More Uses for Integrals

Review. The integral $\int_a^b f(x)\,dx$ has four levels of meaning.

- Physical: Suppose y, z are physical variables determined as continuous functions of an independent variable x, so that $y = f(x)$ and $z = F(x)$. If y is the rate of change of z, i.e. $y = \frac{dz}{dx}$, then the integral of y is the cumulative total change of z between $x = a$ and $x = b$. In Leibnitz notation:*

  $$\int_a^b y\ dx \ = \ z\,\Big|_{x=a}^{x=b} \ .$$

  In Newton notation, $f(x) = F'(x)$, and:

  $$\int_a^b f(x)\,dx \ = \ F(b) - F(a)\,.$$

  This is the *Second Fundamental Theorem of Calculus.*

  If we know an initial value $F(a)$, we have $F(x) = F(a) + \int_a^x f(t)\,dt$, and:

  $$F'(x) \ = \ \frac{d}{dx}\int_a^x f(t)\,dt \ = \ f(x)\,.$$

  This is the *First Fundamental Theorem of Calculus.*

- Geometric: The integral is the area between the graph $y = f(x)$ and the interval $x \in [a, b]$, counting area above the x-axis as positive, area below the x-axis as negative.

- Numerical: To compute the integral, we divide $x \in [a, b]$ into n increments of width $\Delta x = \frac{b-a}{n}$, and choose sample points $x_1, \ldots, x_n$, one in each increment. Then the integral is approximated by a Riemann sum:

  $$\int_a^b f(x)\,dx \ \approx \ \sum_{i=1}^{n} f(x_i)\Delta x \ = \ f(x_1)\Delta x + \cdots + f(x_n)\Delta x\,.$$

  The exact integral is the limit of these approximations as $n \to \infty$, $\Delta x \to 0$.

- Algebraic: If, by reversing derivative formulas, we can find a formula for an antiderivative $F(x)$ with $F'(x) = f(x)$, then we can compute $\int_a^b f(x)\,dx = F(b) - F(a)$ by the Second Fundamental Theorem. Later, we will develop techniques for finding $F(x)$, such as the Substitution Method which reverses the Chain Rule.

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* In Leibnitz notation, a function is denoted by its output variable, such as $z = F(x)$, and its derivative function is $\frac{dz}{dx} = F'(x)$. A particular output value of the function is denoted: $z|_{x=a} = F(a)$; and the change in the value over an interval $x \in [a, b]$ is denoted: $z|_{x=a}^{x=b} = F(b) - F(a)$.

Indefinite integral notation. Since antiderivatives are so closely related to integrals by the Fundamental Theorems, we adopt the integral sign as a notation for the most general antiderivative of a function:

$$\int f(x)\,dx \;=\; F(x)+C \quad \text{for all } C.$$

Here $F(x)$ is a particular antiderivative: $F'(x) = f(x)$; and $F(x)+C$ means the family of all antiderivatives, one for every constant C (§3.9). This family is called the *indefinite integral*, with no specific limits of integration next to the $\int$ sign.

EXAMPLE: Since $\frac{d}{dx}(x^3) = 3x^2$ and $\frac{d}{dx}(\frac{1}{3}x^3) = x^2$, we have the indefinite integral:

$$\int x^2\,dx \;=\; \tfrac{1}{3}x^3 + C\,.$$

EXAMPLE: Suppose a car with position function $s(t)$ lurches forward with velocity $v(t) = 10t + 10\sin(\pi t)$ m/sec. How far does it travel from $t = 0$ to $t = 3$ sec? Making use of the antiderivative table in §3.9, we first find the indefinite integral:

$$\int 10t + 10\sin(\pi t)\,dt \;=\; 5t^2 - \tfrac{10}{\pi}\cos(\pi t) + C$$

Since velocity is the rate of change of position, the total change in position is the definite integral:

$$s(3)-s(1) \;=\; \int_0^3 10t + 10\sin(\pi t)\,dt \;=\; 5t^2 - \tfrac{10}{\pi}\cos(\pi t)\,\Big|_{t=0}^{t=3}$$

$$=\; \big(5(3^2) - \tfrac{10}{\pi}\cos(\pi{\cdot}3)\big) - \big(5(0^2) - \tfrac{10}{\pi}\cos(\pi{\cdot}0)\big) \;=\; 45 + \tfrac{20}{\pi} \;\approx\; 51.4 \text{ meters.}$$

Average of a function. In the numerical definition of integral above, we can rewrite the Riemann sum as:

$$\sum_{i=1}^{n} f(x_i)\,\Delta x \;=\; \sum_{i=1}^{n} f(x_i)\,\frac{b-a}{n} \;=\; (b-a)\,\frac{f(x_1)+\cdots+f(x_n)}{n}\,.$$

This is just the interval length $(b-a)$ times the average of the sample values $f(x_1),\ldots,f(x_n)$. The integral is the limit of this as $n\to\infty$, which becomes $(b-a)$ times the average of a more and more dense set of sample values:

$$\int_a^b f(x)\,dx \;=\; (b-a)\,\lim_{n\to\infty}\frac{f(x_1)+\cdots+f(x_n)}{n}\,.$$

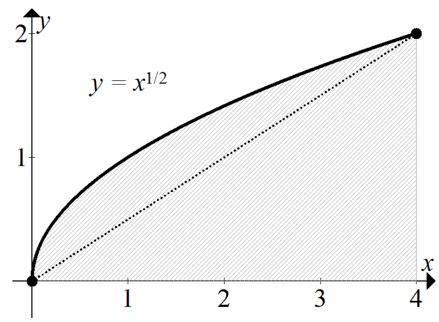
We define the average of $f(x)$ over all $x\in[a,b]$ to be the above limit. Hence:

$$\text{Average of } f(x) \text{ over } [a,b] \;=\; f_{\text{ave}} \;=\; \frac{1}{b-a}\int_a^b f(x)\,dx\,.$$

EXAMPLE: Find the average value of $f(x) = \sqrt{x}$ over $x \in [0,4]$. The indefinite integral is: $\int \sqrt{x}\, dx = \int x^{1/2}\, dx = \frac{2}{3}x^{3/2} + C$. The average is thus:

$$\frac{1}{4-0}\int_0^4 \sqrt{x}\, dx \;=\; \frac{1}{4}\,\frac{2}{3}x^{\frac{3}{2}}\Big|_{x=0}^{x=4} \;=\; \frac{4}{3} \;\approx\; 1.3\ .$$

That is, the function varies between $0 \leq \sqrt{x} \leq 2$ over the interval, but its average value is higher than the halfway point 1.0, because the graph bulges above the straight line from $(0,0)$ to $(4,2)$.



A geometric way to picture the average of a positive $f(x)$ over $[a,b]$ is to think of the area under the curve as a fluid. If we remove the curve and contain the fluid between the walls $x = a$ and $x = b$, then the level of the fluid is the average of the function. In the picture, the fluid under the straight line would fill the container between $x = 0$ and $x = 4$ to the midpoint level $y = 1$; but with extra fluid under $y = \sqrt{x}$ and above the line, the average of $f(x) = \sqrt{x}$ is higher: $y = \frac{4}{3}$.

EXAMPLE: We first discussed average velocity §1.4 as distance traveled divided by time elapsed, then defined instantaneous velocity as a limit of this, leading to the definition of derivative. Having made this definition, we can start with a position function $s(t)$ and its velocity function $s'(t) = v(t)$, and find over time interval $t \in [a,b]$:

$$v_{\text{ave}} \;=\; \tfrac{1}{b-a}\int_a^b v(t)\, dt \;=\; \tfrac{1}{b-a}\int_a^b s'(t)\, dt \;=\; \tfrac{s(b)-s(a)}{b-a}$$

by the Second Fundamental Theorem (§4.3). That is, the average of the velocity function is indeed the distance traveled $s(b)-s(a)$, divided by the time elapsed $b-a$.[†]

Mean Value Theorem for Integrals

> If $f(x)$ is continuous for $x \in [a,b]$, then there is some $c \in (a,b)$ where $f(c)$ equals the average of $f(x)$ over the interval: $f(c) = f_{\text{ave}} = \frac{1}{b-a}\int_a^b f(x)\, dx$.

Proof: Take $F(x) = \int_a^x f(x)\, dx$. The Mean Value Theorem for Derivatives (§3.2) says there is a value $c \in (a,b)$ where the tangent line for $F(x)$ is parallel to the secant line over the interval: $F'(c) = \frac{F(b)-F(a)}{b-a}$. By the First Fundamental Theorem, the left side is $F'(c) = f(c)$; and since $F(a) = 0$, the right side is $\frac{F(b)}{b-a} = \frac{1}{b-a}\int_a^b f(x)\, dx$, as desired.

In our example $f(x) = \sqrt{x}$, there is $c \in (0,4)$ with $f(c) = \sqrt{c} = f_{\text{ave}} = \frac{4}{3}$: i.e. $c = \frac{16}{9}$.

[†] This is not circular reasoning: rather, mathematics aims to turn physical intuitions into abstract definitions which we can reason with independent of intuition, checking whether our abstract objects behave as they intuitively should. If not, we need better definitions, which capture more of our intuition. If so, our non-intuitive mathematical predictions will probably hold in the physical world.