Math 132

Differentiation Formulas

Stewart §2.3

So far, we have seen how various real-world problems (rate of change) and geometric problems (tangent lines) lead to derivatives. In this section, we will see how to solve such problems by computing derivatives (differentiating) algebraically.

Notations. We have seen the Newton notation $f'(x)$ for the derivative of $f(x)$. The alternative Leibniz notation for the derivative is $\frac{df}{dx}$, meant to remind us of the definition of $f'(x)$ as the limit of difference quotients:

$$f'(x) \;=\; \frac{df}{dx} \;=\; \lim_{\Delta x \to 0} \frac{\Delta f}{\Delta x}\,.$$

Here $\Delta f = f(x{+}h) - f(x)$, the difference* in $f(x)$ produced by the difference $\Delta x = (x{+}h){-}x = h$. Also, df and dx are meant to suggest very small Δf and Δx, but $\frac{df}{dx}$ is not literally the quotient of two small quantities, just a complicated symbol meaning the limit of such quotients.

To illustrate: for $f(x) = x^2$, the formula $f'(x) = (x^2)' = 2x$ can be written in Leibniz notation as:

$$\frac{df}{dx} \;=\; \frac{d}{dx}(x^2) \;=\; 2x.$$

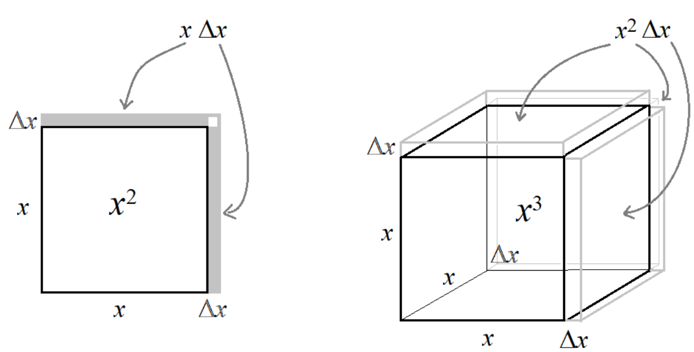
The symbol $\frac{df}{dx}$ means the function $f'(x)$; for a particular value of a derivative at $x = a$, we write $f'(a) = \frac{df}{dx}\Big|_{x=a}$. The notation $f' = Df$ is also used, and $f'(x) = Df(x)$.

Basic Derivatives. To compute derivatives without a limit analysis each time, we use the same strategy as for limits in Notes §1.6: we establish the derivatives of some basic functions, then we show how to compute the derivatives of sums, products, and quotients of known functions.

Theorem: (i) For a constant function $f(x) = c$, we have $\frac{d}{dx}(c) = (c)' = 0$.
(ii) For $f(x) = x$, we have $\frac{d}{dx}(x) = (x)' = 1$.
(iii) For $f(x) = x^p$ with p any real number, we have:

$$\tfrac{d}{dx}(x^p) = (x^p)' = px^{p-1}.$$

The picture below shows the change in the area $f(x) = x^2$ due to increment $h = \Delta x$ is about $\Delta f \cong 2x\Delta x$, so $f'(x) \cong \frac{\Delta f}{\Delta x} \cong 2x$. Similarly for volume $f(x) = x^3$, with $f'(x) \cong 3x^2$.



The same holds for the growth of an n-dimensional cube, giving $f'(x) = nx^{n-1}$, but we can only compute this algebraically, not picture it.

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*Δ is capital letter delta, the Greek D, standing for "difference". The small letter delta is δ.

Proof: (i) and (ii) follow easily from the definition of $f'(x)$. We prove (iii) in stages, for more and more general powers p, relying repeatedly on the family of formulas: $a^n - b^n = (a-b)(a^{n-1} + a^{n-2}b + a^{n-3}b^2 + \cdots + b^{n-1})$, valid for $n = 1, 2, 3, \ldots$ First, we consider a whole number $p = n$, and take $a = x+h$ and $b = x$:

$$(x^n)' \;=\; \lim_{h\to 0} \frac{(x+h)^n - x^n}{h} \;=\; \lim_{h\to 0} \frac{((x+h)-x)\,((x+h)^{n-1}+(x+h)^{n-2}x+\cdots+x^{n-1})}{h}$$

$$= \lim_{h\to 0} (x+h)^{n-1}+(x+h)^{n-2}x+\cdots+x^{n-1} \;=\; (x+0)^{n-1}+(x+0)^{n-2}x+\cdots+x^{n-1} \;=\; nx^{n-1}.$$

Thus, $(x^n)' = nx^{n-1}$, and (iii) holds for $p = n$.

Second, we do a similar calculation for a negative integer $p = -n$, so that $x^p = \frac{1}{x^n}$; in the derivative limit, we combine fractions and apply the $a^n - b^n$ formula with $a = x$ and $b = x+h$. The result simplifies to $-\frac{nx^{n-1}}{x^{2n}} = (-n)x^{(-n)-1}$.

Third, we consider a fraction $p = \frac{n}{m}$ with m a whole number and n an integer, so that $x^p = x^{\frac{m}{n}} = \sqrt[m]{x^n}$. We take the derivative limit with numerator $\sqrt[m]{(x+h)^n} - \sqrt[m]{x^n} = a - b$. As in §2.2 for $\sqrt[3]{x}$, multiplying top and bottom by $a^{m-1} + a^{m-2}b + a^{m-3}b^2 + \cdots + b^{m-1}$ gets rid of the radicals $\sqrt[m]{\ }$, leaving the numerator $a^m - b^m = (x+h)^n - x^n$, which we handled previously. Again, the limit eventually simplifies to formula (iii).

Formula (iii) is also valid for an irrational power like $p = \sqrt{2}$, but this requires more theory: we will have to wait until Calculus II to even state a clear definiton of $x^{\sqrt{2}}$.

Having computed all these limits, we never have to do so again. Just from quoting the Theorem, we get formulas like: $(x^2)' = 2x^1 = 2x$; $(x^{10})' = 10x^9$;

$$\left(\sqrt[3]{x}\right)' \;=\; (x^{1/3})' \;=\; \tfrac{1}{3}x^{-2/3} \;=\; \tfrac{1}{3\sqrt[3]{x^2}}\,; \qquad \left(\tfrac{1}{x}\right)' \;=\; (x^{-1})' \;=\; (-1)x^{-1-1} \;=\; -\tfrac{1}{x^2}\,.$$

Derivative Rules. Suppose the functions $f(x), g(x)$ are differentiable at x, so that $f'(x)$ and $g'(x)$ exist. Then we get the following derivatives:

- *Sum:* $(f(x) + g(x))' = f'(x) + g'(x)$.
- *Difference:* $(f(x) - g(x))' = f'(x) - g'(x)$.
- *Constant Multiple:* $(c\,f(x))' = c\,f'(x)$ for any constant c.
- *Product:* $(f(x)g(x))' = f'(x)g(x) + f(x)g'(x)$.
- *Quotient:* $\left(\dfrac{f(x)}{g(x)}\right)' = \dfrac{f'(x)g(x) - f(x)g'(x)}{g(x)^2}$, where $g(x) \neq 0$.

The first three of these Rules, which express the *linearity* of the derivative operation, are intuitive and easy to prove. For example the Sum Rule:

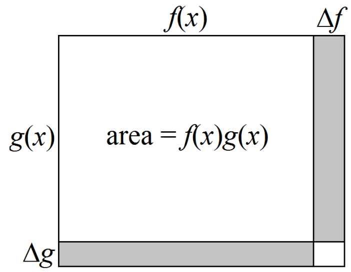
$$(f(x) + g(x))' \;=\; \lim_{h\to 0} \tfrac{(f(x+h)+g(x+h))-(f(x)+g(x))}{h} \;=\; \lim_{h\to 0} \tfrac{f(x+h)-f(x)}{h} + \tfrac{g(x+h)-g(x)}{h}$$

$$= \lim_{h\to 0} \tfrac{f(x+h)-f(x)}{h} + \lim_{h\to 0} \tfrac{g(x+h)-g(x)}{h} \;=\; f'(x) + g'(x).$$

Here the third equality follows from the Sum Law for limits in Notes §1.6.

$$\boxed{\text{Warning: The derivative of a product is NOT the product of derivatives.}}$$

We obtain the correct Product Rule from a geometric model: consider a rectangle with changing sides of lengths $f(x)$ and $g(x)$ depending on some variable x, the upper left rectangle below:



The product $f(x)g(x)$ is the area, and the derivative $(f(x)g(x))'$ is the rate of change of area with respect to a change in x. Suppose small increment $\Delta x = h$ produces some positive increments $\Delta f = f(x{+}h) - f(x)$ and $\Delta g = g(x{+}h) - g(x)$ in the sides, so that the increment of area, $\Delta(f\cdot g) = f(x{+}h)g(x{+}h) - f(x)g(x)$, is the area of the three edge rectangles:†

$$\Delta(f\cdot g) \;=\; (\Delta f)\cdot g(x) + f(x)\cdot(\Delta g) + (\Delta f)\cdot(\Delta g).$$

To get the derivative, we divide by Δx to get the difference quotient, and send $\Delta x = h \to 0$:

$$\begin{aligned}
(f(x)g(x))' &= \lim_{\Delta x\to 0} \tfrac{\Delta(f\cdot g)}{\Delta x} \;=\; \lim_{\Delta x\to 0} \tfrac{(\Delta f)g(x)}{\Delta x} + \tfrac{f(x)(\Delta g)}{\Delta x} + \tfrac{(\Delta f)(\Delta g)}{\Delta x} \\
&= \left(\lim_{\Delta x\to 0} \tfrac{\Delta f}{\Delta x}\right) g(x) \;+\; f(x)\left(\lim_{\Delta x\to 0} \tfrac{\Delta g}{\Delta x}\right) \;+\; \left(\lim_{\Delta x\to 0} \tfrac{\Delta f}{\Delta x}\right)\left(\lim_{\Delta x\to 0} \Delta g\right) \\
&= f'(x)g(x) + f(x)g'(x) + f'(x)(0) \;=\; f'(x)g(x) + f(x)g'(x).
\end{aligned}$$

Note that the vanishing third term corresponds to the tiny bottom right rectangle.

Lastly, we prove the Quotient Rule:

$$\left(\frac{f(x)}{g(x)}\right)' = \lim_{h\to 0} \frac{\frac{f(x+h)}{g(x+h)} - \frac{f(x)}{g(x)}}{h} \;=\; \lim_{h\to 0} \frac{f(x{+}h)g(x) - f(x)g(x{+}h)}{h\, g(x{+}h)\, g(x)}$$

$$= \lim_{h\to 0} \frac{f(x{+}h)g(x) - f(x)g(x) + f(x)g(x) - f(x)g(x{+}h)}{h\, g(x{+}h)\, g(x)}$$

Here, after putting the expression over a common denominator, we have added and subtracted the quantity $f(x)g(x)$ in the numerator, leaving the limit unchanged. Our aim is to factor the first pair and last pair of terms:

$$\left(\frac{f(x)}{g(x)}\right)' = \lim_{h\to 0} \frac{(f(x{+}h){-}f(x))\, g(x) \;+\; f(x)\, (g(x){-}g(x{+}h))}{h\, g(x{+}h)\, g(x)}$$

$$\begin{aligned}
&= \lim_{h\to 0} \frac{1}{g(x{+}h)\, g(x)} \left(\frac{f(x{+}h) - f(x)}{h}\, g(x) - f(x)\, \frac{g(x{+}h) - g(x)}{h} \right) \\
&= \frac{1}{g(x{+}0)\, g(x)} \big(f'(x)g(x) - f(x)g'(x)\big) \;=\; \frac{f'(x)g(x) - f(x)g'(x)}{g(x)^2}.
\end{aligned}$$

We have again used several Limit Laws from Notes §1.6. We could give another proof of the Product Rule in a very similar way.

† We can check this formula algebraically for any $f(x), f(x{+}h), g(x), g(x{+}h)$: just substitute for Δf, Δg.

Derivative computations. By repeatedly using these Rules, we can quickly compute the derivatives of most functions.

EXAMPLE: Find $(\sqrt{x})' = \frac{d}{dx}(\sqrt{x})$. Solution: $(\sqrt{x})' = (x^{1/2})' = \frac{1}{2}x^{(1/2)-1} = \frac{1}{2}x^{-1/2} = \frac{1}{2\sqrt{x}}$, where we used the Basic Derivative $(x^p)' = px^{p-1}$ with $p = \frac{1}{2}$.

EXAMPLE: $(\sqrt{10})' = 0$ since the derivative of *any* constant, even a complicated one, is *zero*.

EXAMPLE: For $f(x) = (5x^2+1)(\sqrt{x}-3)$, find the derivative $f'(x) = \frac{df}{dx}$:

$$\begin{aligned}
\big((5x^2{+}1)(\sqrt{x}{-}3)\big)' &= \big(5x^2{+}1\big)'\,(\sqrt{x}{-}3) + \big(5x^2{+}1\big)\,(\sqrt{x}{-}3)' && \text{by Product Rule} \\
&= \big(5(x^2)'{+}(1)'\big)\,(\sqrt{x}{-}3) + \big(5x^2{+}1\big)\,((\sqrt{x})'{-}(3)') && \text{by Sum \& Const Mult Rules} \\
&= \big(5(2x^1){+}0\big)\,(\sqrt{x}{-}3) + \big(5x^2{+}1\big)\,\big(\tfrac{1}{2}x^{-1/2}{-}0\big) && \text{by Basic Derivatives} \\
&= 10x(\sqrt{x}{-}3) + (5x^2{+}1)\tfrac{1}{2\sqrt{x}} && \text{tidying up}
\end{aligned}$$

Note how we used the derivative from the previous example, $(\sqrt{x})' = \frac{1}{2}x^{-1/2}$.

Another way to find the same derivative would be to multiply out first:

$$f(x) \;=\; (5x^2{+}1)(\sqrt{x}{-}3) \;=\; 5x^2\sqrt{x} - 15x^2 + \sqrt{x} - 3 \;=\; 5x^{5/2} - 15x^2 + x^{1/2} - 3.$$

Then we get the derivative:

$$f'(x) \;=\; 5(\tfrac{5}{2}x^{(5/2)-1}) - 15(2x^1) + \tfrac{1}{2}x^{(1/2)-1} - 0 \;=\; \tfrac{25}{2}x\sqrt{x} - 30x + \tfrac{1}{2\sqrt{x}}.$$

This agrees with our previous answer, multiplied out.

EXAMPLE: Differentiate $g(t) = \frac{t^5+1}{t\sqrt{t}}$. Solution by the Quotient Rule:

$$g'(t) \;=\; \frac{dg}{dt} \;=\; \left(\frac{t^5{+}1}{t\sqrt{t}}\right)' \;=\; \frac{(t^5{+}1)'(t\sqrt{t}) - (t^5{+}1)(t\sqrt{t})'}{(t\sqrt{t})^2} \;=\; \frac{(5t^4)(t\sqrt{t}) - (t^5{+}1)(\frac{3}{2}t^{1/2})}{t^3},$$

where we use $t\sqrt{t} = t^{3/2}$.

Solution by multiplying out: $\frac{1}{t\sqrt{t}} = t^{-3/2}$, so:

$$g(t) \;=\; (t^5{+}1)t^{-3/2} \;=\; t^{7/2} + t^{-3/2} \qquad \text{and} \qquad g'(t) = \tfrac{7}{2}\,t^{5/2} \;-\; \tfrac{3}{2}\,t^{1/2}.$$

EXAMPLE: A block of ice has length 10cm, width 5cm, and height 20cm. Its length and width are melting at a rate of 1cm per hour, but its height is melting at 2cm per hour (because the base is sitting on the warm ground). How fast is the volume decreasing?

Solution: The volume is $V = \ell w h\,\mathrm{cm}^3$, where V, ℓ, w, h are all functions of time t. The rate of change is the derivative. We use the Product Rule twice, considering $\ell w h = (\ell)(wh)$:

$$\frac{dV}{dt} \;=\; V' \;=\; (\ell w h)' \;=\; (\ell)'(wh) + (\ell)(wh)' \;=\; \ell' wh + \ell(w'h + wh') \;=\; \ell' wh + \ell w' h + \ell w h'.$$

We want the melt rate at the current time $t = 0$, and we are given: $\ell(0) = 10$ cm, $\ell'(0) = -1$ cm/hr; and $w(0) = 5$ cm, $w'(0) = -1$ cm/hr; and $h(0) = 20$ cm, $h'(0) = -2$ cm/hr. Thus:

$$\begin{aligned}
V'(0) &= \ell'(0)w(0)h(0) + \ell(0)w'(0)h(0) + \ell(0)w(0)h'(0) \\
&= (-1)(5)(20) + (10)(-1)(20) + (10)(5)(-2) \;=\; -400\,\mathrm{cm}^3/\mathrm{hr}.
\end{aligned}$$

Higher derivatives. Since the derivative operation turns a function $f(x)$ into another function $f'(x)$, we can do it again to $f'(x)$, obtaining yet another function denoted $f''(x) = (f'(x))'$ or $\frac{d^2f}{dx} = \frac{d}{dx}\left(\frac{df}{dx}\right)$, called the *second derivative* of $f(x)$.

In real-world terms, if $f'(x)$ is the rate of change of $f(x)$, then $f''(x)$ is the rate of change of $f'(x)$, namely how much the rate $f'(x)$ is speeding up or slowing down.

EXAMPLE: A stone falls $f(t) = 16t^2$ ft in t seconds. Compute the repeated derivatives of this function, and interpret their physical meaning.

- The first derivative is $f'(t) = (16t^2)' = 16(2t^1) = 32t$ ft/sec. This is the *velocity* $v(t) = f'(t) = 32t$ ft/sec, increasing proportional to time.
- The second derivative is $f''(t) = (32t)' = 32$, with units ft/sec per sec = ft/sec^2. It means the rate of change of velocity, how many ft/sec of speed is gained each second. This is the *acceleration* of the stone, $a(t) = f''(t) = 32$ ft/sec^2, the constant acceleration due to gravity.
- The third derivative is $f'''(t) = (32)' = 0$, meaning the rate of change of a constant acceleration is zero. The physics term for this quantity is the *jerk*, and since the jerk here is zero, we see that gravity does not jerk: it pulls smoothly. All higher derivatives are also zero; these do not have common names.[‡]

[‡]But look up "Snap, crackle, pop (physics)".