Instantaneous velocity. We start our study of the derivative with the *velocity problem:* If a particle moves along a coordinate line so that at time t, it is at position $f(t)$, then compute its velocity or speed† at a given instant.

Velocity means distance traveled, divided by time elapsed (e.g. feet per second). If the velocity changes during the time interval, then this quotient is the *average velocity.* From time $t = a$ to $t = b$, the distance traveled is the change in position $f(b) - f(a)$, and the time elapsed is $b - a$, so the average velocity is:

$$v_{\text{avg}} = \frac{f(b) - f(a)}{b - a}.$$

What do we mean by the instantaneous velocity at time $t = a$? We cannot compute this directly, since the particle does not move at all in an instant. Rather, we find the average velocity from $t = a$ to $t = a+h$, where h is a small time increment, and take the instantaneous velocity v to be the limiting value approached by the average velocities:

$$v = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h},$$

where $\lim_{h \to 0}$ means "the limit as h approaches 0" of the quantity on the right.

Another way to say this is that velocity is the *rate of change* of position with respect to time: how fast the position $f(x)$ is changing per unit change in time t. Thus, v_{avg} is the average rate of change over an interval $t \in [a, b]$, while v is the instantaneous rate of change at a particular $t = a$.

Falling stone example. A stone dropped off a bridge has position approximately $f(t) = 16t^2$ feet below the bridge after falling for t seconds. The average velocity between $t = 3$ and $t = 4$ is:

$$v_{\text{avg}} = \frac{f(4) - f(3)}{4 - 3} = \frac{16(4^2) - 16(3^2)}{1} = 112.$$

That is, the stone has an average velocity of 112 ft/sec, although it starts slower than this at $t = 3$ and speeds up steadily throughout the interval.

Now, what is the instantaneous velocity at $t = 3$? We compute the average velocity over a short time interval from $t = 3$ to $t = 3 + h$, for example $h = 0.1$:

$$v_{\text{avg}} = \frac{f(3.1) - f(3)}{3.1 - 3} = \frac{16(3.1^2) - 16(3^2)}{0.1} = 97.6.$$

---

*Notes by Peter Magyar `magyar@math.msu.edu`, with sections corresponding to James Stewart's *Calculus*, 7th ed.

† *Velocity* can be positive or negative, depending on the direction of motion. *Speed* is the absolute value of velocity.

This is a pretty good estimate of the velocity, but to be more precise we take shorter intervals:

h	1	0.1	0.01	0.001	0.0001	0.00001
v_{avg}	112	97.6	96.16	96.016	96.0016	96.00016

It is pretty clear that as the interval gets shorter and shorter, the average velocity approaches the limiting value $v = 96$, and we define this to be the instantaneous velocity.

Let us prove this algebraically: instead of trying sample values of the time increment h, we let h be a variable:

$$v_{\text{avg}} = \frac{f(3+h) - f(3)}{(3+h) - 3} = \frac{16(3+h)^2 - 16(3^2)}{h} = 16 \cdot \frac{(3+h)^2 - 3^2}{h}$$

$$= 16 \cdot \frac{(3^2 + 2(3h) + h^2) - 3^2}{h} = 16 \cdot \frac{6h + h^2}{h} = 16(6 + h) = 96 + 16h\,.$$

As we take h smaller and smaller, the error term $16h$ approaches zero, and the average velocity approaches the limiting value 96, which by definition is the instantaneous velocity:

$$v = \lim_{h \to 0} \frac{f(3+h) - f(3)}{h} = 96\,.$$

Tangent Slope. We have described velocity on three conceptual levels: as a *physical* quantity, a *numerical* approximation, and an *algebraic* computation. Velocity also has a *geometric* meaning in terms of the graph $y = f(t)$. Consider a *secant line* which cuts the graph at points $(a, f(a))$ and $(b, f(b))$.
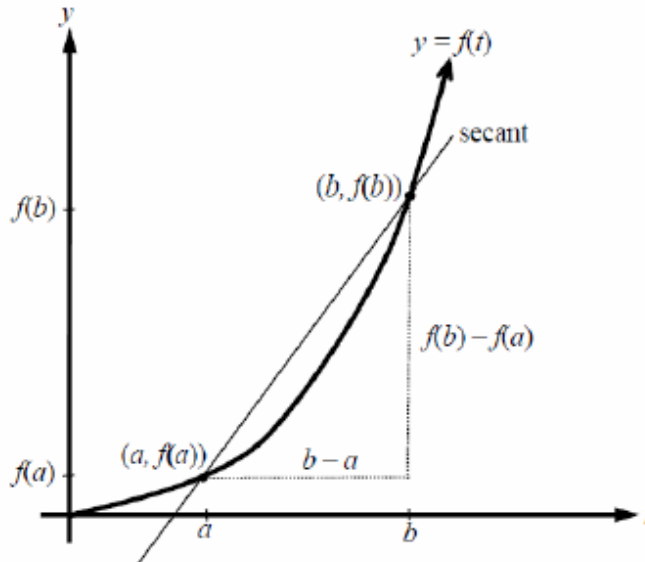

The slope m_{sec} of the secant line is the rise in the graph per unit of horizontal run, which means distance traversed divided by time elapsed, which is the average velocity:

$$m_{\text{sec}} = \frac{f(b) - f(a)}{b - a} = v_{\text{avg}}\,.$$

The reason for this coincidence is that slope is the rate of vertical rise with respect to horizontal run, just as velocity is the rate of change of position (drawn on the vertical axis) with respect to time (on the horizontal axis).

As we move the point $(b, f(b))$ to $(a+h, f(a+h))$, closer and closer to a, the secant lines approach the *tangent line* which touches the curve at the single point $(a, f(a))$.
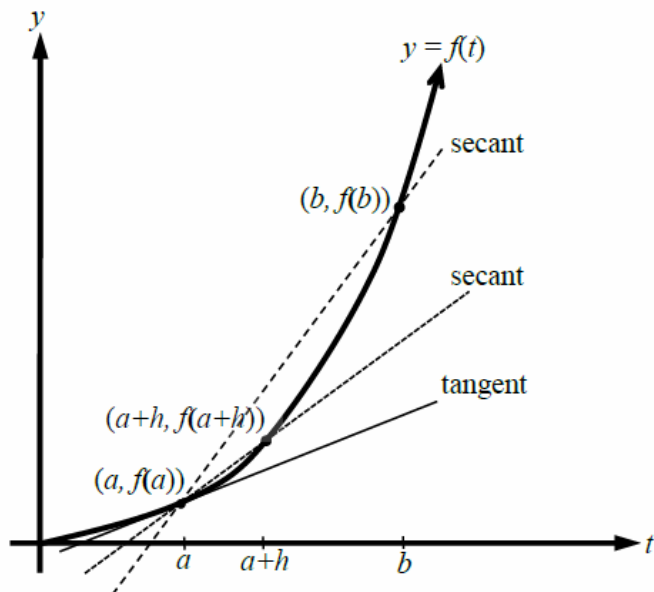
The tangent slope m is the limit of the secant slopes, so it is equal to the instantaneous velocity:

$$m = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} = v \, .$$

Trig Function Example: We model average daily temperature through the year by the sinusoidal function:

$$T(x) = 55 - 40 \cos\left(\tfrac{2\pi}{365} x\right) \text{ degrees F on day } x.$$

How quickly is the weather warming at $x = 100$ (April 10), in degrees/day? We can only approximate the instantaneous rate of change r by looking at average rates near the given $x = 100$, for example over $x \in [100, 110]$:

$$r_1 = \frac{T(110) - T(100)}{110 - 100} = \frac{(55 - 40\cos(\frac{2\pi}{365} 110)) - (55 - 40\cos(\frac{2\pi}{365} 100))}{10} \approx 0.668$$

Or over $x \in [90, 100]$, giving $r_2 = 0.686$. We can get a better approximation for r by averaging the underestimate and the overestimate:

$$r \approx \tfrac{1}{2}(r_1 + r_2) = 0.677 \, .$$

Of course, the accuracy of this model is way less than 3 decimal places, so we should say the model predicts warming of about 0.7 degree F per day.