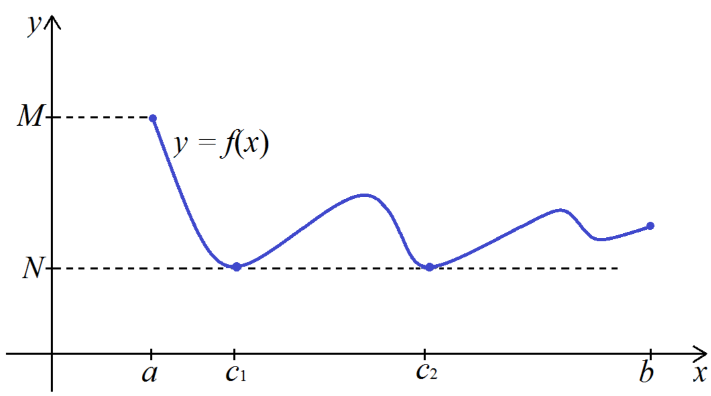
Math 132 # Extreme Values **Stewart §3.1**

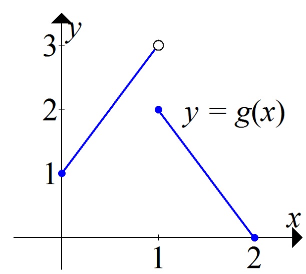
Absolute maxima and minima. In many practical problems, we must find the largest or smallest possible value of a function over a given interval.

> *Definition:* For a function $f(x)$ defined on an interval $x \in [a,b]$, an *absolute maximum* (or *global maximum*) is a point $c \in [a,b]$ such that $f(c) \geq f(x)$ for all $x \in [a,b]$. That is, $f(c)$ is the largest output value of the function at any input point in its domain. We say $x = c$ is a *maximum point* and $f(c)$ is the *maximum value*.

We define an *absolute minimum* similarly, and both maximums and and minimums are *extremums** or *extreme points*. Note that the maximum value M (the largest possible output) is unique, but $f(x)$ could touch this value at several input points $c_1, c_2, \ldots \in [a,b]$, all having $f(c_1) = f(c_2) = \cdots = M$.

EXAMPLE: At left below, the function $y = f(x)$ on the interval $[a,b]$ has one absolute maximum point, the left endpoint $x = a$ with $f(a) = M$, so that $(a, f(a))$ is the highest point on the graph; and it has two absolute minimum points $x = c_1, c_2$ with $f(c_1) = f(c_2) = N$, so that $(c_1, f(c_1))$ and $(c_2, f(c_2))$ are the lowest points on the graph.





> *Extremal Value Theorem:* If $f(x)$ is continuous on the closed, finite interval $x \in [a,b]$, then $f(x)$ possesses at least one maximum point and one minimum point.

A proof valid for all possible continuous functions would require sophisticated Real Analysis concepts as in Math 320. To see that the theorem is not obvious, consider the function $y = g(x)$ graphed at right above. It is not continuous because the graph has a break, so the Theorem does not guarantee an absolute maximum; and indeed there is *no absolute maximum.* Instead, the function approaches $y = 3$ as $x \to 1^-$ (i.e. $x = 1-\delta$ for small $\delta > 0$), but it never actually reaches $y = 3$ because it suddenly drops to $g(1) = 2$. Thus, for any given output $g(c)$, we can find some slightly larger output $g(1-\delta) > g(c)$ for a tiny $\delta > 0$, so no $g(c)$ is largest. However, there is an absolute min point $x = 2$.

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*The Latin plurals of *maximum, minimum, extremum* are *maxima, minima, extrema.*

Local maxima and minima. A broader, but still useful, concept is that of a *local extremum*: this is a point where the graph has a hill or valley, but not necessarily the highest or lowest one.

> *Definition:* For a function $f(x)$ defined on an interval $x \in [a,b]$, a *local maximum* (or *relative maximum*) is a point $c \in [a,b]$ such that $f(c)$ is the largest output value for any input point *nearby* $x = c$.
>
> Formally, there is a small $\delta > 0$ such that $f(c) \geq f(x)$ for all $x \in [c{-}\delta, c{+}\delta]$; or $x \in [a, a{+}\delta]$ if $c = a$; or $x \in [b{-}\delta, b]$ if $c = b$.

Clearly, an absolute maximum must also be a local maximum. To illustrate, the function $f(x)$ in the figure at left above has four local maximum points, the two endpoints and the two hill tops; and it has three local minimum points, all valley bottoms. (The discontinuous $g(x)$ does not have any local maxima.)

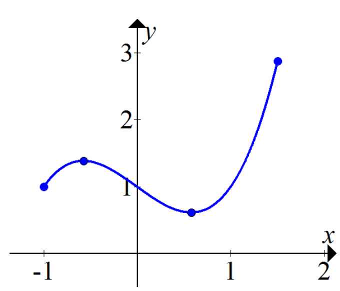
Vanishing derivatives. Calculus makes finding extremums surprisingly easy. We have already seen a real-world example at the end of Notes §2.7, where we asked for the maximum height of a ball whose height at time t is given by the ballistic function $s(t) = 5 + 79t - 16t^2$. The ball is highest at the moment when it passes from rising to falling and its velocity is zero: $t = c$ with $v(c) = 0$, where $v(t) = s'(t) = 79 - 32t$; and we solve to get $c = \frac{79}{32}$. That is, if $t = c$ is the maximum point of $s(t)$, then $s'(c) = 0$.

This also makes sense graphically. If $x = c$ is a local maximum of a function $f(x)$, then $(c, f(c))$ is most likely a hill-top of the graph $y = f(x)$, and the tangent line is horizontal at this point, having zero slope. But the tangent slope is the derivative $f'(c)$, so if $t = c$ is a local maximum, then $f'(c) = 0$. The same goes for a local minimum or valley-bottom.

> *First Derivative Theorem:* If $f(x)$ has a local maximum or minimum at $x = c$, which is not an endpoint of the interval of definition, and $f(x)$ is differentiable at this point, then $f'(c) = 0$.

Proof: Suppose $f(x)$ has a local minimum, so $f(x) \geq f(c)$ for $c{-}\delta \leq x \leq c{+}\delta$. For $c < x \leq c{+}\delta$, we have $\frac{f(x)-f(c)}{x-c} = \frac{+}{+} \geq 0$, so $f'(c) = \lim_{x\to c^+} \frac{f(x)-f(c)}{x-c} \geq 0$. For $c{-}\delta \leq x < c$, we have $\frac{f(x)-f(c)}{x-c} = \frac{+}{-} \leq 0$, so $f'(c) = \lim_{x\to c^-} \frac{f(x)-f(c)}{x-c} \leq 0$. Thus $f'(c)$ is both positive and negative, which can only mean $f'(c) = 0$.

EXAMPLE: We wish to find the maxima and minima, both local and absolute, of $f(x) = x^3 - x + 1$ on the interval $x \in [-1, \frac{3}{2}]$. Since $f(x)$ is continuous (by the Limit Laws), the Extremal Value Theorem guarantees there is at least one of each type of point.



Exactly where are the hill-top and the valley-bottom points? Since $f(x)$ is differentiable at every point, the First Derivative Theorem means that all local maximum and minimum points must be endpoints or solutions of $f'(x) = 0$, namely $3x^2 - 1 = 0$, or $x = \pm\frac{1}{\sqrt{3}} \approx \pm 0.58$. The graph shows that the local maxima are the hill-top $x = -\frac{1}{\sqrt{3}}$ and the right endpoint $x = \frac{3}{2}$, and the one with the larger output is the absolute maximum: $f(-\frac{1}{\sqrt{3}}) \approx 1.4 < f(\frac{3}{2}) \approx 2.9$, so the endpoint $x = \frac{3}{2}$ is the absolute maximum point. Similarly, the local minima are $x = -1$ and $x = \frac{1}{\sqrt{3}}$ with $f(-1) = 1 > f(\frac{1}{\sqrt{3}}) \approx 0.61$, so $x = \frac{1}{\sqrt{3}}$ has the smaller output and is the absolute minimum point.

Critical points. The above example illustrates the method for identifying all relevant candidates for the absolute maximum and minimum: the endpoints and the points where the derivative vanishes, and also possibly where the derivative is not defined because the graph has a corner or a discontinuity.

> *Definition:* For a function $f(x)$, a *critical point* (or *critical number*) is a point $x = c$ where the derivative is either zero or the function is not differentiable: $f'(c) = 0$ or undefined.

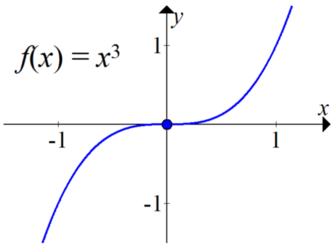
Method for absolute maxima and minima problems.

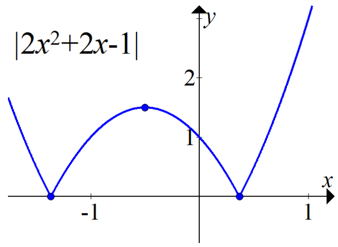
1. Given $f(x)$ on an interval $x \in [a, b]$, determine the critcal points (critical numbers) $x = c$ such that $f'(c) = 0$ or undefined. Be sure to consider only those $c \in [a, b]$, discarding any critical points outside the relevant interval.

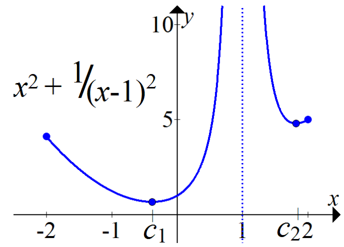
2. If $f(x)$ is continuous, find $f(x)$ for all critical points $x = c$ and for the endpoints $x = a, b$. Those points with the largest output are the absolute maximum points, and those with smallest values are the absolute minima.

3. If $f(x)$ has any discontinuity $x = c$, examine nearby $x \to c^+$ and $x \to c^-$ to see if the outputs $f(x)$ become larger or smaller than in Step 2.

Most functions are continuous and differentiable as in the previous example, and it is enough to perform Step 1 with $f'(c) = 0$, then Step 2. Below we illustrate some more complicated situations.







EXAMPLE: Not every critical point must be a local maximum or minimum. For $f(x) = x^3$, solving $f'(x) = 3x^2 = 0$ gives $x = 0$ as the unique critical point. The graph (above left) has a horizontal slope which is neither a hill-top nor a valley-bottom, but rather a *stationary point*, where the function pauses in its rise. This does not derail the Method, since it only gives an extra candidate for the absolute max/min, which will be discarded because its output value is neither largest nor smallest over any given interval.

EXAMPLE: Let $f(x) = |2x^2 + 2x - 1|$, with graph at center above. Recall that $\frac{d}{dx}|x| = \mathrm{sgn}(x) = \frac{|x|}{x}$, which is undefined when $x = 0$. By the Chain Rule:

$$f'(x) \;=\; \mathrm{sgn}(2x^2{+}2x{-}1)\cdot(2x^2{+}2x{-}\,1)' \;=\; \mathrm{sgn}(2x^2{+}2x{-}1)\cdot(4x{+}2).$$

Since sgn( ) is never zero, we have $f'(x) = 0$ when the second factor vanishes: $4x + 2 = 0$, or $x = \frac{1}{2}$.

But this is not the only critical point, since we must also consider when $f'(x)$ is undefined. This happens when the first factor $\mathrm{sgn}(2x^2 + 2x - 1)$ is undefined, namely when $2x^2 + 2x - 1 = 0$, or $x = \frac{1}{4}(-2\pm\sqrt{12})$ by the Quadratic Formula. These are the corners of the graph sitting on the x-axis: we must not skip them, since they are actually the absolute minimum points.

EXAMPLE: Let $f(x) = x^2 + \frac{1}{(x-1)^2}$on the interval $x \in [-2, 2]$ (above right):

$$f'(x) \;=\; \big(x^2\big)' + \big((x{-}1)^{-2}\big)' \;=\; 2x + (-2)(x{-}1)^{-3}(x{-}2)' \;=\; \frac{2(x^4{-}3x^3{+}3x^2{-}x{-}1)}{(x{-}1)^3}\,.$$

We have $f'(x) = 0$ when the numerator vanishes, $x^4{-}3x^3{+}3x^2{-}x{-}1 = 0$, and graphing this degree 4 polynomial gives approximate solutions $x = c_1 \approx -0.38$ and $c_2 \approx 1.82$ with $f(c_1) \approx 0.67$ and $f(c_2) \approx 4.80$. The endpoints $x = \pm 2$ give $f(-2) = \frac{37}{9} \approx 4.11$ and $f(2) = 5$. We might be tempted to take the largest of these outputs as the absolute maximum, but clearly none of these is the highest point of the graph.

We neglected to consider when $f'(x)$ is undefined: this is when the denominator $(x{-}1)^2 = 0$, or $x = 1$. This is a discontinuity, so by Step 3 we must consider not only $f(1)$, which is undefined, but also a small interval around $x = 1$. In fact, we have a vertical asymptote, and $\lim_{x\to 1^-} f(x) = \lim_{x\to 1^+} f(x) = \infty$.

That is, $f(x)$ can get as large as desired for x close enough to 1. There is *no absolute maximum*. However, the rising asymptotes do not affect the absolute minumum, which is still the smallest of the outputs at the other critical points, namely $f(c_1) \approx 0.67$. Since $f(x)$ is not continuous, the Extremal Value Theorem does not guanrantee an absolute max or min; in fact the max does not exist, but the min does.