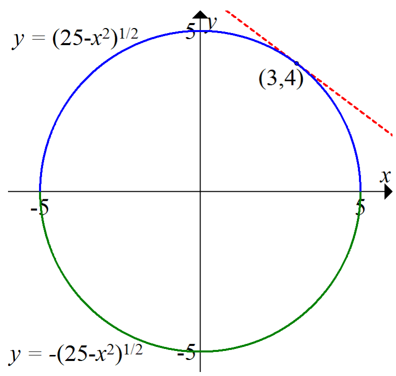
Explicit versus implicit functions. Given the circle defined by the equation $x^2+y^2=25$, suppose we wish to find the tangent line at the point $(x,y)=(3,4)$. Calculus finds a tangent slope of a function graph $y=f(x)$ as a derivative $f'(a)=\frac{df}{dx}|_{x=a}$; but there is no function specified in our problem.

Rather, we must interpret x as an independent variable, which *implicitly* makes y a function of x: to make this *explicit*, we solve the equation for y, giving $y=\pm\sqrt{25-x^2}$. That is, the circle is the union of two function graphs, $y=\sqrt{25-x^2}$ and $y=-\sqrt{25-x^2}$, each over the domain $x\in[-5,5]$.



The given point $(3,4)$ is on the top graph, and we differentiate its explicit function:

$$\frac{dy}{dx} \;=\; \frac{d}{dx}\sqrt{25-x^2} \;=\; \frac{d}{dx}(25-x^2)^{\frac{1}{2}} \;=\; \frac{1}{2}(25-x^2)^{-\frac{1}{2}}\frac{d}{dx}(25-x^2) \;=\; \frac{-x}{\sqrt{25-x^2}}.$$

Here we used the Chain Rule with outside function $(\quad)^{1/2}$. At our point, we have tangent slope $\frac{dy}{dx}|_{x=3}=y'(3)=\frac{-3}{\sqrt{25-3^2}}=-\frac{3}{4}$, and the tangent line $y=-\frac{3}{4}(x-3)+4$.

Implicit differentiation is a smoother way to do this problem. Instead of solving the equation for y, we assume $y=y(x)$ for some unkown function $y(x)$ which satisfies the equation $x^2+y(x)^2=25$. Then we differentiate both sides using the Rules:

$$\begin{aligned} \left(x^2+y(x)^2\right)' &= (25)' \\ \left(x^2\right)'+\left(y(x)^2\right)' &= 0 \\ 2x+2y(x)y'(x) &= 0. \end{aligned}$$

Note that $(x^2)'=2x$ is a Basic Derivative, but for $(y^2)'$, we need the Chain Rule with outside function $(\quad)^2$ and inside function $y=y(x)$. The derivative $y'(x)$ is the unknown we are trying to find, and now we can solve for it: $y'(x)=-\frac{x}{y(x)}$, which was easier than solving for the original $y(x)$. Since are considering the point $(x,y)=(3,4)$, we must have $y(3)=4$, so that $\frac{dy}{dx}|_{x=3}=y'(3)=-\frac{3}{y(3)}=-\frac{3}{4}$, as before.

Note that the formula $y'(x)=-\frac{x}{y(x)}$, or in Leibnitz notation $\frac{dy}{dx}=-\frac{x}{y}$, is valid for both of the functions defining the upper and lower half-circles. Since both functions obey the original equation, they both obey the derivative equation. For example, at $(x,y)=(3,-4)$, the slope is $y'(3)=-\frac{3}{y(3)}=-\frac{3}{-4}=\frac{3}{4}$.

We could even take this one step further to find the second derivative implicitly:

$$y''(x) \;=\; (y'(x))' \;=\; (-\tfrac{x}{y})' \;=\; -\frac{(x)'y-x\,y'}{y^2} \;=\; -\frac{y-x(-\frac{x}{y})}{y^2} \;=\; -\frac{y^2+x^2}{y^3} \;=\; -\frac{25}{y^3}.$$

We used the Quotient Rule, the previous $y'=-\frac{x}{y}$, and the original equation $x^2+y^2=25$.

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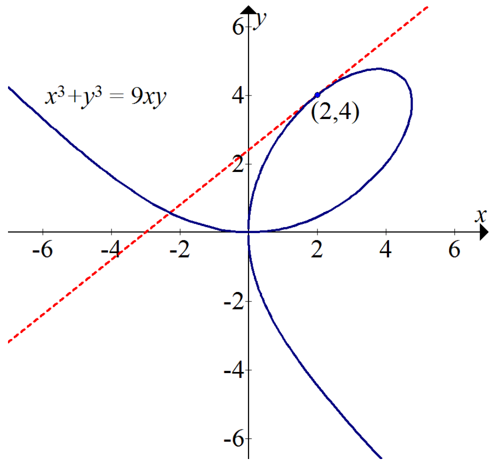
Folium of Descartes. This is a curve discovered by the famous mathematician who gave us Cartesian xy-coordinates. It is defined by a nodal cubic equation: $x^3 + y^3 = 9xy$:*



We want to find the tangent line at the point $(x, y) = (2, 4)$, which is on the curve because $2^3 + 4^3 = 9(2)(4)$. In this case, there is no easy way to solve for y to get an explicit function $y(x)$; indeed, over $x \in [0, \frac{9}{2}]$, the curve is the union of *three* function graphs.

Nevertheless, implicit differentiation works without a hitch: we assume $y = y(x)$ is some unknown function which satisfies the equation, and differentiate both sides (this time in Leibnitz notation):

$$\begin{aligned} \tfrac{d}{dx}\big(x^3 + y^3\big) &= \tfrac{d}{dx}(9xy) \\ \tfrac{d}{dx}(x^3) + \tfrac{d}{dx}(y^3) &= 9\big(\tfrac{d}{dx}(x)\,y \;+\; x\,\tfrac{d}{dx}(y)\big) \\ 3x^2 + 3y^2\tfrac{dy}{dx} &= 9y + 9x\tfrac{dy}{dx}. \end{aligned}$$

Here we used the Sum and Product Rules, then the Chain Rule. Solving for $\frac{dy}{dx}$:

$$3y^2\tfrac{dy}{dx} - 9x\tfrac{dy}{dx} = 9y - 3x^2, \qquad \tfrac{dy}{dx} = \frac{9y-3x^2}{3y^2-9x}.$$

We do not know $y(x)$ explicitly, but our given point $(x, y) = (2, 4)$ means that $y(2) = 4$, so:

$$\tfrac{dy}{dx}\Big|_{x=2} = \frac{9y-3x^2}{3y^2-9x} = \frac{9(4) - 3(2^2)}{3(4^2) - 9(2)} = \tfrac{4}{5}.$$

Thus, the tangent line through the point $(2, 4)$ is: $y = \frac{4}{5}(x-2) + 4$.

Method for implicit differentiation. Given an equation involving variables x and y, we assume x is an independent variable and $y = y(x)$ is a dependent variable. To find the derivative $\frac{dy}{dx}$:

1. Take the derivative of both sides of the equation, using the Chain Rule for expressions involving $y = y(x)$ as the inside function.

2. Solve the derivative equation for the unknown $\frac{dy}{dx}$, in terms of x and y.

3. To get a specific value $y'(a) = \frac{dy}{dx}|_{x=a}$, plug in the known values $x = a$ and $y = y(a)$.

*To find points satisfying this equation, substitute $y = tx$ for a new variable t, and solve for x, giving: $x = \frac{9t}{1+t^3}$ and $y = \frac{9t^2}{1+t^3}$. Then each value of t gives a point (x, y) on the curve: this is called a *parametrization.*