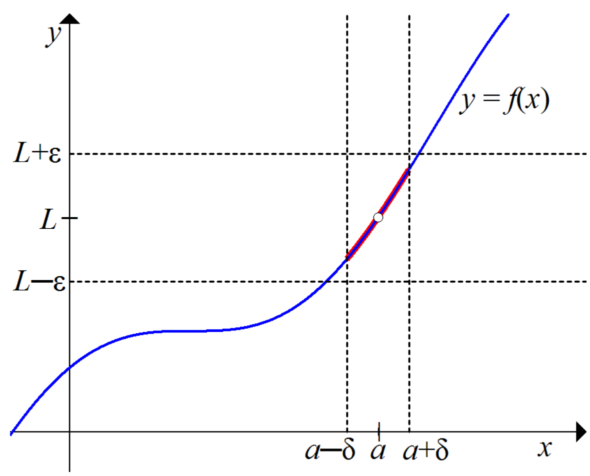
# Limit Definition



Why do we need limits? Because we cannot directly evaluate important quantities like instantaneous velocity or tangent slope, but we can approximate them with arbitrary accuracy. A limit pinpoints the exact value within this cloud of approximations. In this section, we get to the logical core of this concept.

For example, consider the tangent line of $y = x^2$ at $x = 1$, approximated by the secant through $(1, 1)$ and a nearby point $(x, x^2)$, giving the slope: $f(x) = \frac{x^2-1}{x-1}$. There is no defined value for $f(1)$, but as $x$ gets very close to 1, we expect the approximations $f(x)$ to have the exact tangent slope as their "limiting value", $\lim_{x\to 1} f(x) = L$. This means a candidate value $L$ is the correct value if we can force $f(x)$ as close as desired to $L$ (within an error $\varepsilon = \frac{1}{10}$, or $\frac{1}{100}$, or $\frac{1}{1{,}000{,}000}$, or any possible $\varepsilon > 0$), provided we restrict $x$ close enough to 1.

Thus, proving a limit is an *error-control problem* of a type we see in the real world. For example, how accurately must you set the angle of your tennis racket to land the ball within one foot of a given spot (or within one inch)? In the general situation, an input setting $x$ produces an output $f(x)$: how accurate must the input be to ensure a tolerable output error? That is, what allowed difference $\delta$ of $x$ from $a$ will force an error less than $\varepsilon$ of $f(x)$ from $L$?*




In the graph $y = f(x)$, we take the small red piece between the vertical lines $a - \delta < x < a + \delta$ (not including $x = a$). By setting $\delta$ small enough, we try to force this piece between the fixed horizontal lines $L - \varepsilon < y < L + \varepsilon$, for the specified output error $\varepsilon$.

Rewriting $a-\delta < x < a+\delta$ as $|x - a| < \delta$, and $L-\varepsilon < f(x) < L+\varepsilon$ as $|f(x) - L| < \varepsilon$, we get the formal definition of a limit:

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*Here $\delta$ (delta) is a Greek letter $d$, standing for "difference", and $\varepsilon$ (epsilon) is Greek $e$, standing for "error".

> *Definition:* $\lim_{x\to a} f(x) = L$ means that for any output error tolerance $\varepsilon > 0$, there is an input accuracy $\delta > 0$ such that $0 < |x-a| < \delta$ forces $|f(x) - L| < \varepsilon$.

We can define one-sided and infinte limits similarly.[†]

**Proof of Individual Limits.** The precise definition allows us to rigorously prove facts about limits: specific limit computatons, as well as the general Limit Laws, which can then be applied instead of case-by-case proofs.

EXAMPLE: We prove that $\lim_{x\to 5}(3x-2) = 3(5)-2 = 13$. We treat the desired error tolerance $\varepsilon$ as a variable, and we want to guarantee the output error $|f(x) - L| < \varepsilon$, or equivalently $-\varepsilon < f(x) - L < \varepsilon$. We write this out and solve the inequalities for $x$:

$$-\varepsilon < (3x-2) - 13 < \varepsilon \iff 15 - \varepsilon < 3x < 15 + \varepsilon$$

$$\iff \tfrac{1}{3}(15-\varepsilon) < x < \tfrac{1}{3}(15+\varepsilon) \iff 5-\tfrac{1}{3}\varepsilon < x < 5+\tfrac{1}{3}\varepsilon.$$

(Here $\iff$ means "is logically equivalent to".) Finally, we put this in terms of the input accuracy $x - a = x - 5$:

$$-\tfrac{1}{3}\varepsilon < x - 5 < \tfrac{1}{3}\varepsilon.$$

To force this, we are allowed to set any input accuracy $|x - a| < \delta$, or $-\delta < x - a < \delta$. Evidently, $\delta = \frac{1}{3}\varepsilon$ will work.[‡]

EXAMPLE: A harder error-control problem: $\lim_{x\to 3}\sqrt{x} = \sqrt{3}$. We translate the output accuracy requirement $-\varepsilon < f(x) - L < \varepsilon$ into inequalities bounding the input accuracy $x - a$. (Here $x$ is a positive value close to 3, and we take any small error tolerance $1 > \varepsilon > 0$.)

$$\begin{aligned} -\varepsilon < \sqrt{x} - \sqrt{3} < \varepsilon \quad &\iff \quad \sqrt{3} - \varepsilon < \sqrt{x} < \sqrt{3} + \varepsilon \\ &\iff \quad \sqrt{3}^2 - 2\varepsilon\sqrt{3} + \varepsilon^2 < x < \sqrt{3}^2 + 2\varepsilon\sqrt{3} + \varepsilon^2 \\ &\iff \quad -2\varepsilon\sqrt{3} + \varepsilon^2 < x-3 < 2\varepsilon\sqrt{3} + \varepsilon^2 \end{aligned}$$

We need an input accuracy $\delta$ which guarantees the last inequalities above: In general, to guarantee a desired equality of the form $-d_1 < x-a < d_2$, we

[†] $\lim_{x\to a^+} f(x) = L$ means that for any $\varepsilon > 0$, there is some $\delta > 0$ such that $0 < x-a < \delta$ implies $|f(x) - L| < \varepsilon$; and $\lim_{x\to a} f(x) = \infty$ means that for any bound $B$, there is some $\delta > 0$ such that $0 < |x-a| < \delta$ implies $f(x) > B$.

[‡] For the formal proof, we must reverse this logic, and show that the given input accuracy guarantees the desired output accuracy. Given any desired $\varepsilon > 0$, we define $\delta = \frac{1}{3}\varepsilon$, and assume $|x - 5| < \delta$. Then we have:

$$3|x-5| < 3\delta = \varepsilon \implies |3x - 15| < \varepsilon \implies |(3x-2) - 13| < \varepsilon,$$

which is our desired conclusion $|f(x) - L| < \varepsilon$. (Here $\implies$ means "logically implies".)

choose $\delta$ to be the smaller of $d_1$ and $d_2$. Thus we take $\delta = 2\varepsilon\sqrt{3} - \varepsilon^2$. Then $-\delta < x{-}3$ is equivalent to the desired lower bound $-2\varepsilon\sqrt{3} + \varepsilon^2 < x{-}3$; and also $x{-}3 < \delta$ implies the desired upper bound, since:

$$\delta \;=\; 2\varepsilon\sqrt{3} - \varepsilon^2 \;<\; 2\varepsilon\sqrt{3} + \varepsilon^2.$$

*Note:* In evaluating limits, we almost always rely on the Limit Laws and other general theorems, without a specific error analysis. The general results guarantee that the error approaches zero, and this is all we need.

**Proof of Limit Theorems.** All the general Limit Laws of §1.6 can be rigorously proved by error-control analysis. We prove the simplest one:

*Sum Law:* If $\lim_{x\to a} f(x) = L$ and $\lim_{x\to a} g(x) = M$, then:

$$\lim_{x\to a} (f(x) + g(x)) = L + M.$$

*Proof.* Consider any $\varepsilon > 0$. Since we assume $\lim_{x\to a} f(x) = L$ and $\lim_{x\to a} g(x) = M$, we can require the error tolerance $\frac{1}{2}\varepsilon$ for these limits, getting $\delta > 0$ small enough that $0 < |x - a| < \delta$ forces:

$$-\tfrac{1}{2}\varepsilon < f(x){-}L < \tfrac{1}{2}\varepsilon \qquad \text{and} \qquad -\tfrac{1}{2}\varepsilon < g(x){-}M < \tfrac{1}{2}\varepsilon.$$

Adding these inequalities, we find that $0 < |x - a| < \delta$ also forces:

$$-\tfrac{1}{2}\varepsilon - \tfrac{1}{2}\varepsilon \;<\; (f(x){-}L) + (g(x){-}M) \;<\; \tfrac{1}{2}\varepsilon + \tfrac{1}{2}\varepsilon$$

Rewriting, this is just $-\varepsilon \;<\; (f(x){+}g(x)) - (L{+}M) \;<\; \varepsilon$, which is the desired output error bound.

*Squeeze Theorem:* If $f(x) < g(x) < h(x)$ for all values of $x$ near $a$ (except perhaps $x = a$), and $\lim_{x\to a} f(x) = \lim_{x\to a} h(x) = L$, then $\lim_{x\to a} g(x) = L$.

*Proof.* Consider any $\varepsilon > 0$. Since we assume $\lim_{x\to a} f(x) = L$ and $\lim_{x\to a} h(x) = L$, we can find a $\delta > 0$ such that $0 < |x{-}a| < \delta$ forces $-\varepsilon < f(x){-}L < \varepsilon$ and $-\varepsilon < g(x){-}L < \varepsilon$. We also know $f(x) < g(x) < h(x)$ provided $|x - a| < \delta$ restricts $x$ close enough to $a$, so:

$$f(x){-}L < g(x){-}L < h(x){-}L.$$

Then $0 < |x - a| < \delta$ also forces:

$$-\varepsilon < f(x){-}L < g(x){-}L \qquad \text{and} \qquad g(x){-}L < h(x){-}L < \varepsilon,$$

which gives the desired output accuracy for $g(x)$.

**Substitution Theorem:** If $\lim_{x\to b} f(x) = L$ and $\lim_{x\to a} g(x) = b$, and $g(x) \neq b$ for all $x$ close enough to (but unequal to) $a$, then $\lim_{x\to a} f(g(x)) = L$.

*Proof.* For any $\varepsilon > 0$, we must find a number $\delta > 0$ such that $0 < |x-a| < \delta$ forces $|f(g(x)) - L| < \varepsilon$.

Take any $\varepsilon > 0$. Since $\lim_{x\to b} f(x) = L$, there is $\delta_1 > 0$ such that $0 < |y-b| < \delta_1$ forces $|f(y) - L| < \varepsilon$. Also, since $\lim_{x\to a} g(x) = b$, there exists $\delta_2 > 0$ such that $0 < |x-a| < \delta_2$ forces $|g(x) - b| < \delta_1$. Now take $\delta < \delta_2$, and $\delta$ small enough that $0 < |x-a| < \delta$ forces $g(x) \neq b$. Then we know $|x-a| < \delta$ forces $0 < |g(x) - b| < \delta_1$, which in turn forces $|f(g(x)) - L| < \varepsilon$, as required.