + g(x))\,dx \ge (L)\int_a^b f(x)\,dx + (L)\int_a^b g(x)\,dx.$$

b) Prove that

$$(U)\int_a^b f(x)\,dx = (U)\int_a^c f(x)\,dx + (U)\int_c^b f(x)\,dx$$

and

$$(L)\int_a^b f(x)\,dx = (L)\int_a^c f(x)\,dx + (L)\int_c^b f(x)\,dx$$

for $a < c < b$.

5.1.8. **This exercise is used in Sections *5.5, 6.2, and *7.5.**

a) If f is increasing on $[a, b]$ and $P = \{x_0, \dots, x_n\}$ is any partition of $[a, b]$, prove that

$$\sum_{j=1}^{n} (M_j(f) - m_j(f))\,\Delta x_j \le (f(b) - f(a))\,\|P\|.$$

b) Prove that if f is monotone on $[a, b]$, then f is integrable on $[a, b]$. [**Note**: By Theorem 4.19, f has at most countably many (i.e., relatively few) discontinuities on $[a, b]$. This has nothing to do with the proof of part b), but points out a general principle which will be discussed in Section 9.6.]

5.1.9. Let $a < b$ and $0 < c < d$ be real numbers and $f : [a, b] \to [c, d]$. If f is Riemann integrable on $[a, b]$, prove that $\sqrt{f}$ is Riemann integrable on $[a, b]$.

5.1.10. Let f be bounded on a nondegenerate interval $[a, b]$. Prove that f is integrable on $[a, b]$ if and only if given $\varepsilon > 0$ there is a partition P_ε of $[a, b]$ such that

$$P \supseteq P_\varepsilon \quad \text{implies} \quad |U(f, P) - L(f, P)| < \varepsilon.$$

5.2 RIEMANN SUMS

There is another definition of the Riemann integral frequently found in elementary calculus texts.

5.17 Definition.

Let $f : [a, b] \to \mathbf{R}$.

i) A *Riemann sum* of f with respect to a partition $P = \{x_0, \ldots, x_n\}$ of $[a, b]$ generated by samples $t_j \in [x_{j-1}, x_j]$ is a sum

$$\mathcal{S}(f, P, t_j) := \sum_{j=1}^{n} f(t_j)\, \Delta x_j.$$

ii) The Riemann sums of f are said to *converge* to $I(f)$ as $\|P\| \to 0$ if and only if given $\varepsilon > 0$ there is a partition P_ε of $[a, b]$ such that

$$P = \{x_0, \ldots, x_n\} \supseteq P_\varepsilon \quad \text{implies} \quad \left|\mathcal{S}(f, P, t_j) - I(f)\right| < \varepsilon$$

for all choices of $t_j \in [x_{j-1}, x_j]$, $j = 1, 2, \ldots, n$. In this case we shall use the notation

$$I(f) = \lim_{\|P\| \to 0} \mathcal{S}(f, P, t_j) := \lim_{\|P\| \to 0} \sum_{j=1}^{n} f(t_j)\, \Delta x_j.$$

The following result shows that this definition of the Riemann integral is the same as the one using upper and lower integrals.

5.18 Theorem. *Let $a, b \in \mathbf{R}$ with $a < b$, and suppose that $f : [a, b] \to \mathbf{R}$. Then f is Riemann integrable on $[a, b]$ if and only if*

$$I(f) = \lim_{\|P\| \to 0} \sum_{j=1}^{n} f(t_j)\, \Delta x_j$$

exists, in which case $I(f) = \int_a^b f(x)dx$.

Proof. Suppose that f is integrable on $[a, b]$ and that $\varepsilon > 0$. By the Approximation Property, there is a partition P_ε of $[a, b]$ such that

$$L(f, P_\varepsilon) > \int_a^b f(x)\,dx - \varepsilon \quad \text{and} \quad U(f, P_\varepsilon) < \int_a^b f(x)\,dx + \varepsilon. \qquad (4)$$

Let $P = \{x_0, x_1, \dots, x_n\} \supseteq P_\varepsilon$. Then (4) holds with P in place of P_ε. But $m_j(f) \le f(t_j) \le M_j(f)$ for any choice of $t_j \in [x_{j-1}, x_j]$. Hence,

$$\int_a^b f(x)\,dx - \varepsilon < L(f, P) \le \sum_{j=1}^n f(t_j)\,\Delta x_j \le U(f, P) < \int_a^b f(x)\,dx + \varepsilon;$$

that is, $-\varepsilon < \sum_{j=1}^n f(t_j)\,\Delta x_j - \int_a^b f(x)dx < \varepsilon$. We conclude that

$$\left| \sum_{j=1}^n f(t_j)\,\Delta x_j - \int_a^b f(x)\,dx \right| < \varepsilon$$

for all partitions $P \supseteq P_\varepsilon$ and all choices of $t_j \in [x_{j-1}, x_j]$, $j = 1, 2, \dots, n$.

Conversely, suppose that the Riemann sums of f converge to $I(f)$. Let $\varepsilon > 0$ and choose a partition $P = \{x_0, x_1, \dots, x_n\}$ of $[a, b]$ such that

$$\left| \sum_{j=1}^n f(t_j)\,\Delta x_j - I(f) \right| < \frac{\varepsilon}{3} \qquad (5)$$

for all choices of $t_j \in [x_{j-1}, x_j]$. Since f is bounded on $[a, b]$ (see Exercise 5.2.11), use the Approximation Property to choose $t_j, u_j \in [x_{j-1}, x_j]$ such that $f(t_j) - f(u_j) > M_j(f) - m_j(f) - \varepsilon/(3(b-a))$. By (5) and telescoping, we have

$$\begin{aligned}
U(f, P) - L(f, P) &= \sum_{j=1}^n (M_j(f) - m_j(f))\,\Delta x_j \\
&< \sum_{j=1}^n (f(t_j) - f(u_j))\,\Delta x_j + \frac{\varepsilon}{3(b-a)} \sum_{j=1}^n \Delta x_j \\
&\le \left| \sum_{j=1}^n f(t_j)\,\Delta x_j - I(f) \right| \\
&\quad + \left| I(f) - \sum_{j=1}^n f(u_j)\,\Delta x_j \right| + \frac{\varepsilon}{3(b-a)} \sum_{j=1}^n \Delta x_j \\
&< \frac{2\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon.
\end{aligned}$$

Therefore, f is integrable on $[a, b]$. ∎

The next two results show that Riemann integrals of complicated functions can be broken into simpler pieces.

5.19 Theorem. **[LINEAR PROPERTY].**
If f, g are integrable on $[a, b]$ and $\alpha \in \mathbf{R}$, then $f + g$ and αf are integrable on $[a, b]$. In fact,

$$\int_a^b (f(x) + g(x))\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx \tag{6}$$

and

$$\int_a^b (\alpha f(x))\,dx = \alpha \int_a^b f(x)\,dx. \tag{7}$$

Proof. Let $\varepsilon > 0$ and choose P_ε such that for any partition $P = \{x_0, x_1, \dots, x_n\} \supseteq P_\varepsilon$ of $[a, b]$ and any choice of $t_j \in [x_{j-1}, x_j]$, we have

$$\left| \sum_{j=1}^n f(t_j)\,\Delta x_j - \int_a^b f(x)\,dx \right| < \frac{\varepsilon}{2}$$

and

$$\left| \sum_{j=1}^n g(t_j)\,\Delta x_j - \int_a^b g(x)\,dx \right| < \frac{\varepsilon}{2}.$$

By the Triangle Inequality,

$$\left| \sum_{j=1}^n f(t_j)\Delta x_j + \sum_{j=1}^n g(t_j)\Delta x_j - \int_a^b f(x)\,dx - \int_a^b g(x)\,dx \right| < \varepsilon$$

for any choice of $t_j \in [x_{j-1}, x_j]$. Hence, (6) follows directly from Theorem 5.18.

To prove (7), we may suppose that $\alpha \neq 0$. Choose P_ε such that if $P = \{x_0, \dots, x_n\}$ is finer than P_ε, then

$$\left| \sum_{j=1}^n f(t_j)\,\Delta x_j - \int_a^b f(x)\,dx \right| < \frac{\varepsilon}{|\alpha|}$$

for any choice of $t_j \in [x_{j-1}, x_j]$. Multiplying this inequality by $|\alpha|$, we obtain

$$\left| \sum_{j=1}^n \alpha f(t_j)\,\Delta x_j - \alpha \int_a^b f(x)\,dx \right| < |\alpha| \frac{\varepsilon}{|\alpha|} = \varepsilon$$

for any choice of $t_j \in [x_{j-1}, x_j]$. We conclude by Theorem 5.18 that (7) holds. ■

5.20 Theorem. *If f is integrable on* $[a, b]$, *then f is integrable on each subinterval* $[c, d]$ *of* $[a, b]$. *Moreover,*

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx \tag{8}$$

for all $c \in (a, b)$.

Proof. We may suppose that $a < b$. Let $\varepsilon > 0$ and choose a partition P of $[a, b]$ such that

$$U(f, P) - L(f, P) < \varepsilon. \tag{9}$$

Let $P' = P \cup \{c\}$ and $P_1 = P' \cap [a, c]$. Since P_1 is a partition of $[a, c]$ and P' is a refinement of P, we have by (9) that

$$U(f, P_1) - L(f, P_1) \le U(f, P') - L(f, P') \le U(f, P) - L(f, P) < \varepsilon.$$

Therefore, f is integrable on $[a, c]$. A similar argument proves that f is integrable on any subinterval $[c, d]$ of $[a, b]$.

To verify (8), suppose that P is any partition of $[a, b]$. Let $P_0 = P \cup \{c\}$, $P_1 = P_0 \cap [a, c]$, and $P_2 = P_0 \cap [c, b]$. Then $P_0 = P_1 \cup P_2$ and by definition

$$\begin{aligned} U(f, P) &\ge U(f, P_0) = U(f, P_1) + U(f, P_2) \\ &\ge (U)\int_a^c f(x)\,dx + (U)\int_c^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx. \end{aligned}$$

(This last equality follows from the fact that f is integrable on both $[a, c]$ and $[c, b]$.) Taking the infimum of

$$U(f, P) \ge \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

over all partitions P of $[a, b]$, we obtain

$$\int_a^b f(x)\,dx = (U)\int_a^b f(x)\,dx \ge \int_a^c f(x)\,dx + \int_c^b f(x)\,dx.$$

A similar argument using lower integrals shows that

$$\int_a^b f(x)\,dx \le \int_a^c f(x)\,dx + \int_c^b f(x)\,dx.$$ ■

Using the conventions

$$\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx \quad \text{and} \quad \int_a^a f(x)\,dx = 0,$$

it is easy to see that (8) holds whether or not c lies between a and b, provided f is integrable on the union of these intervals.

5.21 Theorem. **[Comparison Theorem for Integrals].** *If f, g are integrable on $[a, b]$ and $f(x) \le g(x)$ for all $x \in [a, b]$, then*

$$\int_a^b f(x)\,dx \le \int_a^b g(x)\,dx.$$

In particular, if $m \le f(x) \le M$ for $x \in [a, b]$, then

$$m(b-a) \le \int_a^b f(x)\,dx \le M(b-a).$$

Proof. Let P be a partition of $[a, b]$. By hypothesis, $M_j(f) \le M_j(g)$ whence $U(f, P) \le U(g, P)$. It follows that

$$\int_a^b f(x)\,dx = (U)\int_a^b f(x)\,dx \le U(g, P)$$

for all partitions P of $[a, b]$. Taking the infimum of this inequality over all partitions P of $[a, b]$, we obtain

$$\int_a^b f(x)\,dx \le \int_a^b g(x)\,dx.$$

If $m \le f(x) \le M$, then (by what we just proved and by Theorem 5.16)

$$m(b-a) = \int_a^b m\,dx \le \int_a^b f(x)\,dx \le \int_a^b M\,dx = M(b-a). \quad \blacksquare$$

We shall use the following result nearly every time we need to estimate an integral.

5.22 Theorem. *If f is (Riemann) integrable on $[a, b]$, then $|f|$ is integrable on $[a, b]$ and*

$$\left|\int_a^b f(x)\,dx\right| \le \int_a^b |f(x)|\,dx.$$

Proof. Let $P = \{x_0, x_1, \dots, x_n\}$ be a partition of $[a, b]$. We claim that

$$M_j(|f|) - m_j(|f|) \le M_j(f) - m_j(f) \tag{10}$$

holds for $j = 1, 2, \ldots, n$. Indeed, let $x, y \in [x_{j-1}, x_j]$. If $f(x)$, $f(y)$ have the same sign, say both are nonnegative, then

$$|f(x)| - |f(y)| = f(x) - f(y) \le M_j(f) - m_j(f).$$

If $f(x)$, $f(y)$ have opposite signs, say $f(x) \ge 0 \ge f(y)$, then $m_j(f) \le 0$ and, hence,

$$|f(x)| - |f(y)| = f(x) + f(y) \le M_j(f) + 0 \le M_j(f) - m_j(f).$$

Thus in either case, $|f(x)| \le M_j(f) - m_j(f) + |f(y)|$. Taking the supremum of this last inequality for $x \in [x_{j-1}, x_j]$ and then the infimum as $y \in [x_{j-1}, x_j]$, we see that (10) holds, as promised.

Let $\varepsilon > 0$ and choose a partition P of $[a, b]$ such that $U(f, P) - L(f, P) < \varepsilon$. Since (10) implies $U(|f|, P) - L(|f|, P) \le U(f, P) - L(f, P)$, it follows that

$$U(|f|, P) - L(|f|, P) < \varepsilon.$$

Thus $|f|$ is integrable on $[a, b]$. Since $-|f(x)| \le f(x) \le |f(x)|$ holds for any $x \in [a, b]$, we conclude by Theorem 5.21 that

$$-\int_a^b |f(x)|\,dx \le \int_a^b f(x)\,dx \le \int_a^b |f(x)|\,dx. \qquad \blacksquare$$

By Theorem 5.19, the sum of integrable functions is integrable. What about the product?

5.23 Corollary. *If f and g are (Riemann) integrable on $[a, b]$, then so is fg.*

Proof. Suppose for a moment that the square of any integrable function is integrable. Then, by hypothesis, f^2, g^2, and $(f + g)^2$ are integrable on $[a, b]$. Since

$$fg = \frac{(f+g)^2 - f^2 - g^2}{2},$$

it follows from Theorem 5.19 that fg is integrable on $[a, b]$.

It remains to prove that f^2 is integrable on $[a, b]$. Let P be a partition of $[a, b]$. Since $M_j(f^2) = (M_j(|f|))^2$ and $m_j(f^2) = (m_j(|f|))^2$, it is clear that

$$\begin{aligned} M_j(f^2) - m_j(f^2) &= (M_j(|f|))^2 - (m_j(|f|))^2 \\ &= (M_j(|f|) + m_j(|f|))(M_j(|f|) - m_j(|f|)) \\ &\le 2M(M_j(|f|) - m_j(|f|)), \end{aligned}$$

where $M = \sup |f|([a, b])$; that is, $|f(x)| \le M$ for all $x \in [a, b]$. Multiplying the displayed inequality by Δx_j and summing over $j = 1, 2, \ldots, n$, we have

$$U(f^2, P) - L(f^2, P) \le 2M(U(|f|, P) - L(|f|, P)).$$

Hence, it follows from Theorem 5.22 that f^2 is integrable on $[a, b]$. ∎

We close this section with two integral analogues of the Mean Value Theorem.

5.24 Theorem. [FIRST MEAN VALUE THEOREM FOR INTEGRALS]. *Suppose that f and g are integrable on $[a, b]$ with $g(x) \geq 0$ for all $x \in [a, b]$. If*

$$m = \inf f[a, b] \quad \text{and} \quad M = \sup f[a, b],$$

then there is a number $c \in [m, M]$ such that

$$\int_a^b f(x)g(x)\,dx = c\int_a^b g(x)\,dx.$$

In particular, if f is continuous on $[a, b]$, then there is an $x_0 \in [a, b]$ which satisfies

$$\int_a^b f(x)g(x)\,dx = f(x_0)\int_a^b g(x)\,dx.$$

Proof. Since $g \geq 0$ on $[a, b]$, Theorem 5.21 implies

$$m\int_a^b g(x)\,dx \leq \int_a^b f(x)g(x)\,dx \leq M\int_a^b g(x)\,dx.$$

If $\int_a^b g(x)dx = 0$, then $\int_a^b f(x)g(x)dx = 0$ and there is nothing to prove. Otherwise, set

$$c = \frac{\int_a^b f(x)g(x)\,dx}{\int_a^b g(x)\,dx}$$

and note that $c \in [m, M]$. If f is continuous, then (by the Intermediate Value Theorem) we can choose $x_0 \in [a, b]$ such that $f(x_0) = c$. ∎

Before we state the Second Mean Value Theorem, we introduce an idea that will be used in the next section to prove the Fundamental Theorem of Calculus. If f is integrable on $[a, b]$, then f can be used to define a new function

$$F(x) := \int_a^x f(t)\,dt, \qquad x \in [a, b].$$

5.25 *EXAMPLE.*

Find $F(x) = \int_0^x f(t)\,dt$ if

$$f(x) = \begin{cases} 1 & x \geq 0 \\ -1 & x < 0. \end{cases}$$

Solution. By Theorem 5.16,

$$F(x) = \int_0^x f(t)\,dt = \begin{cases} x & x \geq 0 \\ -x & x < 0. \end{cases}$$

Hence, $F(x) = |x|$. ■

Notice in Example 5.25 that the integral F of f is continuous even though f itself is not. The following result shows that this is a general principle.

5.26 Theorem. *If f is (Riemann) integrable on $[a, b]$, then $F(x) = \int_a^x f(t)\,dt$ exists and is continuous on $[a, b]$.*

Proof. By Theorem 5.20, $F(x)$ exists for all $x \in [a, b]$. To prove that F is continuous on $[a, b]$, it suffices to show that $F(x+) = F(x)$ for all $x \in [a, b)$ and $F(x-) = F(x)$ for all $x \in (a, b]$. Fix $x_0 \in [a, b)$. By definition, f is bounded on $[a, b]$. Thus, choose $M \in \mathbf{R}$ such that $|f(t)| \leq M$ for all $t \in [a, b]$. Let $\varepsilon > 0$ and set $\delta = \varepsilon/M$ where $x_0 + \delta < b$. If $0 \leq x - x_0 < \delta$, then by Theorem 5.22,

$$|F(x) - F(x_0)| = \left| \int_{x_0}^x f(t)\,dt \right| \leq \int_{x_0}^x |f(t)|\,dt \leq M|x - x_0| < \varepsilon.$$

Hence, $F(x_0+) = F(x_0)$. A similar argument shows that $F(x_0-) = F(x_0)$ for all $x_0 \in (a, b]$. ■

5.27 Theorem. **[SECOND MEAN VALUE THEOREM FOR INTEGRALS].** *Suppose that f, g are integrable on $[a, b]$, that g is nonnegative on $[a, b]$, and that m, M are real numbers which satisfy $m \leq \inf f([a, b])$ and $M \geq \sup f([a, b])$. Then there is a $c \in [a, b]$ such that*

$$\int_a^b f(x)g(x)\,dx = m \int_a^c g(x)\,dx + M \int_c^b g(x)\,dx.$$

In particular, if f is also nonnegative on $[a, b]$, then there is a $c \in [a, b]$ which satisfies

$$\int_a^b f(x)g(x)\,dx = M \int_c^b g(x)\,dx.$$

Proof. The second statement follows from the first since we may use $m = 0$ when $f \geq 0$. To prove the first statement, set

$$F(x) = m \int_a^x g(t)\,dt + M \int_x^b g(t)\,dt$$

for $x \in [a, b]$, and observe by Theorem 5.26 that F is continuous on $[a, b]$. Since g is nonnegative, we also have $mg(t) \leq f(t)g(t) \leq Mg(t)$ for all $t \in [a, b]$. Hence, it follows from the Comparison Theorem (Theorem 5.21) that

$$F(b) = m \int_a^b g(t)\,dt \le \int_a^b f(t)g(t)\,dt \le M \int_a^b g(t)\,dt = F(a).$$

Since F is continuous, we conclude by the Intermediate Value Theorem that there is an $c \in [a, b]$ such that

$$F(c) = \int_a^b f(t)g(t)\,dt.$$

■

When $g(x) = 1$ and $f(x) \ge 0$, these mean value theorems have simple geometric interpretations. Indeed, let A represent the area bounded by the curves $y = f(x)$, $y = 0$, $x = a$, and $x = b$. By the First Mean Value Theorem, there is a $c \in [m, M]$ such that the area of the rectangle of height c and base $b - a$ equals A (see Figure 5.3). And by the Second Mean Value Theorem, if M is greater than or equal to the maximum value of f on $[a, b]$, then there is an $c \in [a, b]$ such that the area of the rectangle of height M and base $b - c$ equals A (see Figure 5.4).

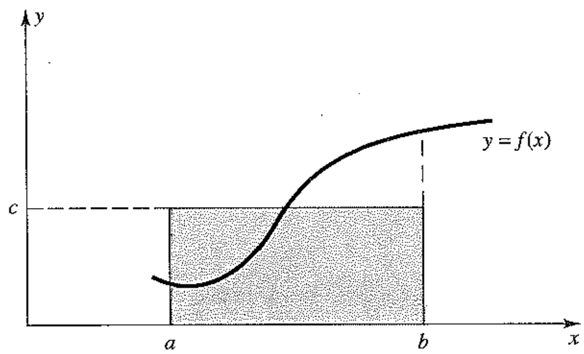

FIGURE 5.3

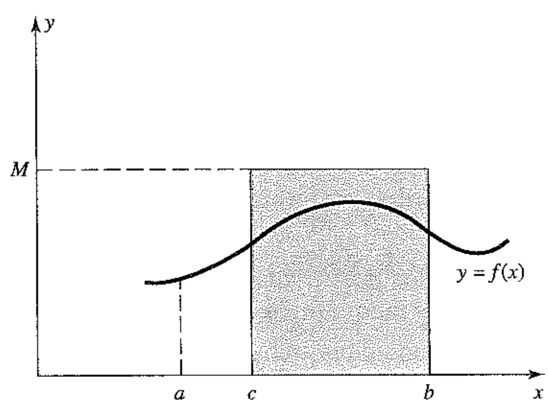

FIGURE 5.4

EXERCISES

5.2.0. Suppose that $a < b$. Decide which of the following statements are true and which are false. Prove the true ones and give counterexamples for the false ones.

a) If f and g are Riemann integrable on $[a, b]$, then $f - g$ is Riemann integrable on $[a, b]$.

b) If f is Riemann integrable on $[a, b]$ and P is any polynomial on $\mathbf{R}$, then $P \circ f$ is Riemann integrable on $[a, b]$.

c) If f and g are nonnegative real functions on $[a, b]$, with f continuous and g Riemann integrable on $[a, b]$, then there exist $x_0, x_1 \in [a, b]$ such that

$$\int_a^b f(x)g(x)\,dx = f(x_0)\int_{x_1}^b g(x)\,dx.$$

d) If f and g are Riemann integrable on $[a, b]$ and f is continuous, then there is an $x_0 \in [a, b]$ such that

$$\int_a^b f(x)g(x)\,dx = f(x_0)\int_a^b g(x)\,dx.$$

5.2.1. Using the connection between integrals and area, evaluate each of the following integrals.

a) $$\int_{-2}^{2} |x+1|\,dx$$

b) $$\int_{-2}^{2} (|x+1| + |x|)\,dx$$

c) $$\int_{-a}^{a} \sqrt{a^2 - x^2}\,dx, \qquad a > 0$$

d) $$\int_0^2 (5 + \sqrt{2x + x^2})\,dx$$

5.2.2. a) Suppose that $a < b$ and $n \in \mathbf{N}$ is even. If f is continuous on $[a, b]$ and $\int_a^b f(x)x^n dx = 0$, prove that $f(x) = 0$ for at least one $x \in [a, b]$.

b) Show that part a) might not be true if n is odd.

c) Prove that part a) does hold for odd n when $a \geq 0$.

5.2.3. Use Taylor polynomials with three or four nonzero terms to verify the following inequalities.

a) $$0.3095 < \int_0^1 \sin(x^2)\,dx < 0.3103$$

(The value of this integral is approximately 0.3102683.)

b)
$$1.4571 < \int_0^1 e^{x^2}\,dx < 1.5704$$

(The value of this integral is approximately 1.4626517.)

5.2.4. Suppose that $f : [0,\infty) \to [0,\infty)$ is integrable on every closed interval $[a,b] \subset [0,\infty)$. If

$$F(x) := \int_0^x e^{-y^2} f(y)\,dy, \quad x \in [0,\infty),$$

then there is a function $g : [0,\infty) \to [0,\infty)$ such that $F(x) = \int_{g(x)}^x f(y)\,dy$ for all $x \in [0,\infty)$.

5.2.5. Prove that if f is integrable on $[0,1]$ and $\beta > 0$, then

$$\lim_{n\to\infty} n^\alpha \int_0^{1/n^\beta} f(x)\,dx = 0$$

for all $\alpha < \beta$.

5.2.6. a) Suppose that $g_n \ge 0$ is a sequence of integrable functions which satisfies

$$\lim_{n\to\infty} \int_a^b g_n(x)\,dx = 0.$$

Show that if $f : [a,b] \to \mathbf{R}$ is integrable on $[a,b]$, then

$$\lim_{n\to\infty} \int_a^b f(x) g_n(x)\,dx = 0.$$

b) Prove that if f is integrable on $[0,1]$, then

$$\lim_{n\to\infty} \int_0^1 x^n f(x)\,dx = 0.$$

5.2.7. Suppose that f is integrable on $[a,b]$, that $x_0 = a$, and that x_n is a sequence of numbers in $[a,b]$ such that $x_n \uparrow b$ as $n \to \infty$. Prove that

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{k=0}^{n} \int_{x_k}^{x_{k+1}} f(x)\,dx.$$

5.2.8. Let f be continuous on a closed, nondegenerate interval $[a,b]$ and set

$$M = \sup_{x\in[a,b]} |f(x)|.$$

a) Prove that if $M > 0$ and $p > 0$, then for every $\varepsilon > 0$ there is a nondegenerate interval $I \subset [a, b]$ such that

$$(M - \varepsilon)^p |I| \le \int_a^b |f(x)|^p \, dx \le M^p (b - a).$$

b) Prove that

$$\lim_{p \to \infty} \left(\int_a^b |f(x)|^p \, dx \right)^{1/p} = M.$$

5.2.9. Let $f : [a, b] \to \mathbf{R}$, $a = x_0 < x_1 < \cdots < x_n = b$, and suppose that $f(x_k+)$ exists and is finite for $k = 0, 1, \ldots, n-1$ and $f(x_k-)$ exists and is finite for $k = 1, \ldots, n$. Show that if f is continuous on each subinterval (x_{k-1}, x_k), then f is integrable on $[a, b]$ and

$$\int_a^b f(x) \, dx = \sum_{k=1}^n \int_{x_{k-1}}^{x_k} f(x) \, dx.$$

5.2.10. Prove that if f and g are integrable on $[a, b]$, then so are $f \vee g$ and $f \wedge g$ (see Exercise 3.1.8).

5.2.11. Suppose that $f : [a, b] \to \mathbf{R}$.

a) If f is not bounded above on $[a, b]$, then given any partition P of $[a, b]$ and $M > 0$, there exist $t_j \in [x_{j-1}, x_j]$ such that $\mathcal{S}(f, P, t_j) > M$.

b) If the Riemann sums of f converge to a finite number $I(f)$, as $\|P\| \to 0$, then f is bounded on $[a, b]$.

5.3 THE FUNDAMENTAL THEOREM OF CALCULUS

Let f be integrable on $[a, b]$ and $F(x) = \int_a^x f(t)\, dt$. By Theorem 5.26, F is continuous on $[a, b]$. The next result shows that if f is continuous, then F is continuously differentiable. Thus "indefinite integration" improves the behavior of the function.

5.28 Theorem. **[Fundamental Theorem of Calculus].**
Let $[a, b]$ be nondegenerate and suppose that $f : [a, b] \to \mathbf{R}$.

i) *If f is continuous on $[a, b]$ and $F(x) = \int_a^x f(t)\, dt$, then $F \in \mathcal{C}^1[a, b]$ and*

$$\frac{d}{dx} \int_a^x f(t) \, dt := F'(x) = f(x)$$

for each $x \in [a, b]$.

ii) *If f is differentiable on $[a, b]$ and f' is integrable on $[a, b]$, then*

$$\int_a^x f'(t) \, dt = f(x) - f(a)$$

for each $x \in [a, b]$.