NOTES ON MINORS AND PLUCKER COORDINATES

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We list for reference several identities involving Plucker coordinates of subspaces. Cf. Fulton's *Young Tableaux*, §8.1, §9.1. Note that Weyman has given conceptual proofs of most of these results by decomposing Schur functors into irreducibles, Cauchy's identity, etc.

1 Minors

For a whole number n, denote $[n] := \{1, 2, \ldots, n\}$. For an $n \times l$ matrix $X = (x_{ij}) \in M_{n\times l}$, and subsets $I \subset [n]$ and $J \subset [l]$ of equal size, we have the submatrix $X_{IJ} = (x_{ij})_{i\in I,\, j\in J}$ and the minor $\Delta_I^J(M) := \det(M_{IJ})$.

Proposition 1.1 For any matrices X, Y which can be multiplied, we have:

$$\Delta_I^J(XY) = \sum_K \Delta_I^K(X)\Delta_K^J(Y).$$

This generalizes the multiplicativity of the determinant.

Proposition 1.2 (Sylvester) Let $X = [x^1, \ldots, x^n],\ Y = [y^1, \ldots, y^n]$ be two $n\times n$ matrices of column vectors. Fix a set $\{j_1 < \ldots < j_k\} \subset [n]$. Then we have:

$$\det X \det Y = \sum_{i_1 < \cdots < i_k} \det[x^1, \ldots, y^{j_1}, \ldots, y^{j_k}, \ldots, x^n]\ \det[y^1, \ldots, x^{i_1}, \ldots, x^{i_k}, \ldots, y^n]$$

where $[x^1, \ldots, y^{j_1}, \ldots, y^{j_k}, \ldots, x^n]$ denotes the matrix X with columns $x^{i_1}, \ldots, x^{i_k}$ replaced by the columns $y^{j_1}, \ldots, y^{j_k}$ of Y, and vice versa for the other factor.

Proposition 1.3 (Laplace) Let X be an $l\times m$ matrix, $I \subset [l],\ J \subset [m]$; and fix a partition $I = I' \sqcup I''$. Then we have:

$$\Delta_I^J(X) = \sum_{J = J' \sqcup J''} (-1)^{\mathrm{inv}(I'|I'') + \mathrm{inv}(J'|J'')} \Delta_{I'}^{J'}(X)\Delta_{I''}^{J''}(X),$$

where the sum runs over all partitions J of appropriate size, and $\mathrm{inv}(A|B) = \#\{(a, b) \in A\times B \mid a > b\}$. The same formula holds for a fixed partition of J, summed over all partitions of I.

This formula follows easily from Prop. 1.2, and it generalizes the usual expansion of a determinant by minors. Note that the symmetry of the formula with respect to switching I' and I'' is guaranteed by the relation: $\mathrm{inv}(A|B) + \mathrm{inv}(B|A) = \#A \cdot \#B$.

2 Plucker embedding

All our spaces will lie inside an n-dimensional vector space E with standard basis $e_1, \ldots, e_n$. We will often think of an l-dimensional subspace $V \subset E$ as an $n \times l$ matrix $[v^1, \ldots, v^l]$ of column vectors $v^j = (v^j_1, \ldots, v^j_n)^T$ which span V. This matrix is not unique, but we do have a well-defined Plucker embedding

$$\begin{array}{rcl} \mathrm{Gr}(l,E) & \to & \mathbf{P}(\wedge^l E) \\ V=[v^1,\ldots,v^l] & \mapsto & v^1\wedge\cdots\wedge v^l. \end{array}$$

The projective coordinate corresponding to the basis vector $e_{i_1}\wedge\cdots\wedge e_{i_k}$ is:

$$\Delta_I(V) := \Delta_I^{[l]}[v_1,\ldots,v_l],$$

where $I=\{i_1,\ldots,i_k\}$. The Plucker coordinates are defined up to simultaneous multiplication by an arbitrary non-zero scalar.

Proposition 2.1 Let $V\subset E$ a subspace and $A{\cdot}V$ its translation by a matrix $A\in GL(E)$. Then $\Delta_I(A{\cdot}V)=\sum_{J\subset[n]}\Delta_I^J(A)\,\Delta_J(V)$.

This follows immediately from Prop. 1.1.

Let us also define the Plucker coordinate of an ordered list $(i_1,\ldots,i_l)$ by the relations:

$$\Delta_{(i_1,\ldots,i_j,i_{j+1},\ldots,i_l)} = -\Delta_{(i_1,\ldots,i_{j+1},i_j,\ldots,i_l)}$$

and

$$\Delta_{(i_1,\ldots i_l)} = \Delta_{\{i_1,\ldots i_l\}} \quad \text{if} \quad i_1<\cdots<i_l\,.$$

In particular, $\Delta_{(i_1,\ldots,i_l)}=0$ if any subscript appears twice.

Proposition 2.2 The image of the Plucker embedding $\mathrm{Gr}(l,E)\to\mathbf{P}(\wedge^l E)$ is a projective variety whose vanishing ideal is generated by the following polynomials in the Plucker coordinates. For any $k\le l$, and any $(p_1,\ldots,p_l)$, $(q_1,\ldots,q_l)$, lists of distinct elements in $[n]$, we have the quadratic polynomial (Plucker relation):

$$\Delta_{(p_1,\ldots,p_l)}\Delta_{(q_1,\ldots,q_m)} - \sum_{i_1<\cdots<i_k}\Delta_{(p_1,\ldots,q_1,\ldots,q_k,\ldots,p_l)}\Delta_{(p_{i_1},\ldots,p_{i_k},q_{k+1},\ldots,q_l)},$$

where $(p_1,\ldots,q_1,\ldots,q_k,\ldots,p_l)$ denotes the list $(p_1,\ldots,p_l)$ with the entries $p_{i_1},\ldots,p_{i_k}$ replaced by $q_1,\ldots,q_k$, and vice versa for the other factor.

Proposition 2.3 Let $V,W\subset E$ be subspaces with $\dim V=l<\dim W=m$. Then $V\subset W$ if and only if all the following polynomials vanish. For any $k\le l$, and any $(p_1,\ldots,p_l)$, $(q_1,\ldots,q_m)$, lists of distinct elements in $[n]$, we have the polynomial:

$$\Delta_{(p_1,\ldots,p_l)}\Delta_{(q_1,\ldots,q_l)} - \sum_{i_1<\cdots<i_k}\Delta_{(p_1,\ldots,q_1,\ldots,q_k,\ldots,p_l)}\Delta_{(p_{i_1},\ldots,p_{i_k},q_{k+1},\ldots,q_m)},$$

where we use the same notation as before.

3 Orthogonals and intersections

If $r\le n$, and X is a generic matrix in $M_{r\times n}$, then $U=\mathrm{Ker}\,X$, the kernel of the map $u\mapsto X\cdot u$, is a subspace of dimension $n-r$ in n-space. If we think of the row vectors

$v_1, \dots, v_r$ of X as elements of E^*, spanning a space $V = \mathrm{Span}(v_1, \dots, v_r) \subset E^*$, then $U = V^\perp = \{u \in E \mid \langle v, u\rangle = 0 \ \ \forall\, v \in V\}$.

Proposition 3.1 The Plucker coordinates of $U = \mathrm{Ker}\, X = V^\perp$ are given by:

$$\Delta_I(U) = (-1)^{\ell(I)} \Delta^{\bar I}_{[n]}(X) = (-1)^{\ell(I)} \Delta^{\bar I}(V),$$

where $I \sqcup \bar I = [n]$, and for $I = \{i_1 < \cdots < i_{n-r}\}$, we define $\ell(I) := \mathrm{inv}(I|\bar I) = \sum_{j=1}^{n-r}(i_j - j)$.

Proposition 3.2 Let $V, W \subset E$ be subspaces with $\dim V = l,\ \dim W = m,\ \dim V \cap W = k = l + m - n \geq 0$. Then the Plucker coordinates of the intersection are given by:

$$\Delta_K(V \cap W) = \sum_{\substack{L,M \\ K \sqcup L \sqcup M = [n]}} (-1)^{\mathrm{inv}(L|M)}\, \Delta_{K\cup L}(V)\, \Delta_{K \cup M}(W)\,,$$

where the sum is over all partitions of $[n] \setminus K$ into disjoint subsets L, M with $\#(K \cup L) = l$, $\#(K \cup M) = m$.

Proof. Consider matrices $V = [v_1, \dots, v_l]$, $W = [w_1, \dots, w_m]$, and $[V, W] = [v_1, \dots, v_l, w_1, \dots w_m] \in M_{n \times (l+m)}$. Let

$$Y = \begin{pmatrix} Y_V \\ Y_W \end{pmatrix} = \mathrm{Ker}[V, W] \in M_{(l+m)\times k}$$

Then $V \cap W = V \cdot Y_V \in M_{k\times n}$. Thus, using Props. 1.1, 1.3, and 3.1, we have:

$$\begin{aligned}
\Delta_K(V \cap W) &= \Delta^{[k]}_K(V \cdot Y_V) \\
&= \sum_{J \subset [l]} \Delta^J_K(V)\, \Delta^{[k]}_J(Y_V) \\
&= \sum_{J \subset [l]} (-1)^{\ell(J)} \Delta^J_K(V)\, \Delta^{\bar J \sqcup [l+1, l+m]}_{[n]}([V, W]) \\
&= \sum_{J \subset [l]} (-1)^{\ell(J)} \Delta^J_K(V) \\
&\qquad \sum_{K' \sqcup K'' = [n]} (-1)^{\mathrm{inv}(K'|K'') + \mathrm{inv}(\bar J|[l+1, l+m])} \\
&\qquad\qquad \Delta^{\bar J}_{K'}([V, W])\, \Delta^{[l+1, l+m]}_{K''}([V, W]) \\
&= \sum_{\substack{J \subset [l] \\ K' \sqcup K'' = [n]}} (-1)^{\ell(J) + \ell(K')} \Delta^J_K(V)\, \Delta^{\bar J}_{K'}(V)\, \Delta^{[m]}_{K''}(W) \\
&= \sum_{\substack{K \cap K' = \emptyset \\ K' \sqcup K'' = [n]}} (-1)^{\ell(K') + \mathrm{inv}(K|K')} \Delta^{[l]}_{K \cup K'}(V)\, \Delta^{[m]}_{K''}(W) \\
&= (-1)^{k(n-m)} \sum_{K \sqcup L \sqcup M = [n]} (-1)^{\mathrm{inv}(L|M)} \Delta_{K \cup L}(V)\, \Delta_{K \cup M}(W)\,.
\end{aligned}$$

We may remove the sign factor on the left, since it is independent of K. QED