Math 881 HW 2/10 Solutions Spring 2006

We have defined several measures of graph connectivity which are less sensitive to small-scale phenomena than the classical connectivity $\kappa_V(G)$ or $\kappa_E(G)$, and which better capture the features we seek in a robustly connected network. Let G = (V,E) be a graph with n vertices. Let $S \subset V$ be any set of vertices, $S \neq \emptyset, V$, and let $\overline{S} = V - S$ be the complement.

- *Vertex Cheeger constant (restricted):*

$$g'(G) := \min_{|S| \leq n/2} \frac{|\delta(S)|}{|S|} .$$

  Here the vertex boundary $\delta(S) := N(S) - S$ is the set of neighbors of S not including S itself. This is the most important measure.

- *Vertex Cheeger constant (unrestricted):*

$$g(G) := \min_{S} \frac{|\delta(S)|}{\min(|S|, |\overline{S}|)} .$$

- *Edge Cheeger constant:*

$$\begin{aligned} h(G) &:= \min_{|S| \leq n/2} \frac{|E(S, \overline{S})|}{|S|} \\ &= \min_{S} \frac{|E(S, \overline{S})|}{\min(|S|, |\overline{S}|)} . \end{aligned}$$

  Here the edge boundary $E(S, \overline{S})$ is the set of edges between S and $\overline{S}$. In this case there is no difference between the restricted and unrestricted version, since $E(\overline{S}, S) = E(S, \overline{S})$.

1. We compute these measures for some of our favorite graphs. In each case $g(G) = g'(G)$, except for the complete graph K_n with n odd, in which case $g(G) = 1$ and $g'(G) = \frac{n+1}{n-1} = 1 + \frac{2}{n-1}$.

Graph	S for g	$g(G)$	S for h	$h(G)$	$\kappa_V(G)$	$\kappa_E(G)$
Cycle C_n						
n even	$\frac{n}{2}$ interval	$\frac{4}{n}$	$\frac{n}{2}$ interval	$\frac{4}{n}$	2	2
n odd	$\frac{n-1}{2}$ interval	$\frac{4}{n-1}$	$\frac{n-1}{2}$ interval	$\frac{4}{n-1}$	2	2
Complete K_n						
n even	$\lvert S\rvert = \frac{n}{2}$	1	$\lvert S\rvert = \frac{n}{2}$	$\frac{n}{2}$	$n-1$	$n-1$
n odd	$\lvert S\rvert = \frac{n+1}{2}$	1	$\lvert S\rvert = \frac{n-1}{2}$	$\frac{n+1}{2}$	$n-1$	$n-1$
Tetrahedron K_4		1.0		2.0	3	3
Octahedron	triangle	$\frac{3}{3} = 1.0$	triangle	$\frac{6}{3} = 2.0$	4	4
Icosahedron	$N(v)$	$\frac{5}{6} = 0.83$	$N(v)$	$\frac{10}{6} = 1.67$	5	5
Cube	$N(v)$	$\frac{3}{4} = 0.75$	square	$\frac{4}{4} = 1.0$	3	3
Dodecahedron	$N(\text{pentagon})$	$\frac{5}{10} = 0.5$	$N(N(v))$	$\frac{6}{10} = 0.6$	3	3

Here $N(S)$ means S together with its neighbors. Note that $\kappa_V(G) = \kappa_E(G) = d(v)$ in each case, so these are governed only by the degree at each vertex. However $g(G)$ and $h(G)$ suggest that globally, the icosahedron is more connected than the cube, which is more connected that the dodecahedron.

2. Consider the wrapped grid graph $G_m = C_m \times C_m$, consisting of m^2 vertices (i, j) mod m having the 4 neighbors $(i\pm1, j), (i, j\pm1)$. As $m \to \infty$, the computation of the minimizing set S with

$$g'(m) := g'(G_m) = \frac{|\delta(S)|}{|S|}$$

approximates the *isomperimetric problem* in plane geometry. That is, find the plane region $R \subset \mathbf{R}^2$ with a fixed area $A = A(R)$ that has the minimal perimeter $P(R)$.

For a given shape R, dilating by a scalar $t \geq 0$ multiplies the perimeter by t and the area by t^2, so we have a formlua:

$$P(tR) = i(R)\, A(tR)^{1/2}$$

where $i(R)$ is the *isoperimetric constant* of R. The optimal shape will have the smallest isoperimetric constant.

If the boundary curve of R is parametrized by $(x(t), y(t))$ for $a \leq t \leq b$, the usual perimeter is:

$$P(R) := \int_a^b \sqrt{x'(t)^2 + y'(t)^2}\, dt\,.$$

The optimal shape in this case is $R = C$, the circle defined by: $\sqrt{x^2 + y^2} \leq r$ for $r = \sqrt{A/\pi}$. This gives:

$$P(C) = 2\sqrt{\pi}\ A(R)^{1/2} \cong 3.5\ A(R)^{1/2}\,.$$

However, in our discrete case, the vertex boundary approaches a different measure of perimeter from the usual geometric one:

$$P_V(R) := \int_a^b \max(|x'(t)|, |y'(t)|)\, dt\,.$$

In this case, the minimum P_V for a given area is attained by $R = D$ a diamond (rotated square) defined by: $|x| + |y| \leq r$ for $r = \sqrt{A/2}$. This gives:

$$P_V(D) = 2\sqrt{2}\ A(D)^{1/2} \cong 2.8\ A(D)^{1/2}\,;$$

whereas the circle gives:

$$P_V(C) = 4\sqrt{\tfrac{2}{\pi}}\ A(C)^{1/2} \cong 3.2\ A(C)^{1/2}\,.$$

That is, $i_V(C) \cong 4.2$, about 10% below $i(C) \cong 4.5$

To compute $g'(m)$ we set $A = n/2 = m^2/2$, so that for the optimal shape $R = D$,

$$g'(m) \cong \frac{P_V(D)}{A(D)} = \frac{i_V(D)}{(m^2/2)^{1/2}} = \frac{\sqrt{2}\ i_V(D)}{m} = 4\, m^{-1}\,.$$

Using the non-optimal $R = C$ gives:

$$g'(m) \leq \frac{\sqrt{2}\ i_V(C)}{m} = \frac{8}{\sqrt{\pi}}\ m^{-1} \cong 4.5\, m^{-1}\,.$$