Slope in polar coordinates. We have seen that round, turny shapes are more simply described by polar $r \theta$-equations than by rectangular $x y$-equations. In this section, we use polar equations to compute geometric information.

Thus, we consider a polar curve $r=f(\theta)$ over $\theta \in[a, b]$. We split the interval $\theta \in[a, b]$ into a large number $n$ of increments, each of length $\Delta \theta=\frac{b-a}{n}$, with sample points $\theta_{1}, \ldots, \theta_{n}$. Here is a typical increment of the curve over $\theta \in\left[\theta_{i}, \theta_{i+1}\right]$, showing the corresponding increments in the coordinates:


Our first problem is to find the slope of this curve at a given $\theta$. It is not the derivative $f^{\prime}(\theta)=\frac{d r}{d \theta}$, which is the rate of change of the radius with respect to the angle. Rather, slope is the rate of change of $y=r \sin (\theta)=f(\theta) \sin (\theta)$ with respect to $x=r \cos (\theta)=$ $f(\theta) \cos (\theta)$. That is:

$$
\text { (slope at } \theta)=\frac{d y}{d x}=\frac{\frac{d y}{d \theta}}{\frac{d x}{d \theta}}=\frac{(f(\theta) \sin (\theta))^{\prime}}{(f(\theta) \cos (\theta))^{\prime}}=\frac{f^{\prime}(\theta) \sin (\theta)+f(\theta) \cos (\theta)}{f^{\prime}(\theta) \cos (\theta)-f(\theta) \sin (\theta)} \text {. }
$$

Area in polar coordinates. Assume $r=f(\theta) \geq 0$ for $\theta \in[a, b]$ to avoid complications with signs, and consider the region inside the curve, defined by $0 \leq r \leq f(\theta)$ for $\theta \in[a, b]$. Apply Slice Analysis (§5.2), splitting the area $A$ into $n$ thin wedges $\Delta A_{i}$ over $\left[\theta_{i}, \theta_{i+1}\right]$ :


We must compute the wedge area $\Delta A_{i}$. Since $\Delta \theta$ is tiny, the small curve segments are very close to straight lines, and $\Delta A_{i}$ is a very thin triangle. Neglecting the small piece with

[^0]radius larger that $r_{i}$, the slice $\Delta A_{i}$ is approximately an isosceles triangle with height $r_{i}$ and base $r_{i} \Delta \theta$.* Thus:
$$
\Delta A_{i} \approx \frac{1}{2}(\text { base }) \times(\text { height }) \approx \frac{1}{2}\left(r_{i} \Delta \theta\right) r_{i}=\frac{1}{2} r_{i}^{2} \Delta \theta
$$

The total area is the sum of these pieces, which is clearly a Riemann sum for an integral:

$$
A=\lim _{n \rightarrow \infty} \sum_{i=1}^{n} \Delta A_{i}=\lim _{n \rightarrow \infty} \sum_{i=1}^{n} \frac{1}{2} r_{i}^{2} \Delta \theta=\lim _{n \rightarrow \infty} \sum_{i=1}^{n} \frac{1}{2} f\left(\theta_{i}\right)^{2} \Delta \theta=\int_{a}^{b} \frac{1}{2} f(\theta)^{2} d \theta
$$

That is, the area inside a polar graph $r=f(\theta)$ is given by an integral formula, but a different integral from the area under a rectangular graph $y=f(x)$.

Arclength in polar coordinates. Finally, we compute the length of the curve $r=f(\theta)$ for $\theta \in[a, b]$. The length $L$ is a sum of $n$ increments $\Delta L_{i}$ :


Each increment $\Delta L_{i}$ is approximately a straight line segment. Next to it is the radial segment $\Delta r$ and the tiny circular arc with length $r_{i} \Delta \theta$, which is also approximately a straight line. We get an approximate right triangle with hypotentuse $\Delta L_{i}$ and legs $r_{i} \Delta \theta$ and $\Delta r$, so the Pythagorean Theorem gives:

$$
\Delta L_{i} \approx \sqrt{\left(r_{i} \Delta \theta\right)^{2}+(\Delta r)^{2}}=\sqrt{\frac{\left(r_{i} \Delta \theta\right)^{2}+(\Delta r)^{2}}{(\Delta \theta)^{2}}} \Delta \theta=\sqrt{r_{i}^{2}+\left(\frac{\Delta r}{\Delta \theta}\right)^{2}} \Delta \theta .
$$

Therefore the total arclength is:

$$
\begin{gathered}
L=\lim _{n \rightarrow \infty} \sum_{i=1}^{n} \Delta L_{i}=\lim _{n \rightarrow \infty} \sum_{i=1}^{n} \sqrt{r_{i}^{2}+\left(\frac{\Delta r}{\Delta \theta}\right)^{2}} \Delta \theta \\
=\lim _{n \rightarrow \infty} \sum_{i=1}^{n} \sqrt{f\left(\theta_{i}\right)^{2}+\left(\frac{\Delta f\left(\theta_{i}\right)}{\Delta \theta}\right)^{2}} \Delta \theta=\int_{a}^{b} \sqrt{f(\theta)^{2}+f^{\prime}(\theta)^{2}} d \theta .
\end{gathered}
$$

We could also deduce this from our previous parametric arclength formula (§10.3) by applying it to $(x(t), y(t))=(f(t) \cos (t), f(t) \sin (t))$.

[^1]Example: Area of intersections. Consider the polar curve $r=f(\theta)=1-\cos (\theta)$. We picture the abstract function $f$ by its rectangular graph in $\theta r$ parameter space (end $\S 10.3$ ):


The polar graph is a cardioid (heart-shape), which we draw along with the circle $r=\frac{1}{2}$.

prob: Find the area of the crescent-shaped region inside the cardioid \& outside the circle. We must first determine the intersection points of the two curves, where:

$$
r=1-\cos (\theta)=\frac{1}{2} \quad \Longrightarrow \quad \cos (\theta)=\frac{1}{2} \quad \Longrightarrow \quad \theta= \pm \frac{\pi}{3}+2 n \pi,
$$

where $n$ is any integer. Since the whole cardioid is traced by $\theta \in[0,2 \pi]$, we can take all intersection points in this range: $\theta=\frac{\pi}{3}$ and $\theta=-\frac{\pi}{3}+2 \pi=\frac{5 \pi}{3}$. Now we take the area inside the cardioid $r=f(\theta)=1-\cos (\theta)$, minus the area inside the circle $r=g(\theta)=\frac{1}{2}$ :

$$
\begin{aligned}
A & =\int_{a}^{b} \frac{1}{2} f(\theta)^{2}-\frac{1}{2} g(\theta)^{2} d \theta=\int_{\pi / 3}^{5 \pi / 3} \frac{1}{2}(1-\cos (\theta))^{2}-\frac{1}{2}\left(\frac{1}{2}\right)^{2} d \theta \\
& =\left[\frac{5}{8} \theta-\sin (\theta)+\frac{1}{8} \sin (2 \theta)\right]_{\theta=\pi / 3}^{\theta=5 \pi / 3}=\frac{7}{8} \sqrt{3}+\frac{5 \pi}{6} \approx 4.1 .
\end{aligned}
$$

To do the integral, expand $(1-\cos (\theta))^{2}$ and use $\cos ^{2}(\theta)=\frac{1}{2}+\frac{1}{2} \cos (2 \theta)$ (see $\S 7.2$ ).

Review example: Exponential spiral. Consider a snail-shell spiral curve which doubles in radius with each turn:


This is the polar graph $r=f(\theta)=c a^{\theta}=c e^{b \theta}$ of a general exponential function (§6.4). Assuming $f(0)=1, f(2 \pi)=2$, allows us to solve for $c=1$ and $a=2^{1 / 2 \pi}=e^{\ln (2) / 2 \pi}$ to get:

$$
r=2^{\theta / 2 \pi}=e^{b \theta} \quad \text { for } \quad b=\frac{\ln (2)}{2 \pi} .
$$

What is the length of this curve, from the point $(r, \theta)=(1,0)$ all the way to the center, that is, for $\theta \in(-\infty, 0]$ ? We have $f^{\prime}(\theta)=\left(e^{b \theta}\right)^{\prime}=b e^{b \theta}$, so the arclength formula gives:

$$
\begin{gathered}
L=\int_{-\infty}^{0} \sqrt{f(\theta)^{2}+f^{\prime}(\theta)^{2}} d \theta=\int_{-\infty}^{0} \sqrt{1+b^{2}} e^{b \theta} d \theta=\left[\frac{1}{b} \sqrt{1+b^{2}} e^{b \theta}\right]_{\theta=-\infty}^{\theta=0} \\
=\frac{1}{b} \sqrt{1+b^{2}}-\lim _{N \rightarrow \infty} \frac{1}{b} \sqrt{1+b^{2}} e^{-N}=\sqrt{1+\frac{4 \pi^{2}}{\ln ^{2}(2)}} \approx 9.12 .
\end{gathered}
$$

Or we could use geometry to show that these infinitely many turns have finite length. Let $L_{1}$ be the length of the first turn $\theta \in[-2 \pi, 0]$, and $L_{2}$ the length of the second turn, etc. The exponential spiral is scale invariant: each turn inward is the $\frac{1}{2}$ dilation of the previous turn, with half the length, so the total is a geomteric series $\sum_{n=1}^{\infty} c r^{n-1}=\frac{c}{1-r}(\S 11.2)$ :

$$
L=L_{1}+L_{2}+L_{3}+L_{4}+\cdots=L_{1}+\frac{1}{2} L_{1}+\frac{1}{2^{2}} L_{1}+\frac{1}{2^{3}} L_{1}+\cdots=\frac{L_{1}}{\left(1-\frac{1}{2}\right)}=2 L_{1} .
$$

Harmonic Spiral. From the above, we may say that the inward spiral $r=1 / 2^{\theta}$ has finite arclength as $\theta \rightarrow \infty$ because the geometric series $\sum_{n=1}^{\infty} \frac{1}{2^{n}}$ is convergent. Let us instead model an inward spiral on the divergent harmonic series $\sum \frac{1}{n}=\infty$, namely $r=\frac{1}{\theta}$ for $\theta \geq 1$. Then this should have infinite arclength:

$$
L=\int_{1}^{\infty} \sqrt{\left(\frac{1}{\theta}\right)^{2}+\left(-\frac{1}{\theta^{2}}\right)^{2}} d \theta=\int_{1}^{\infty} \frac{1}{\theta} \sqrt{1+\frac{1}{\theta^{2}}} d \theta=\int_{1}^{\infty} \frac{\sqrt{1+\theta^{2}}}{\theta^{2}} d \theta
$$

Since the integrand (in the second integral) is clearly positive and decreasing, the Integral Test (§11.3) tells us that this diverges whenever the corresponding series diverges, namely :

$$
\sum_{n=1}^{\infty} \frac{1}{n} \sqrt{1+\frac{1}{n^{2}}} \approx \sum_{n=1}^{\infty} \frac{1}{n}=\infty \text { (divergent) }
$$

This can be justified by the Direct Comparison Test (§11.4), since $\frac{1}{n} \sqrt{1+\frac{1}{n^{2}}}>\frac{1}{n}$; or the Limit Comparison Test: the ratio $\frac{a_{n}}{b_{n}}=\sqrt{1+\frac{1}{n^{2}}} \rightarrow 1$, so they have the same divergence.

Alternatively, we can directly integrate, switching the variable to $\int \frac{\sqrt{1+x^{2}}}{x^{2}} d x$. Since we have $\sqrt{1+x^{2}}$ (§7.3), we try the trig substitution $x=\tan (t), \sqrt{1+x^{2}}=\sec (t), d x=\sec ^{2}(t) d t$ :

$$
\int \frac{\sqrt{1+x^{2}}}{x^{2}} d x=\int \frac{\sec (t)}{\tan ^{2}(t)} \sec ^{2}(t) d t=\int \frac{1}{\sin ^{2}(t) \cos (t)} d t=\int \frac{1}{\sin ^{2}(t) \cos ^{2}(t)} \cos (t) d t
$$

As in $\S 7.2$, we do the substitution $u=\sin (t), 1-u^{2}=\cos ^{2}(t), d u=\cos (t) d t$ :

$$
\int \frac{1}{\sin ^{2}(t) \cos ^{2}(t)} \cos (t) d t=\int \frac{1}{u^{2}\left(1-u^{2}\right)} d u=\int \frac{A}{u^{2}}+\frac{B}{u}+\frac{C}{1+u}+\frac{D}{1-u} d u .
$$

Here the result is a rational function, expanded by partial fracitions (§7.4). Then:

$$
\frac{1}{u^{2}\left(1-u^{2}\right)}=\frac{1-u^{2}}{u^{2}\left(1-u^{2}\right)}+\frac{u^{2}}{u^{2}\left(1-u^{2}\right)}=\frac{1}{u^{2}}+\frac{1}{(1-u)(1+u)}=\frac{1}{u^{2}}+\frac{C}{1+u}+\frac{D}{1-u} .
$$

We can find the remaining coefficients by clearing denominators to get

$$
1=C(1-u)+D(1+u) .
$$

Substituting $u=1$ gives $D=\frac{1}{2}$, and $u=-1$ gives $C=\frac{1}{2}$. The integral becomes:

$$
\int \frac{1}{u^{2}}+\frac{\frac{1}{2}}{1+u}+\frac{\frac{1}{2}}{1-u} d u=-\frac{1}{u}+\frac{1}{2} \ln (1+u)-\frac{1}{2} \ln (1-u)=-\frac{1}{u}+\frac{1}{2} \ln \left(\frac{1+u}{1-u}\right) .
$$

Now we need to restore the original variable $x=\tan (t)$ from $u=\sin (t)$. The standard triangle for $x=\tan (t)$ implies $u=\sin (t)=\frac{x}{\sqrt{1+x^{2}}}$. After simplification, the final answer is:

$$
\int \frac{\sqrt{1+x^{2}}}{x^{2}} d x=-\frac{\sqrt{1+x^{2}}}{x}+\frac{1}{2} \ln \left(\frac{\sqrt{1+x^{2}}+x}{\sqrt{1+x^{2}}-x}\right) .
$$

Therefore the total arclength is:

$$
L=\int_{1}^{\infty} \frac{\sqrt{1+x^{2}}}{x^{2}} d x=\lim _{x \rightarrow \infty} \frac{1}{2} \ln \left(\frac{\sqrt{1+x^{2}}+x}{\sqrt{1+x^{2}}-x}\right)+K
$$

for a constant $K$. In the fraction $\frac{\sqrt{1+x^{2}}+x}{\sqrt{1+x^{2}}-x}$, we cannot use L'Hopital's Rule ( $\S 6.8$ ), since the numerator clearly approaches $\infty$, but the denominator does not. Substitute $x=1 / z$ :

$$
\lim _{x \rightarrow \infty} \sqrt{1+x^{2}}-x=\lim _{z \rightarrow 0^{+}} \sqrt{1+\frac{1}{z^{2}}}-\frac{1}{z}=\lim _{z \rightarrow 0^{+}} \frac{\sqrt{z^{2}+1}-1}{z} .
$$

This is a $\frac{0}{0}$ limit, so we can apply L'Hopital to get:

$$
\lim _{x \rightarrow \infty} \sqrt{1+x^{2}}-x=\lim _{z \rightarrow 0^{+}} \frac{\left(\sqrt{z^{2}+1}-1\right)^{\prime}}{(z)^{\prime}}=\lim _{z \rightarrow 0^{+}} \frac{\frac{z}{\sqrt{z^{2}+1}}}{1}=0^{+} .
$$

After all that, we obtain the expected arclength:

$$
L=\frac{1}{2} \ln \left(\frac{\infty}{0^{+}}\right)+K=\frac{1}{2} \ln (\infty)+K=\infty .
$$


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[^1]:    * On a circle of radius $r$, and arc of $\theta$ radians has length $r \theta$ : this is the definition of radian measure.

