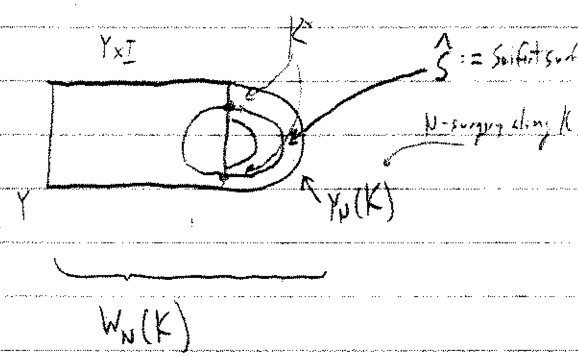
2/15/11 Matt Hedden HFH

Class Tomorrow: A304, 4:10-5:30

$\chi(\widehat{HFK}) = \Delta_K$.

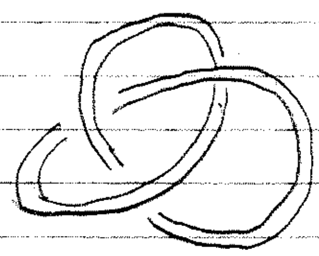
Last Time:



$\hat{S}$:= Seifert surface for K, capped off by core of the 4-dim'l 2-handle.

$(I = [0,1])$

$\nu(K) \cong S^1 \times D^2$



Attaching a 4-dim'l 2-handle

i.e. $D^2 \times D^2$ attached to $Y \times I$ by

$$Y \times I \sqcup D^2 \times D^2$$

$$\underset{\partial(Y\times I)}{\mathrm{nbhd}(K) \subseteq Y \times \{1\}} \xleftrightarrow[\text{framing}]{} S^1 \times D^2 \subseteq \partial(D^2 \times D^2)$$

$$\underset{Y\times\{1\}}{K} = \partial F^2 \qquad \underset{\text{core}}{D^2 \times \{0\}} \subseteq D^2 \times D^2$$

$$F \cup \partial(D^2 \times \{0\}) =: \hat{S}.$$

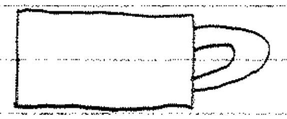
Exercise: Show $[\hat{S}] \cdot [\hat{S}] = N$

$\hat{H}_2(W_N(K); \mathbb{Z})$

Def. Given $s \in Spin^c(Y)$, ~~let~~ $s_i \in Spin^c(Y_N(K))$, $i \in \mathbb{Z}/N\mathbb{Z}$ be the unique $Spin^c$-structure s.t.

(1) s_i extends over $W_N(K)$ to a $Spin^c$-structure t_i restricting to s.

(2) $\langle c_1(t_i), [\hat{S}] \rangle + N \equiv 2i \pmod{2N}$)



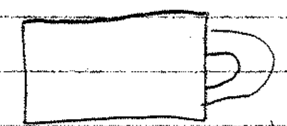
$-Y$ $\quad Y_N(K)$

$W_N(K)$

$$\partial W_N(K) = -Y \sqcup Y_N(K)$$

If N is negative, then $W_N(K)$ is negative definite (Good, if we want map induced by $W_N(K)$ to be an isomorphism)

If N is positive,



Y $\quad -Y_N(K)$

$-W_N(K)$

$$\partial(-W_N(K)) = -Y_N(K) \sqcup Y$$

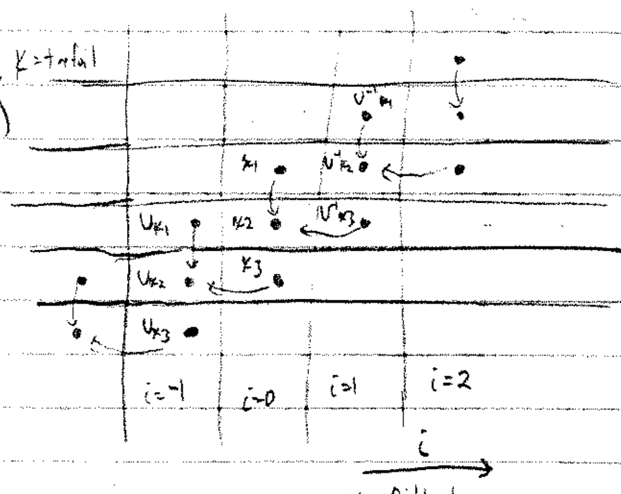
Def. An oriented cobordism from Y_1^n to Y_2^n is an oriented W^{n+1} mnfld. with $\partial(W^{n+1}) = -Y_1^n \sqcup Y_2^n$.

So $-W_N(K)$ is a cobordism from either $-Y$ to $-Y_N(K)$ or $Y_N(K)$ to Y.

Taking latter perspective, denote $-W_N(K)$ by $W_N'(K)$

Ex: $S^3 = Y$, K = trefoil

$CFK^\infty(Y,K)$



$j = 1 = A$

$j = 0 = A$

$j = -1 = A$

Pick $-\frac{N}{2} < m < \frac{N}{2}$

Thm. Let $CFK^\infty(Y,K,s)$ be the $\mathbb{Z}\oplus\mathbb{Z}$-filtered complex associated to $K \subset Y$ & $s \in Spin^c(Y)$. Let $C\left(\max\{i, j-m\} \begin{matrix} \geq 0 \\ < 0 \\ = 0 \end{matrix}\right)$ denote the $\begin{matrix}\text{quotient} \\ \text{sub} \\ \text{sub quotient}\end{matrix}$ complex

~~of $CFK^\infty(Y,K,s)$ consisting of triples $[\vec{x}, i, j]$ whose (i,j)-filtration~~

coordinates satisfy the constraint

$$\max\{i, j-m\} \begin{matrix} \geq 0 \\ < 0 \\ = 0 \end{matrix}.$$

Then, $\forall N \gg 0$,

$$H_*\left(C\left(\max\{i, j-m\} \begin{matrix} \geq 0 \\ < 0 \\ = 0 \end{matrix}\right), \partial^{\text{induced}} \right) \underset{\Phi_*^{+/-/\wedge}}{\cong} \begin{matrix} HF^+(Y_N(K), s_m) \\ HF^-(Y_N(K), s_m) \\ \widehat{HF}(Y_N(K), s_m) \end{matrix}.$$

Moreover,

$$\begin{tikzcd} 0 \arrow[r] & \widehat{CF}(Y_N(K), s_m) \arrow[r] \arrow[d, "\Phi^\wedge"] & CF^+(Y_N(K), s_m) \arrow[r, "U"] \arrow[d, "\Phi^+"] & CF^+(Y_N(K), s_m) \arrow[r] \arrow[d, "\Phi^+"] & 0 \\ 0 \arrow[r] & C(\max\{i, j-m\} = 0) \arrow[r] & C(\max\{i, j-m\} \geq 0) \arrow[r, "U"] & C(\max\{i, j-m\} \geq 0) \arrow[r] & 0 \end{tikzcd}$$

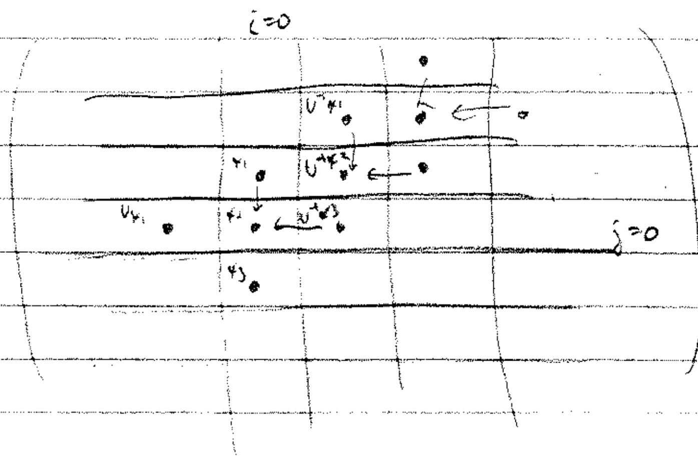
ψ

$$\begin{tikzcd} 0 \arrow[r] & CF^- \arrow[r] \arrow[d, "\Phi^-"] & CF^\infty \arrow[r] \arrow[d, "Id."] & CF^+ \arrow[r] \arrow[d, "\Phi^+"] & 0 \\ & C(\max < 0) \arrow[r] & CFK^\infty \arrow[r] & C(\max \geq 0) & \end{tikzcd}$$

Consider $S^3_N(\text{trefoil})$

$$HF^+(S^3_N(\text{tref.}), s_0) \cong H_*(C(\max\{i, j-0\} \geq 0))$$

H_*



Cycles are Ux_1, x_3 $(U^i x_2 \quad i \geq 0)$

$(U^i x_1 + U^{i-1} x_3, \quad i \leq 0)$

$$U^i x_2 = \partial U^i x_1 = \partial U^{i-1} x_3$$

As a $\mathbb{Z}$-module

So $H_*\left(C(\max\{i,j\}\geq 0)\right) \cong HF^+\left(S^3_N(\text{trefoil}), s_0\right)$

$$\cong \mathbb{Z}\langle Ux_1\rangle \oplus \mathbb{Z}\langle x_3\rangle \bigoplus_{i\leq 0} \mathbb{Z}\langle U^i x_1 + U^{i-1}x_3\rangle$$

Notice, $U\left(U^i x_1 + U^{i-1}x_3\right) = U^{i+1}x_1 + U^i x_3$

As a U-module,

$$\frac{\mathbb{Z}[U,U^{-1}]}{\mathbb{Z}[U]}\langle Ux_1 + x_3\rangle \oplus \mathbb{Z}\langle Ux_1\rangle$$

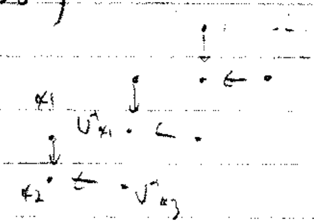
$HF^+\left(S^3_N(K), s_m\right)$ $\quad$ $C_*(\max\{i, j-m\}\geq 0)$

mnfld varies w/ N $\quad$ varies w/ m $\quad$ Independent of N.

<u>Thm.</u> Let $CFK^\infty(Y,K)$ be as before. Then $\forall N \gg 0$, we have the following isomorphisms:

$$H_*\left(C(\min\{i, j-m\}) \begin{matrix}\geq 0\\ <0 \\ =0\end{matrix}\right) \cong \begin{matrix} HF^+\left(S^3_{-N}(K), s_m\right) \\ HF^-\left(S^3_{-N}(K), s_m\right) \\ \widehat{HF}\left(S^3_{-N}(K), s_m\right)\end{matrix}$$

<u>Ex:</u> $HF^+\left(S^3_{-N}(K), s_0\right)$



$$\cong \frac{\mathbb{Z}[U,U^{-1}]}{U\mathbb{Z}[U]}\langle x_1 + U^{-1}x_3\rangle$$

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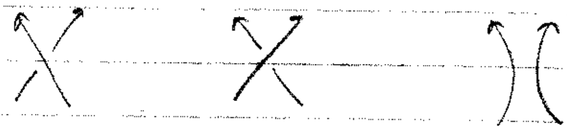
Fox Calculus Knot Floer Homology & Alexander Polynomial

Thm. $$\sum_{i \in \mathbb{Z}} \chi\left(\widehat{HFK}_*(K,i)\right) \cdot T^i = \Delta_K(T).$$



K_- K_+ K_0

Alexander poly. is the unique polynomial taking value 1 on unknot & satisfying the skein relation.
This was the main fact in the original proof of above thm.

Alternately, Fox Calculus definition of Alexander Polynomial (see Lickorish - Ch. 11 "Fundamental Group")

Suppose $\pi_1(S^3 \setminus K) = \langle x_1, \dots, x_n \mid r_1, \dots, r_k \rangle$

Define Fox Derivative:
(1) $$\frac{\partial x_i}{\partial x_j} := \delta_{ij}$$
(2) $$\frac{\partial (w \cdot m)}{\partial x} = \frac{\partial w}{\partial x} + w \cdot \frac{\partial m}{\partial x}$$
(3) $$\frac{\partial x_i^{-1}}{\partial x_j} = -\delta_{ij} x_i^{-1}$$

Check: $$\frac{\partial (x_i \cdot x_i^{-1})}{\partial x_i} = \frac{\partial 1}{\partial x_i} = 0$$
$$\frac{\partial x_i}{\partial x_i} + x_i \frac{\partial x_i^{-1}}{\partial x_i} = 1 + x_i \frac{\partial x_i^{-1}}{\partial x_i}$$
mult. by x_i^{-1} $$\Rightarrow 0 = x_i^{-1} + 1 \cdot \frac{\partial x_i^{-1}}{\partial x_i}$$
$$\Rightarrow \frac{\partial x_i^{-1}}{\partial x_i} = -x_i^{-1}.$$

Fox Matrix

$$F := \left[\frac{\partial r_i}{\partial x_j} \right]_{ij}$$

Consider the minors (i.e., determinants of maximal square submatrices of F.)

$$\pi_1(S^3 - K) = \langle x_1, \dots, x_n \mid r_1, \dots, r_k \rangle$$
$$\downarrow \phi$$
$$H_1(S^3 - K) \cong \mathbb{Z}\langle \mu \rangle$$

ϕ induces:

$$F = \left[\frac{\partial r_i}{\partial x_j} \right]_{i,j} \xrightarrow{\phi} \phi(F)$$

Consider the minors of this matrix.

Thm. The ideal generated by minors of $\phi(F)$ is principal (in $\mathbb{Z}[t, t^{-1}]$), i.e. generated by $p(t) \in \mathbb{Z}[t, t^{-1}]$. Moreover,

$$p(t) \doteq \Delta_K(t)$$

up to mult. by t^k, $k \in \mathbb{Z}$.

$$\mathbb{Z}[\pi_1(S^3 - K)]$$
$$\downarrow$$
$$\mathbb{Z}[H_1(S^3 - K)] \cong \mathbb{Z}[\mathbb{Z}\langle \mu \rangle] \cong \mathbb{Z}[t, t^{-1}]$$

Idea behind all this

From presentation of $\pi_1(S^3 - K)$, we can build a CW-complex homotopically equivalent to $S^3 - K$ w/ no ≥ 3-cells.

The homomorphism $\phi : \pi_1(S^3 - K) \to H_1(S^3 - K)$

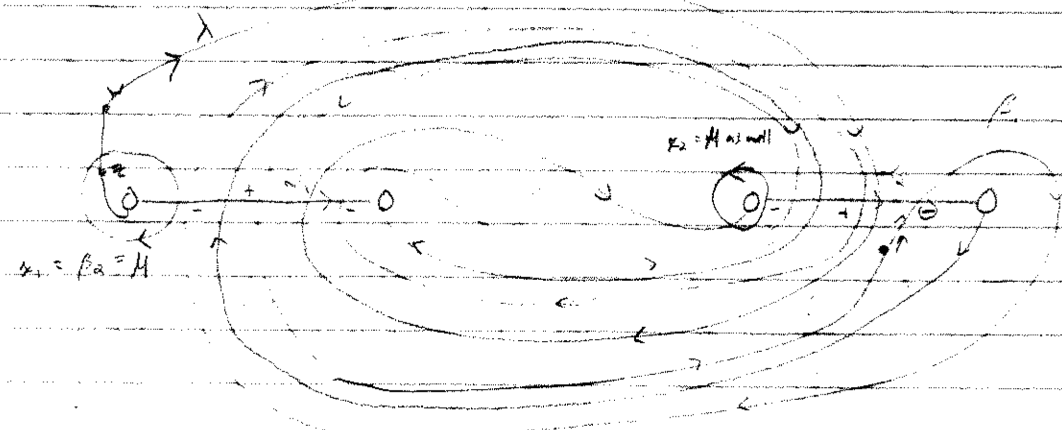
specifies the infinite cyclic cover of $S^3 - K$, X_∞,

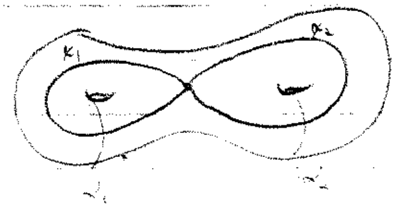
& we can lift the CW-decomposition of $S^3 - K$ to X_∞.

The Fox Matrix $\phi(F)$ is the CW boundary map

$$C_2(X_\infty; \mathbb{Z}[t, t^{-1}]) \xrightarrow{\phi(F)} C_1(X_\infty; \mathbb{Z}[t, t^{-1}])$$

Ex. (Trefoil)





$x_i = \alpha_i^*$

From β_1

$$\pi_1 = \langle x_1, x_2 \mid \underbrace{x_2^{-1} x_1^{-1} x_2 x_1^{-1} x_2^{-1} x_1}_{R} = 1, \ \cancel{x_1 = 1} \ \rangle$$

Don't want S^3

We want $S^3 \setminus K$.

ϕ

$$\left[\frac{\partial R}{\partial x_1} \;\middle|\; \frac{\partial R}{\partial x_2} \right] = \left[-x_2^{-1}x_1^{-1} - x_2^{-1}x_1^{-1} + x_2^{-1}x_1^{-1}x_2x_1^{-1}x_2^{-1} \;\middle|\; -x_2^{-1} + x_2^{-1}x_1^{-1} - x_2^{-1}x_1^{-1}x_2x_1^{-1}x_2^{-1} \right]$$

$\to \mathbb{Z}\langle \mu \rangle$

$$\phi(x_1) = \phi(x_2^{-1}) = 1 \cdot t$$

$$\phi[F] = \left[-1 - t t^{-1} t^{-1} t^{-1} + t t^{-1} t^{-1} t^{-1} t \;\middle|\; -t + t \cdot t^{-1} - t \cdot t^{-1} t^{-1} \cdot t^{-1} \cdot t \right]$$

$$= \left[-1 - t^{2} + t^{-1} \;\middle|\; -t + 1 - t^{-1} \right].$$

$$\Delta_K(t) = t - 1 + t^{-1}$$ (Normalized so that $\Delta_K(1) = 1$, $\Delta_K(t) = \Delta(t^{-1})$.)

__Observe:__ (1) Pts. in $\alpha_i \cap \beta_j \overset{1-1}{\longleftrightarrow}$ Appearances of x_i in R_j (relator from β_j)

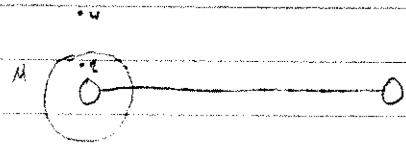
$\overset{1-1}{\longleftrightarrow}$ Terms in $\dfrac{\partial R_j}{\partial x_i}$

For higher genus diagrams:

$$\langle x_1, \ldots, x_g \mid R_1, \ldots, R_{g-1} \rangle$$

Ex. for $g=3$

$$\begin{bmatrix} \frac{\partial R_1}{\partial x_1} & \frac{\partial R_1}{\partial x_2} & \frac{\partial R_1}{\partial x_3} \\ \frac{\partial R_2}{\partial x_1} & \frac{\partial R_2}{\partial x_2} & \frac{\partial R_2}{\partial x_3} \end{bmatrix} \sim \begin{bmatrix} p \in \beta_1 \cap \alpha_1 & \beta_1 \cap \alpha_2 & \beta_1 \cap \alpha_3 \\ \beta_2 \cap \alpha_1 & \beta_2 \cap \alpha_2 & \beta_2 \cap \alpha_3 \end{bmatrix}$$



μ only intersects α_1.

F is $(g-1)\times g$ matrix

Consider the minor that excludes the $\frac{\partial}{\partial x_1}$ derivatives (first column)

<u>Observation (2)</u> (Uncombined) Terms in the minor not including $\frac{\partial R_i}{\partial x_1}$

$$\Updownarrow 1\text{-}1$$

Generators for $\widehat{CFK}$

$$\Updownarrow$$

$$\sum_{\sigma \in S_{g-1}} \operatorname{sign}(\sigma) \cdot \frac{\partial R_1}{\partial x_{\sigma(1)}} \cdot \frac{\partial R_2}{\partial x_{\sigma(2)}} \cdot \dots \cdot \frac{\partial R_{g-1}}{\partial x_{\sigma(g-1)}}$$

(Where, for our silly notational reasons, $S_{g-1} : \{1,\dots,g-1\} \to \{2,\dots,g\}$)

$$\phi\left(\det\left(\left[\frac{\partial R_i}{\partial x_j}\right]_{\substack{j=2,\dots,g \\ i=1,\dots,g-1}}\right)\right) = \Delta_K(t)$$

$$\|$$

$$\sum_{\vec{x}\in \mathbb{T}_\alpha \cap \mathbb{T}_\beta} \left(\text{sum of powers of } t \text{ with sign}\right)(\vec{x})$$

Relative gradings

Maslov grading $\quad M(\vec{x}) - M(\vec{y}) = \mu(\phi) - 2n_w(\phi)$

Alexander grading $\quad A(\vec{x}) - A(\vec{y}) = n_z(\phi) - n_w(\phi)$

$$\sum_{i\in\mathbb{Z}} \chi\left(\widehat{HFK}(K,i)\right) \cdot T^i$$

$$(-1)^{*}\, T^{A(\vec{x})}$$

$$\sum_{\vec{x} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta} \xrightarrow{\text{Floer homology}} (-1)^{*}\, T^{A(\vec{x})}$$

$$\xrightarrow{\text{Fox Calculus}} (-1)^{*'}\, T^{F(\vec{x})}$$

where $F(\vec{x})$ is the power of T it gets assigned, viewing it as a term in expansion of $\phi\left(\det\left(\frac{\partial R}{\partial x}\right)\right)$

Sketch that $\ast = \ast' + c$

Follows from fact that $\ast^{\text{Fiber}}(\vec{x}) = \text{sign of intersection of } \mathbb{T}_\alpha \cap \mathbb{T}_\beta \text{ at } \vec{x} \pmod 2$

$$= \operatorname{sign}(\sigma) \cdot (-1)^{\operatorname{sign} x_1 \in \beta_1 \cap \alpha_{\sigma(1)}} \cdot (-1)^{\operatorname{sign} x_2 \in \beta_2 \cap \alpha_{\sigma(2)}} \cdots \cdot (-1)^{\operatorname{sign} x_{g-1} \in \beta_{g-1} \cap \alpha_{\sigma(g-1)}} \cdot (-1)^{\operatorname{sign}(x_{\text{special}})}$$

α_1 intersects Meridian exactly once.

$$= \text{Sign of } \vec{x} \text{ given by Fox } (\ast'). \pmod 2$$

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Conclusion of $\sum_i \chi(\widehat{HFK}(Y,K,i))\cdot T^i = \Delta_K(T)$

We've seen bijection between terms in the expansion of $\det\left(\phi(F)_{g-1\times g-1}\right) \leftrightarrow \vec{x} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$.

$$\Delta_K(T) \qquad \sum_i \chi(\widehat{CFK}(Y,K,i))T^i = \sum \chi(\widehat{HFK}\cdots)$$

$$\sum_{\vec{x}\in \mathbb{T}_\alpha\cap\mathbb{T}_\beta} (-1)^{*(\vec{x})}\cdot T^{A(\vec{x})}$$

Remains to show (A) $(-1)^{*(\vec{x})}$ = the sign assigned to $\vec{x}$ via Fox determinant,

and (B) $A(\vec{x})$ = power of T associated to $\vec{x}$ " " "

Part A $\quad *(\vec{x}) \bmod 2$ = sign of intersection of $\mathbb{T}_\alpha\cap\mathbb{T}_\beta$, where $Sym^g(\Sigma)$ is oriented by Σ.

* $\mathbb{T}_\alpha, \mathbb{T}_\beta$ oriented by choice of orientation for $\vec{\alpha}, \vec{\beta}$.

$$\text{sign}(\vec{x}\in \mathbb{T}_\alpha\cap\mathbb{T}_\beta) = (-1)^{\text{sign}(\sigma)}\cdot(-1)^{\text{sign}\, x_1\in\alpha_1\cap\beta_{\sigma(1)}}\cdots(-1)^{\text{sign}\, x_g\in\alpha_g\cap\beta_{\sigma(g)}}$$

$$= \text{sign of the monomial corresponding to } \vec{x} \text{ in } \det\left(\phi(F)_{g-1\times g-1}\right)$$

b/c

~~$*$ (mod 2) = $\mu(\phi)$ (mod 2)~~

Exercise: Check $\mu(\phi)$ = sign $\vec{x}\in\mathbb{T}_\alpha\cap\mathbb{T}_\beta$ (mod 2) using

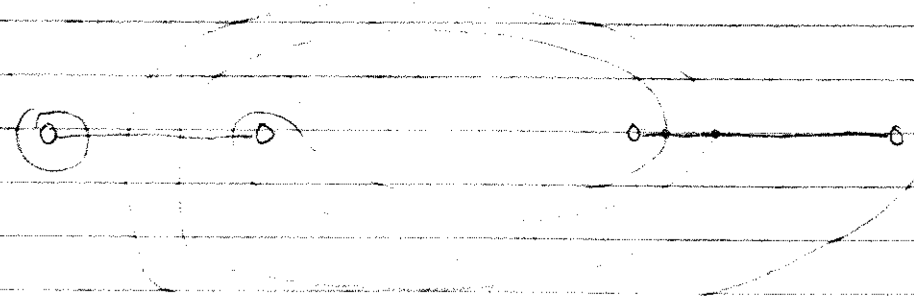
$$\mu(\phi) = \hat{\chi}(D(\phi)) + n_{\vec{x}}(\phi) + n_{\vec{y}}(\phi).$$

Part B $\quad$ ETS: $A(\vec{x}) - A(\vec{y}) = T_{Fox}(\vec{x}) - T_{Fox}(\vec{y}) \quad \forall \vec{x}, \vec{y}$

^ Power of T assigned to monomial in $\det\left(\phi(F)_{g-1\times g-1}\right)$ by Fox determinant

Recall: $A(\vec{x}) - A(\vec{y}) = n_z(\phi) - n_w(\phi)$ for $\phi\in\pi_2(\vec{x},\vec{y})$

= # times meridian (generator of $H_1(Y\setminus K)$) appears in the boundary of $D(\phi)$.



Pretend genus 3, for more general insight:

$$\begin{bmatrix} \frac{\partial R_1}{\partial x_1} & \frac{\partial R_1}{\partial x_2} & \frac{\partial R_1}{\partial x_3} \\ \frac{\partial R_2}{\partial x_1} & \frac{\partial R_2}{\partial x_2} & \frac{\partial R_2}{\partial x_3} \end{bmatrix} \rightsquigarrow \begin{bmatrix} \beta_1 \cap \alpha_1 & \beta_1 \cap \alpha_2 & \beta_1 \cap \alpha_3 \\ \beta_2 \cap \alpha_1 & \beta_2 \cap \alpha_2 & \beta_2 \cap \alpha_3 \end{bmatrix}$$

$$\{r_1, r_2\} \qquad r_i \in \beta_i \cap \alpha_{\sigma(i)}$$

$$\epsilon\left(\{r_1^i, r_2\}, \{r_1^j, r_2\}\right)$$

Recall: K fibered $\Rightarrow$ $\Delta_K(T)$ monic.

Look for the corresponding property in HFK:

K fibered $\Leftrightarrow$ $HFK(K, g(\text{fiber})) \cong \mathbb{Z}/2$.

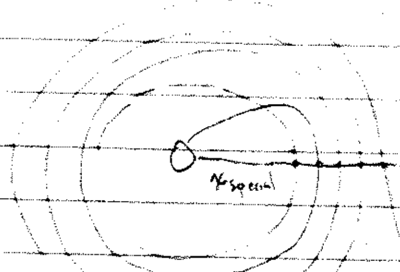
Coming back to surgery formula

Thm. $K \subseteq Y$ knot. Then $\forall N >> 0$,

(A) $$H_*\left(C\left(\max\{i, j-m\} \begin{smallmatrix} \geq 0 \\ < 0 \\ = 0 \end{smallmatrix}\right)\right) \cong \begin{smallmatrix} HF^+ \\ HF^- \\ \widehat{HF} \end{smallmatrix} \left(Y^3_N(K), s_m\right)$$

(B) $$H_*\left(C\left(\min\{i, j-m\} \begin{smallmatrix} \geq 0 \\ < 0 \\ = 0 \end{smallmatrix}\right)\right) \cong \begin{smallmatrix} HF^+ \\ HF^- \\ \widehat{HF} \end{smallmatrix} \left(Y^3_{-N}(K), s_m\right)$$

Pf.



$$\{x_{special}, \vec{y}_{(g-1 \text{ tuple})}\} \xrightarrow{1:N} \{x_i, \vec{y}_{(g \cdot N \text{ tuple})}\}$$

~~intersection # in~~

winding region

$$\pi_2\left(\{x, \vec{y}\}, \{x, \vec{r}\}\right)$$

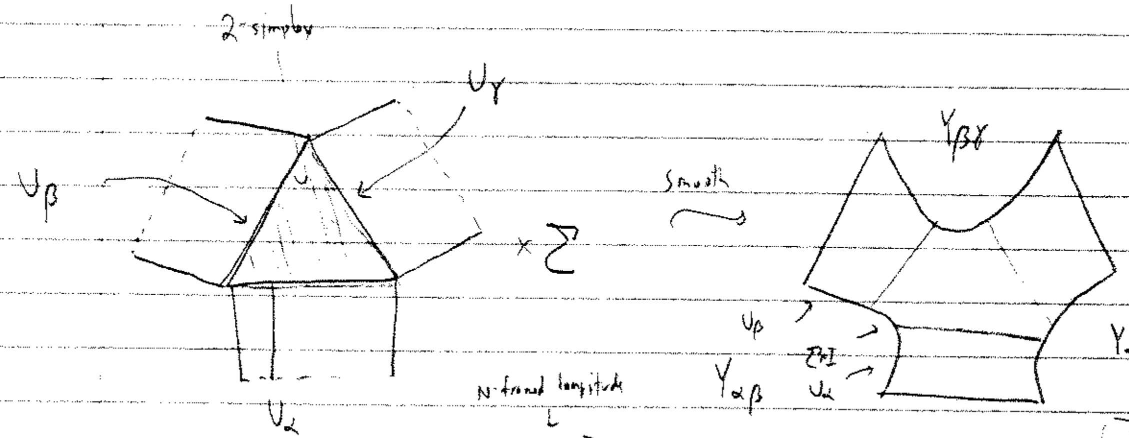
$$\phi \longrightarrow \tilde{\phi} \in \pi_2\left(\{x_i, \vec{y}\}, \{x_{i+k}, \vec{r}\}\right)$$

$$n_z(\phi) - n_w(\phi) = k$$

Pf. of (A)

Define $\widehat{\Phi}_{t_m} : \widehat{CF}(Y_N(K), \mathfrak{s}_m) \longrightarrow C(\max\{i, j-m\} = 0)$

by considering $(\Sigma, \vec{\alpha}, \vec{\beta}, \vec{\gamma}, w)$



$=: W_{\alpha,\beta,\gamma}$

$\partial W_{\alpha,\beta,\gamma} \cong -\check{Y}_{\alpha\beta} \sqcup -\check{Y}_{\beta\gamma} \sqcup \check{Y}_{\alpha\gamma}$

$W_{\alpha,\beta,\gamma} \cong -W_N'(K) - \text{nbhd.}\left(\bigvee_{g-1 \text{ circles}}\right)$

N-framed longitude

$\{\beta_1, \ldots, \beta_{g-1}, \lambda\}$

where

$(\Sigma, \vec{\alpha}, \vec{\beta}, w) \longrightarrow Y_{\alpha\beta} \cong Y_N(K)$

$(\Sigma, \vec{\alpha}, \vec{\gamma}, w) \longrightarrow Y_{\alpha\gamma} \cong Y$

$\vec{\gamma} = \{\tilde{\beta}_1, \ldots, \tilde{\beta}_{g-1}, \mu\}$

slight perturbations for transversality ~~+ admissibility~~

$(\Sigma, \vec{\beta}, \vec{\gamma}) \longrightarrow Y_{\beta\gamma} \cong \#^{g-1} S^1 \times S^2$

Recall, $\widehat{HF}(Y_{\beta\gamma}) \cong H_*^{\text{Sing.}}(T^{g-1}) \ni \Theta_{top.}$ (Top dim'l gen.)