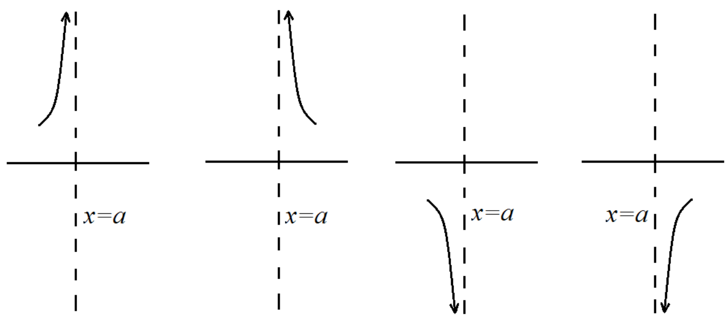
Vertical asymptotes. We say a curve has a line as an *asymptote* if, as the curve runs outward to infinity, it gets closer and closer to the line. "Closer and closer" reminds us of limits, and indeed we have seen that $x = a$ is a vertical asymptote of $y = f(x)$ whenever one of the following holds:



$$\lim_{x\to a^-} f(x) = \infty \quad \lim_{x\to a^+} f(x) = \infty \quad \lim_{x\to a^-} f(x) = -\infty \quad \lim_{x\to a^+} f(x) = -\infty.$$

As we saw in §1.5, ∞ has no meaning by itself; rather, the whole equation means that, as x gets closer to (but unequal to) a, the output $f(x)$ eventually becomes higher than any given bound B, such as $B = 100$ or 1000 or 1 billion. Similarly, a limit equals $-\infty$ when $f(x)$ becomes lower than $-B$ for any large B.

At the end of §3.3, we saw how a sign chart for $f'(x)$ can classify vertical asymptotes. We could do this with a sign chart for $f(x)$ itself, with no derivatives.

EXAMPLE: Let:

$$f(x) = \frac{x^2-6x+9}{x^3-6x^2+11x-6} = \frac{(x-3)^2}{(x-1)(x-2)(x-3)} = \frac{x-3}{(x-1)(x-2)}.$$

(To determine vertical asymptotes and intercepts, we always want $f(x)$ in factored* form.) In the original form, the denominator vanishes at $x = 3$, but we work with the cancelled form at right.

The function can only change its sign at points where $f(x) = 0$ (numerator $=$ 0) or $f(x)$ is not defined (denominator $= 0$), that is, $x = 1, 2, 3$. In the interval $x \in (-\infty, 1)$, the sign is given by a sample point like $f(0) = \frac{-2}{(-1)(-3)} = -\frac{2}{3} < 0$, so $f(x)$ is negative; and similarly for the other intervals.

x		1		2		3	
$f(x)$	$-$	$\pm\infty$	$+$	$\pm\infty$	$-$	0	$+$

Each time x passes one of the sign-change candidates $x = a$, a factor $(x-a)$ changes from negative to positive, and $f(x)$ does indeed change sign.

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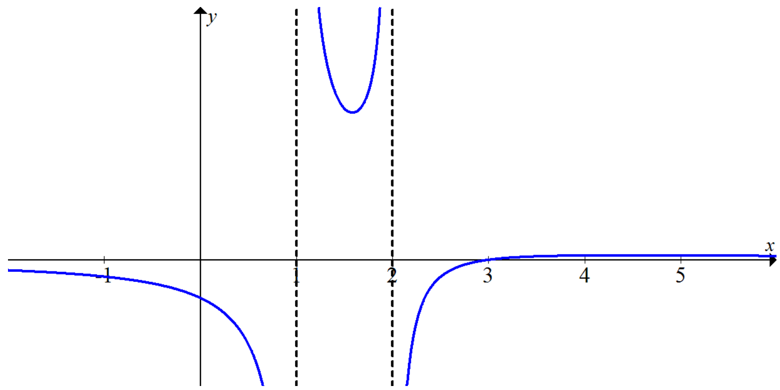
* To factor the bottom, we try linear factors $x - \frac{m}{n}$, where m is an integer factor of the constant coefficient 6, and n is an integer factor of the highest coefficient 1, so $n = \pm1, \pm2, \pm3, \pm6$ and $m = \pm1$. Trying $\frac{m}{n} = 1$, we find $x-1$ is a factor, since polynomial long division gives $x^3-6x^2+11x-6 = (x-1)(x^2-5x+6)$, and the quadratic is easy to factor. For a review of polynomial long division, see Khan Academy: www.khanacademy.org/math/algebra2/polynomial_and_rational/dividing_polynomials/v/polynomial-division.

Here $f(x) = \pm\infty$ just means the denominator vanishes and there is a vertical asymptote. The signs on each side of the asymptote show whether the graph shoots upward or downward: we have $\lim_{x\to 1^-} f(x) = -\infty$, $\lim_{x\to 1^+} f(x) = \infty$, $\lim_{x\to 2^-} f(x) = \infty$, $\lim_{x\to 2^+} f(x) = -\infty$.



Horzontal asymptotes. To understand the behavior of the graph over the left and right ends of the x-axis, we will need a new kind of limit in which x becomes larger and larger.

Definition:

- $\lim_{x\to\infty} f(x) = L$ means that $f(x)$ can be forced arbitrarily close to L, closer than any given $\varepsilon > 0$, by making $x > B$ for some B.
- $\lim_{x\to-\infty} f(x) = L$ means that $f(x)$ can be forced arbitrarily close to L, closer than any given $\varepsilon > 0$, by making $x < -B$ for some B.

Graphically, $\lim_{x\to\infty} f(x) = L$ means that toward the right of the x-axis, the graph $y = f(x)$ approaches the horizontal asymptote $y = L$; and similarly for $\lim_{x\to-\infty} f(x) = L$ toward the left. We can even have $\lim_{x\to\infty} f(x) = \infty$, which means that the graph goes off toward the upper right of the xy-plane in an unspecified way.

The most basic $x \to \infty$ limits are the power funcitons: for a positive real number power $p > 0$, we have:[†]

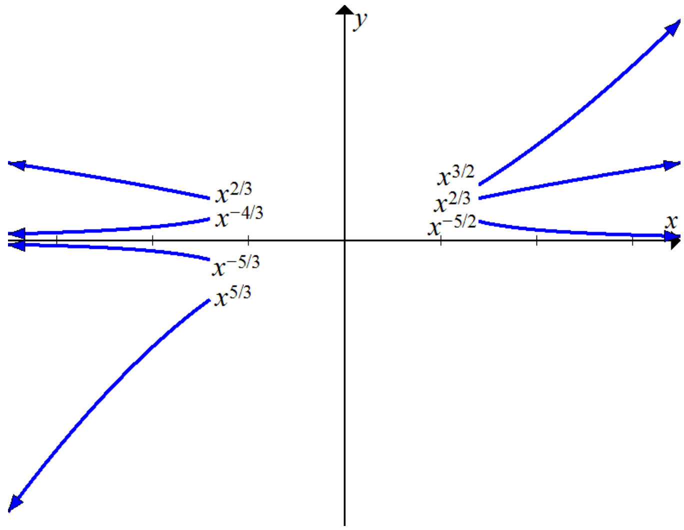
$$\lim_{x\to\infty} x^p = \infty, \qquad \lim_{x\to\infty} \frac{1}{x^p} = 0.$$

For $x \to -\infty$, consider the rational power $p = \frac{m}{n}$ where m, n are positive integers with n odd (perhaps $n = 1$); then:

$$\lim_{x\to-\infty} x^{m/n} = \begin{cases} \infty & \text{for } m \text{ even} \\ -\infty & \text{for } m \text{ odd,} \end{cases} \qquad \lim_{x\to-\infty} \frac{1}{x^{m/n}} = 0.$$

[†] *Proof:* For any large bound C, we can force $x^p > C$ if we take x so large that $x > C^{1/p}$. For any small error tolerance $\varepsilon > 0$, we can force $|\frac{1}{x^p} - 0| < \varepsilon$ if we take x so large that $x > (\frac{1}{\varepsilon})^{1/p}$.

For example:



Based on these, we can deduce the horizontal asymptotes for any rational function (quotient of polynomials).

EXAMPLE: Continuing $f(x) = \frac{x^2-6x+9}{x^3-6x^2+11x-6}$, does $y = f(x)$ have a horizontal asymptote? Informally, we can reason as follows. For large x (positive or negative), the value of x^2-6x+9 is relatively close to x^2: say for $x = 1000$, compare $x^2-6x+9 = 9{,}994{,}009$ and $x^2 = 1{,}000{,}000$. Thus we can approximate $x^2-6x+9 \approx x^2$, which we call the *highest term* of the polynomial. Also doing this for the denominator:

$$f(x) \;=\; \frac{x^2-6x+9}{x^3-6x^2+11x-6} \;\approx\; \frac{x^2}{x^3} \quad \text{for large } x.$$

Thus, $\lim\limits_{x\to\pm\infty} f(x) = \lim\limits_{x\to\pm\infty} \frac{x^2}{x^3} = \lim\limits_{x\to\pm\infty} \frac{1}{x} = 0$, and $y = f(x)$ has the horizontal asymptote $y = 0$ for $x \to \infty$ and $x \to -\infty$. In the graph we drew previously, the left and right ends do indeed approach the x-axis.

Formally, we can show this from the Limit Laws by dividing numerator and denominator by the highest term in the denominator:

$$\lim_{x\to\infty} f(x) \;=\; \lim_{x\to\infty} \frac{x^2-6x+9}{x^3-6x^2+11x-6} \;=\; \lim_{x\to\infty} \frac{x^2-6x+9}{x^3-6x^2+11x-6} \cdot \frac{\frac{1}{x^3}}{\frac{1}{x^3}}$$

$$= \lim_{x\to\infty} \frac{\frac{1}{x} - \frac{6}{x^2} + \frac{9}{x^3}}{1 - \frac{6}{x} + \frac{11}{x^2} - \frac{6}{x^3}} \;=\; \frac{0 - 6(0) + 9(0)}{1 - 6(0) + 11(0) - 6(0)} \;=\; 0.$$

Warning: The informal argument is the easiest way to understand these limits, but the formal argument (dividing by the highest term) might be required for full credit on a quiz or test.

EXAMPLE: For $f(x) = \frac{3x^2-x+9}{5x^2+2x-6}$, we take highest terms to get:

$$\lim_{x\to\infty} f(x) \;=\; \lim_{x\to\infty} \frac{3x^2-x+9}{5x^2+2x-6} \;=\; \lim_{x\to\infty} \frac{3x^2}{5x^2} \;=\; \frac{3}{5}.$$

Thus, $y = f(x)$ has horizontal asymptote $y = \frac{3}{5}$ toward the right. We similarly deduce $\lim\limits_{x\to-\infty} f(x) = \frac{3}{5}$, which means the same horizontal asymptote toward the left.

EXAMPLE: For

$$f(x) = \frac{x^2 + 3x^{7/2} - x^{-5}}{9x\sqrt{x} + 4x^2\sqrt{x}},$$

the terms in the denominator are $9xx^{1/2} = 9x^{3/2}$ and $4x^2x^{1/2} = 4x^{5/2}$, so the second is the highest term. Thus:

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{x^2 + 3x^{7/2} - x^{-5}}{9x\sqrt{x} + 4x^2\sqrt{x}} = \lim_{x\to\infty} \frac{3x^{7/2}}{4x^{5/2}}$$

$$= \lim_{x\to\infty} \frac{3}{4}x^{7/2-5/2} = \lim_{x\to\infty} \frac{3}{4}x = \infty,$$

which means $y = f(x)$ has no horizontal asymptote. However, the approximation $f(x) \approx \frac{3}{4}x$ implies that the right end of the graph looks like a line with slope $\frac{3}{4}$. (See slant asymptotes in §3.5.) This function is not defined for $x < 0$, so there is no left end.