Curve Sketching

Man vs machine. In this section, we learn methods of drawing graphs by hand. The computer can do this much better simply by plotting many points, so why bother with our piddly sketches? One reason is that calculus tells us the critical areas of the graph to look at: the computer might default to showing us some uninteresting region which misses the key features. Another reason is to be able to check the answer for yourself.

This is part of a great danger for anyone who uses mathematics. If you let the computer do the thinking, not just the calculating, you are ready to blindly accept any bizarre wrong answer. Then one typo error can escalate until your scientific paper has to be retracted, your company's expenses are ten times what you predicted, your bridge collapses, your rocket crashes. Don't let it happen! Before you rely on the computer's answer, you must check it against reasonable expectations, qualitatively through a story or sketch, and quantitatively by plotting sample points.

Slant asymptote. This means a diagonal line $y = mx + b$ which is approached by a graph $y = f(x)$.* For example, consider the function:

$$f(x) \;=\; \frac{x^3 - 6x^2 + 11x - 6}{2x^2 - 8x}\,.$$

Recall from §3.3 that to find the large-scale behavior of $f(x)$ as $x \to \pm\infty$, we can approximate by the highest term in numerator and denominator: $f(x) \approx \frac{x^3}{2x^2} = \frac{1}{2}x$. Thus, the right and left ends of the graph look like lines with slope $\frac{1}{2}$.

However, the graph does not actually approach the line $y = \frac{1}{2}x$: there is a vertical shift, $y = \frac{1}{2}x + b$. To approximate better, and find the exact slant asymptote of $y = f(x)$, we perform polynomial long division:

$$\begin{array}{rl}
 & \tfrac{1}{2}x - 1 \;\text{ rem }\; 3x - 6 \\
2x^2 - 8x \,\big) & x^3 - 6x^2 + 11x - 6 \\
 & -\underline{(x^3 - 4x)} \\
 & \quad -2x^2 + 11x - 6 \\
 & -\underline{(-2x^3 + \;8x)} \\
 & \qquad 3x - 6
\end{array}$$

This means:

$$x^3 - 6x^2 + 11x - 6 \;=\; (\tfrac{1}{2}x{-}1)(2x^2{-}8x) + (3x{-}6),$$

so that:

$$f(x) \;=\; \frac{(\frac{1}{2}x{-}1)(2x^2{-}8x) + (3x{-}6)}{2x^2{-}8x} \;=\; \tfrac{1}{2}x - 1 + \frac{3x{-}6}{2x^2{-}8x}.$$

That is, we have the approximation $f(x) \approx \frac{1}{2}x - 1$ with error term $\frac{3x-6}{2x^2-8x}$; but this term gets vanishingly small:

$$\lim_{x\to\pm\infty} \frac{3x{-}6}{2x^2{-}8x} \;=\; \lim_{x\to\pm\infty} \frac{3x}{2x^2} \;=\; 0.$$

Notes by Peter Magyar `magyar@math.msu.edu`

*That is, the difference between them vanishes as x gets large: $\lim_{x\to\pm\infty} f(x) - (mx{+}b) = 0$.

That is, as x gets larger and larger, the error term gets smaller and smaller, and the graph $y = f(x)$ gets closer and closer to the line $y = \frac{1}{2}x - 1$. This is what we mean by a slant asymptote.

For a general rational function $f(x) = \frac{g(x)}{h(x)}$, a quotient of polynomials $g(x), h(x)$, we use polynomial long division to get $g(x) = q(x)h(x) + r(x)$ for a quotient polynomial $q(x)$ and a remainder polynomial $r(x)$ having lower powers of x than $h(x)$. Thus:

$$f(x) \;=\; \frac{g(x)}{h(x)} \;=\; \frac{q(x)h(x) + r(x)}{h(x)} \;=\; q(x) + \frac{r(x)}{h(x)}\,.$$

Since the numerator $r(x)$ is smaller than the denominator $h(x)$, we have $\lim_{x\to\pm\infty} \frac{r(x)}{h(x)} = 0$, and $y = f(x)$ gets closer and closer to the curve $y = q(x)$. If $q(x) = mx + b$, then $y = mx + b$ is a slant asymptote; otherwise, $y = q(x)$ is an *asymptotic curve* of $y = f(x)$.

Rational function example. Referring to the Method for Graphing at the end of this section, we apply the steps to the above function:

$$f(x) \;=\; \frac{x^3 - 6x^2 + 11x - 6}{2x^2 - 8x} \;=\; \frac{(x-1)(x-2)(x-3)}{2x(x-4)}\,.$$

1. We have:

$$f'(x) \;=\; \frac{x^4 - 8x^3 + 13x^2 + 12x - 24}{2x^2(x-4)^2}\,, \qquad f''(x) \;=\; -\frac{3(x^3 - 6x^2 + 24x - 32)}{x^3(x-4)^3}.$$

   The domain of $f(x)$ is all real numbers $x \neq 1, 4$, namely:

$$x \in (-\infty, 1) \cup (1, 4) \cup (4, \infty).$$

2. There is no neat way to solve $f'(x) = 0$. If we computer-plot the numerator $x^4 - 8x^3 + 13x^2 + 12x - 24$, we see 4 roots, which we can name $x = a_1, \ldots, a_4$, approximately at:

$$a_1 \approx -1.26, \quad a_2 \approx 1.39, \quad a_3 \approx 2.61, \quad a_4 \approx 5.26\,.$$

   In §3.8, we will learn Newton's Method to zero in on such approximate solutions when algebraic ones are not available.

   The other critical points are solutions of $f'(x) =$ undefined, namely the roots of the denominator $x = 0$ and 4.

3. The sign chart is:

x		a_1 -1.26		0		a_2 1.39		a_3 2.61		4		a_4 5.26	
$f'(x)$	$+$	0	$-$	$\pm\infty$	$-$	0	$+$	0	$-$	$\pm\infty$	$-$	0	$+$
$f(x)$	$\nearrow$	-2.37	$\searrow$	$\pm\infty$	$\searrow$	-0.05	$\nearrow$	0.05	$\searrow$	$\pm\infty$	$\searrow$	2.37	$\nearrow$
		max		asym		min		max		asym		min	

4. To solve $f''(x) = 0$, a computer-plot of the numerator $x^3-6x^2+24x-32$ appears to show a single root at $x = 2$. To check this, we do polynomial long division $(x^3-6x^2+24x-32) \div (x-2)$ to show the numerator is $(x-2)(x^2-4x+16)$,

where the quadratic factor has no real number roots. Thus $x = 2$ is the only inflection point. (Solving $f''(x) = $ undef just gives the vertical asymptotes.)

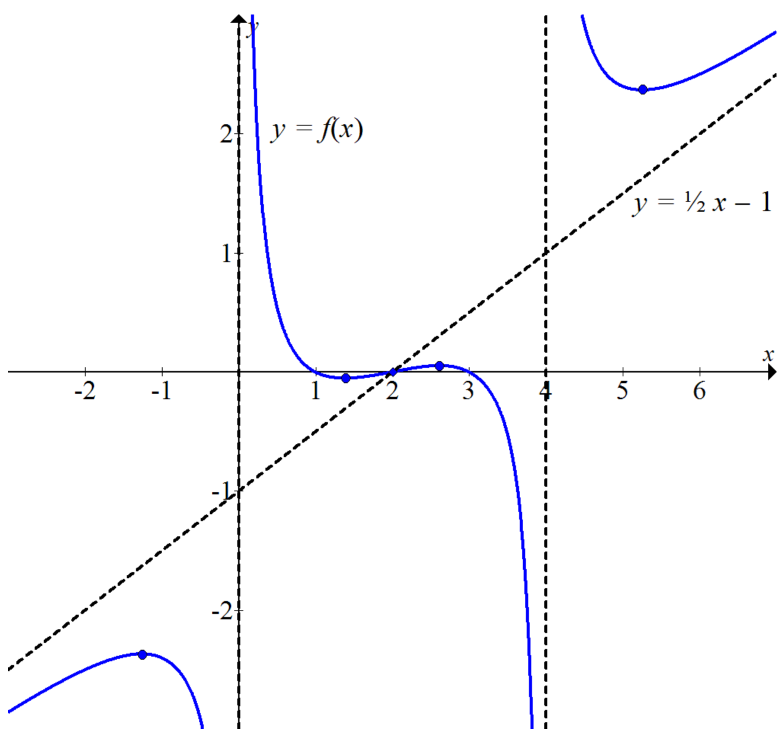
We do not need a sign chart for $f''(x)$, since the concavity seen in the picture below is forced by the known critical and inflection points: anything else would lead to more wiggles.

5. Solving $f(x) = 0$ gives the x-intercepts $x = 1, 2, 3$. There is no y-intercept, since the y-axis is a vertical asymptote.

6. The slant asymptote is $y = \frac{1}{2}x - 1$, computed at the beginning of this section.

7. This function does not have any of the standard symmetries in the Method. However, the graph reveals a 180° rotation symmetry around the point $(2, 0)$. This is equivalent to the equation $f(4-x) = -f(x)$, which can be shown from the factored form.

8. The graph is:

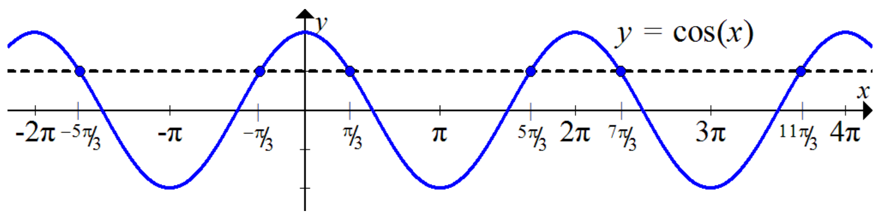


Trigonometric example. Apply the Method at the end to: $s(x) = x - 2\sin(x)$.

1. We have: $s'(x) = 1 - 2\cos(x)$ and $s''(x) = 2\sin(x)$. (See §2.4.)

   The domain is all real numbers: $x \in (\infty, \infty)$.

2. The critical points are solutions of $s'(x) = 1 - 2\cos(x) = 0$, so $\cos(x) = \frac{1}{2}$, and $x = 60° = \frac{1}{3}\pi$, or $x = 2\pi - \frac{1}{3}\pi = \frac{5}{3}\pi$, or any shift of these by a multiple of 2π:
$$x \;=\; \tfrac{1}{3}\pi \pm 2n\pi \quad \text{and} \quad \tfrac{5}{3}\pi \pm 2n\pi \quad \text{for } n = 0, 1, 2 \ldots.$$
   You can see this on the graph of $\cos(x)$:



   There are no points with $s'(x) =$ undefined.

3. The sign chart for $s'(x)$ is periodic (repeating):

x	$\cdots$		$-\frac{5}{3}\pi$		$-\frac{1}{3}\pi$		$\frac{1}{3}\pi$		$\frac{5}{3}\pi$		$\frac{7}{3}\pi$		$\frac{11}{3}\pi$		$\cdots$
$s'(x)$	$\cdots$	$-$	0	$+$	0	$-$	0	$+$	0	$-$	0	$+$	0	$-$	$\cdots$
$s(x)$	$\cdots$	$\searrow$	$-\frac{5}{3}\pi-\sqrt{3}$	$\nearrow$	$-\frac{1}{3}\pi+\sqrt{3}$	$\searrow$	$\frac{1}{3}\pi-\sqrt{3}$	$\nearrow$	$\frac{5}{3}\pi+\sqrt{3}$	$\searrow$	$\frac{7}{3}\pi-\sqrt{3}$	$\nearrow$	$\frac{11}{3}\pi+\sqrt{3}$	$\searrow$	$\cdots$
	$\cdots$		min		max		min		max		min		max		$\cdots$

4. The inflection points are solutions of $s''(x) = 2\sin(x) = 0$, or $x = n\pi$ for any integer n. Every multiple of π is an inflection point of $y = s(x)$.

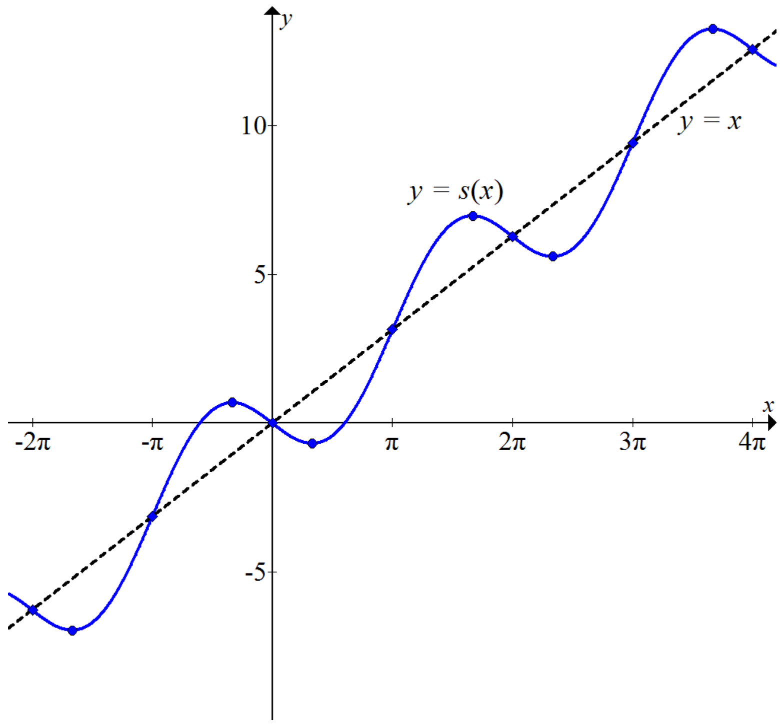
5. The point $(0, s(0)) = (0, 0)$ is an x and y-intercept. From the graph, we can see that there are two more x-intercepts, but we have no way to find them exactly. (We can approximate by Newton's Method §3.8.)

6. The large-scale behavior can be approximated by taking the highest or largest term: $s(x) \approx x$. However, the line $y = x$ is not a slant asymptote, because $s(x)$ oscillates above and below this line, without getting closer and closer.

7. This is an odd function, since:
$$s(-x) = (-x) - 2\sin(-x) = -x + 2\sin(x) = -s(x).$$
   Thus, the graph has 180° rotation symmetry around the origin.

   This function is not periodic, since $s(x+2\pi) \neq s(x)$, so the graph does not have the shift-sideways translation symmetry. However, we do have $s(x+2\pi) = s(x)+2\pi$, so the graph can be moved to itself by shifting sideways and up!

8. The graph is:



Method for Graphing (detailed)

1. Determine the derivatives $f'(x)$ and $f''(x)$ with Derivative Rules.
   Determine the domain of $f(x)$: for what x the formula makes sense.
2. Solve $f'(x) = 0$ and $f'(x) =$ undef to find the critical points.
3. Make a sign table for $f'(x)$ to classify each critical point $x = a$:

		$x<a$	$x=a$	$x>a$
local max $\frown$	$f'(x)$	$+$	0	$-$
	$f(x)$	$\nearrow$	$f(a)$	$\searrow$
local min $\smile$	$f'(x)$	$-$	0	$+$
	$f(x)$	$\searrow$	$f(a)$	$\nearrow$
local max $\wedge$	$f'(x)$	$+$	undef	$-$
	$f(x)$	$\nearrow$	$f(a)$	$\searrow$
local min $\vee$	$f'(x)$	$-$	undef	$+$
	$f(x)$	$\searrow$	$f(a)$	$\nearrow$
vert asymp $\nearrow\mid\nwarrow$	$f'(x)$	$+$	$\frac{1}{0}$	$-$
	$f(x)$	$+$	$\frac{1}{0}$	$+$
vert asymp $\nearrow\mid\swarrow$	$f'(x)$	$+$	$\frac{1}{0}$	$+$
	$f(x)$	$+$	$\frac{1}{0}$	$-$
vert asymp $\searrow\mid\nwarrow$	$f'(x)$	$-$	$\frac{1}{0}$	$-$
	$f(x)$	$-$	$\frac{1}{0}$	$+$
vert asymp $\searrow\mid\swarrow$	$f'(x)$	$-$	$\frac{1}{0}$	$+$
	$f(x)$	$-$	$\frac{1}{0}$	$-$

   Here $f(a)$ means the output value is defined; and $\frac{1}{0}$ means a zero denominator at $x = a$ produces $\pm\infty$ values. There other possibilities if $x = a$ is a discontinuity (see §1.8).

4. Solve $f''(x) = 0$ or undef to find inflection points $x = a$; we also require that $f'(a)$ exists and is a local max/min of $f'(x)$. Make a sign table for $f''(x)$ if concavity is needed: $f''(x) > 0$ means concave up (smiling), $f''(x) < 0$ means concave down (frowning).
5. Solve $f(x) = 0$ to find the x-intercepts; and compute the y-intercept $(0, f(0))$.
6. Find the behavior as $x \to \pm\infty$.
   - Approximate by highest terms on top and bottom to get $f(x) \approx cx^p$.
   - For a better approximation of a rational function $f(x) = \frac{g(x)}{h(x)}$, use polynomial long division to get $f(x) = q(x) + \frac{r(x)}{h(x)}$.
     If $f(x) = mx + b + \frac{r(x)}{h(x)}$, then $y = mx + b$ is a slant asymptote.
     In general, $y = f(x)$ asymtotically approaches $y = q(x)$ as $x \to \pm\infty$.
7. Check for symmetries: ways to move the graph onto itself.
   - Side-to-side reflection symmetry for even function $f(-x) = f(x)$.
     EXAMPLES: $x^2{+}3$, x^4, $\cos(x)$
   - $180°$ rotation symmetry for odd function $f(-x) = -f(x)$.
     EXAMPLES: $2x$, x^3, $\sin(x)$
   - Shift-sideways translation symmetry for periodic $f(x{+}c) = f(x)$.
     EXAMPLES: $\cos(x{+}2\pi) = \cos(x)$, $\tan(x{+}\pi) = \tan(x)$.
8. Draw all the above features on the graph.