Rates of Change

Conceptual levels. Mathematics solves problems partly with technical tools like the differentiation rules, but its most powerful method is to translate between different levels of meaning, transforming the problems to make them accessible to our tools. Problems often originate at the physical or geometric levels, and we translate to the numerical or algebraic levels to solve them, then we translate the answer back to the original level.

Our key concept so far has been the *derivative*, with the following meanings:

- Physical: For a function $y = f(x)$, the derivative $\frac{dy}{dx} = f'(x)$ is the rate of change of y with respect to x, near a particular value of x. For a a particular input, $f'(a)$ means how fast $f(x)$ changes from $f(a)$ per unit change in x away from a. This is the main importance of derivatives.

- Geometric: For a graph $y = f(x)$, the derivative $f'(a)$ is the slope of the tangent line at the point $(a, f(a))$.

- Numerical: We approximate the derivative by the difference quotient:

$$f'(a) \;\cong\; \frac{\Delta f}{\Delta x} \;=\; \frac{f(a+h)-f(a)}{h}.$$

  The right side is the average rate of change of $f(x)$ from $x = a$ to $x = a+h$, for some small increment such as $h = 0.1$. As $\Delta x = h \to 0$, the difference quotient approaches the instantaneous rate of change, the derivative $f'(a)$.

- Algebraic: We can easily compute the derivative of almost any function defined by a formula. Basic Derivatives like $(x^p)' = px^{p-1}$, $\sin'(x) = \cos(x)$, and $\cos'(x) = -\sin(x)$ are combined using the Sum, Product, Quotient, and Chain Rules for Derivatives. Occasionally, we must go back to the definition $f'(a) = \lim\limits_{h\to 0} \frac{f(a+h)-f(a)}{h}$.

Functions of motion. We consider the basic physical quantities describing motion. These are all functions of time t. (See end of §2.3.)

- *Position* or *displacement* s, the distance of an object past a reference point, in feet, at time t seconds.

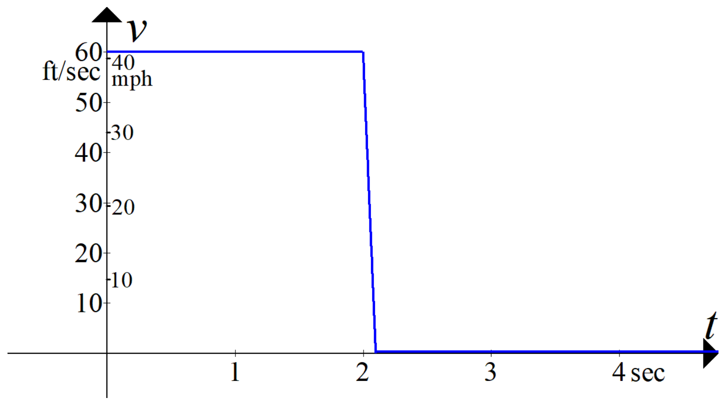
- *Velocity* $v = \frac{ds}{dt}$ or $v(t) = s'(t)$, how fast the position is increasing per second (ft/sec); this is negative if position is decreasing. The *speed* is the magnitude $|v|$.

- *Acceleration* $a = \frac{dv}{dt} = \frac{d^2s}{dt^2}$ or $a(t) = v'(t) = s''(t)$, how fast the velocity is increasing, the number of ft/sec gained each second (ft/sec^2). Equivalently, this is how fast the object is speeding up (positive) or slowing down (negative). A practical unit is the *gee* ≈ 32 ft/sec^2, the acceleration due to gravity in freefall (near the Earth's surface).

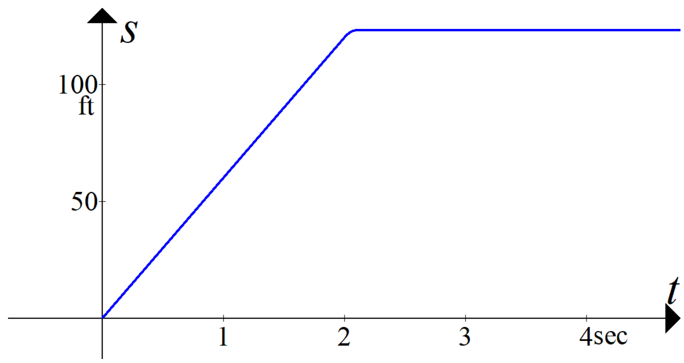
- *Jerk* $j = \frac{da}{dt} = \frac{d^3s}{dt^3}$ or $j(t) = a'(t) = s'''(t)$, rate of change of acceleration (ft/sec^3).

Notes by Peter Magyar `magyar@math.msu.edu`

Car stopping. Consider the following velocity data from a car's speedometer.

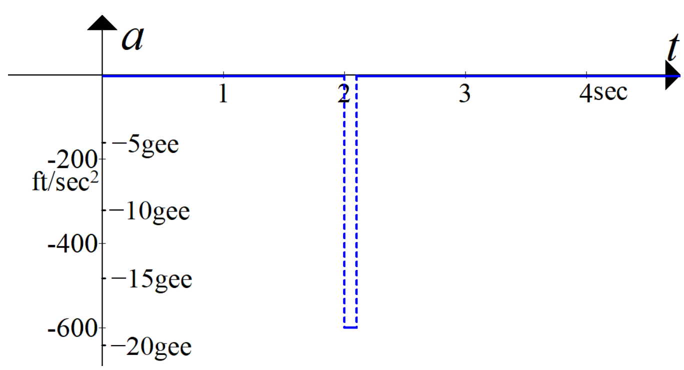


The velocity $v(t) = s'(t)$ is the derivative of $s(t)$, the distance driven or odometer reading, so the *level* of the velocity graph is the *slope* of the distance graph. Thus $s(t)$ has constant slope 60 ft/sec until $t = 2$, then suddenly zero slope: the car has stopped.



The acceleration is the derivative $a(t) = v'(t)$, so the *slope* of the velocity graph is the *level* of the acceleration graph: this slope is zero, except at the very steep transition between 2 and 2.1 sec, which makes an average slope of:

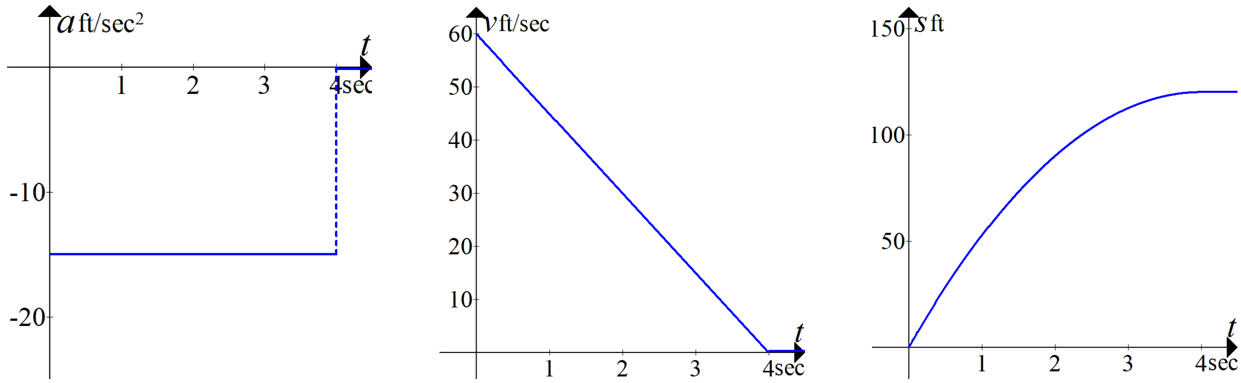
$$a \approx \frac{\Delta v}{\Delta t} = \frac{0-60}{2.1-2} = -600\text{ft/sec}^2.$$



What physical story do these graphs tell? The car went from about 40 mph to zero in 0.1 sec. Brakes cannot decelerate so quickly: this is a car crash. The deceleration graph is zero for most of the time, but for a split second it is almost 20 gees, twenty times your weight pressing you into the seatbelt.

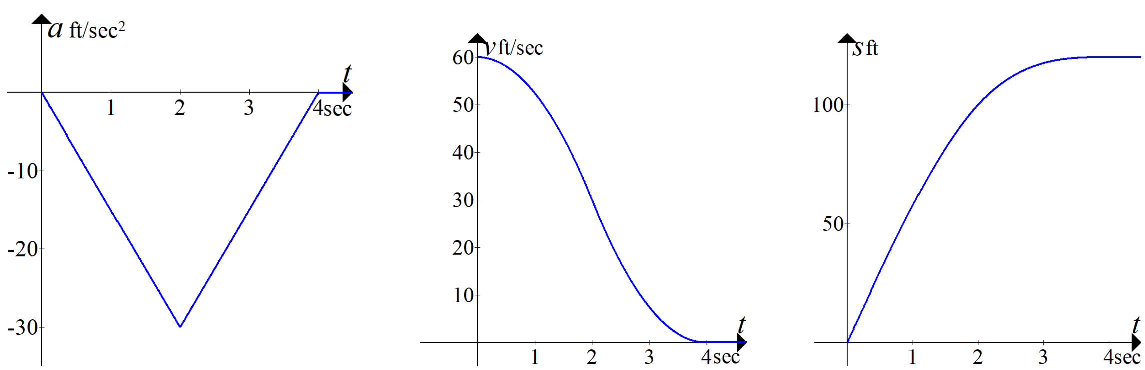
In this analysis, we translated conceptually from graphical (geometric) to physical; and also (for the gee calculation) from graphical to numerical to physical.

Braking techniques. Now imagine braking steadily at a traffic light, slowing at a constant rate until you reach a full stop. This time the deceleration is: $\frac{\Delta v}{\Delta t} = \frac{0-60}{4-0} = -15\,\mathrm{ft/sec}^2$, less than half a gee, but still a pretty hard stop:



Even though the deceleration is not very great, it changes suddenly (instantaneously in our picture), so the derivative of acceleration (the jerk) is very large for a split second, giving a noticeable jerk or jolt at the moment of stopping, not dangerous but annoying.

Is there a braking technique which will eliminate the jerk? To prevent the sudden change in acceleration, squeeze the brake slowly down, then let it slowly up:



Now the stopping time of 4 sec requires a peak deceleration of -30 ft/sec^2 $\approx$ 1 gee, which would send the car skidding helter-skelter. You would need double the time to do this technique safely, starting to brake much earlier.

Ballistic equation. This is the formula giving the height $s(t)$ for a projectile (cannon ball) launched from initial height s_0, straight upward with initial velocity v_0, pulled down by a constant gravitational acceleration g:

$$s(t) \;=\; s_0 + v_0 t - \tfrac{1}{2} g t^2.$$

To justify this equation, note that the initial height is indeed $s(0) = s_0 + v_0(0) - \frac{1}{2}g(0^2) = s_0$. Also, s_0, v_0, g are constants, so:

$$v(t) = s'(t) = (s_0)' + (v_0 t)' - (\tfrac{1}{2} g t^2)' = v_0 - gt,$$

and the initial velocity is $v(0) = v_0$. The acceleration is $a(t) = v'(t) = -g$, which is the desired constant in the correct (downward) direction. Finally, the jerk is $j(t) = a'(t) = 0$, which is correct because gravity pulls steadily and never jerks.

EXAMPLE: Given standard gravity of 32 ft/sec^2 and initial height $s_0 = 5$ ft, how fast to throw a ball upward so that it stays airborne for 4 sec? The equation becomes $s(t) = 5 + v_0 t - 16t^2$, with the throw velocity v_0 an unknown constant. Landing at 4 sec means $s(4) = 0$, that is $5 + v_0(4) - 16(4^2) = 0$, and we can solve for $v_0 = 62.75$ ft/sec. (This is $62.75/1.47 \cong 43$ mph, which would require a strong arm.)

How high will the ball go from such a throw? The velocity is:

$$v(t) = (5 + 62.75t - 16t^2)' \;=\; 62.75 - 32t.$$

At the instant $t = t_1$ when the ball reaches the top of its arc, its velocity is zero. That is: $v(t_1) = 62.75 - 32t_1 = 0$ and $t_1 \cong 1.96$ sec. (This is not quite half the 4 sec interval, because the ball started out at $s_0 = 5$ ft.) The height at this instant is $s(1.96) \cong 66.5$ ft.

For $t < t_1$, the height $s(t)$ increases, the velocity $v(t) = 79 - 32t$ is positive, the ball moves upward; for $t > t_1$, $s(t)$ decreases, $v(t)$ is negative, the ball moves downward.

Note that the graph $s = 5 + 62.75t - 16t^2$ is a downward-curving parabola, but this is not the trajectory of the ball, since this model ignores horizontal motion: the ball might be going straight up and down.