MIN-MAX FORMULAS FOR NONLOCAL ELLIPTIC OPERATORS

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ABSTRACT. In this work, we give a characterization of Lipschitz operators on spaces of $C^2(M)$ functions (also $C^{1,1}$, $C^{1,\gamma}$, C^1, C^γ) that obey the global comparison property– i.e. those that preserve the global ordering of input functions at any points where their graphs may touch, often called "elliptic" operators. Here M is a complete Riemannian manifold. In particular, we show that all such operators can be written as a min-max over linear operators that are a combination of drift-diffusion and integro-differential parts. In the *linear* (and nonlocal) case, these operators had been characterized in the 1960's, and in the *local, but nonlinear* case– e.g. local Hamilton-Jacobi-Bellman operators– this characterization has also been known and used since approximately since 1960's or 1970s. Our main theorem contains both of these results as special cases. It also shows any nonlinear scalar elliptic equation can be represented as an Isaacs equation for an appropriate differential game. Our approach is to "project" the operator to one acting on functions on large finite graphs that approximate the manifold, use non-smooth analysis to derive a min-max formula on this finite dimensional level, and then pass to the limit in order to lift the formula to the original operator. This is the Director's cut, and it contains extra details for our own sanity.

1. INTRODUCTION AND BACKGROUND

Consider a Lipschitz map $I : C_b^2(\mathbb{R}^d) \to C_b(\mathbb{R}^d)$ with the property that given any functions $u, v \in C_b^2(\mathbb{R}^d)$ and a fixed $x \in \mathbb{R}^d$ such that $u \leq v$ everywhere with $u(x) = v(x)$, then

$$I(u, x) \leq I(v, x).$$

Such a map is said to satisfy the *global comparison property* (GCP). Some of the most basic and frequently encountered maps with the GCP might be

$$I(u, x) = |\nabla u(x)|, \ I(u, x) = \Delta u(x), \text{ and } I(u, x) = \max_{A^a \geq 0} \left(\text{tr}(A^a D^2 u(x)) \right)$$

in the local case, or

$$I(u, x) = \int_{\mathbb{R}^d} \left(u(x + h) - u(x) \right) K(x, h) dh, \text{ with } K(x, h) \geq 0,$$

in the nonlocal case. For these and similar operators (e.g. general integro-differential or drift diffusion operators), it is straightforward to confirm the GCP because it follows immediately from their explicit formulas.

In this work, we prove a result in the reverse direction, i.e. we show that any (nonlinear) Lipschitz map I with the GCP, plus minor and reasonable technical assumptions, has a representation

as a min-max of Lévy operators similar to those mentioned above, as presented in Theorem 1.6. That is, I can be written as

$$I(u, x) = \min_a \max_b \{f^{ab}(x) + L^{ab}(u, x)\},\tag{1.1}$$

where each L^{ab} is an operator of Lévy type, meaning that

$$L^{ab}(u, x) = \text{tr}(A^{ab}(x)D^2u(x)) + B^{ab}(x) \cdot \nabla u(x) + C^{ab}(x)u(x)$$
$$+ \int_{\mathbb{R}^d \setminus \{0\}} u(x + y) - u(x) - \mathbb{1}_{B_{r_0}(0)}(y)\nabla u(x) \cdot y \, \mu_x^{ab}(dy),$$

where $f^{ab}, A^{ab}, B^{ab}, C^{ab} \in L^\infty(\mathbb{R}^d)$ are Borel functions (with norms uniform in ab), $A^{ab} \geq 0$, and μ_x^{ab} are Borel measures on $\mathbb{R}^d \setminus \{0\}$ such that

$$\sup_{ab} \sup_x \int_{\mathbb{R}^d \setminus \{0\}} \min\{1, |y|^2\}\mu_x^{ab}(dy) < \infty.$$

The setting of the main result is more general, and it covers operators $I : C_b^2(M) \to C_b(M)$ where M is a complete Riemannian manifold, see Section 1.1 for a full description. Such a min-max characterization for nonlocal, nonlinear operators has been relatively widely known as an open problem in the field of nonlocal equations for a few years, and min-max representations play a fundamental role in many results, which we mention in the Background and Existing Results, Sections 1.2, 1.3, and 1.4.

Example (Dirichlet to Neumann Maps) An important class of examples is given by the Dirichlet to Neumann maps for fully nonlinear elliptic equations. Consider, for instance, a bounded domain Ω with a C^2 boundary. Under mild assumptions on F, the Dirichlet problem

$$\begin{cases} F(D^2U) & = 0 \quad \text{in } \Omega \\ U & = u \quad \text{on } \partial\Omega \end{cases}$$

has a unique viscosity solution $U \in C^{1,\alpha}(\bar{\Omega})$, whenever $u \in C^2(\partial\Omega)$ (for some $\alpha > 0$ independent of u). This defines a map

$$I : C^2(\partial\Omega) \to C(\partial\Omega)$$

obtained by setting $I(u, x) := (\nabla U(x), n(x))$ (i.e. $\partial_n U$), where n is the inner normal to $\partial\Omega$ at x. Using the comparison principle for F, it is straightforward to see that this map I has the global comparison property, and boundary regularity theory for U shows that the mapping is indeed Lipschitz. In particular, our main result applies to the Dirichlet to Neumann map, even for nonlinear equations. In a forthcoming paper, the min-max formula and boundary estimates for elliptic equations are used to analyze these operators in detail.

Example (Isaacs-Bellman equations) Given linear operators $\{L^{ab}\}_{ab}$ each satisfying the global comparison property, one may consider equations of the form

$$I(u, x) = 0, \quad \text{where } I(u, x) := \min_a \max_b L^{ab}(u, x).$$

These are known as Isaacs-Bellman equations, and they arise in stochastic control (e.g Bellman, [5] for first order equations), or zero sum games (e.g. Isaacs [29] or Elliott-Kalton [16]). The original references dealt mainly with first order equations, but second order examples quickly followed; see e.g. [24]. It is easy to see that such an operator must satisfy the global comparison property, as it is preserved from L^{ab} through the min-max. Our main result can be seen as the converse assertion: we show that every Lipschitz operator for which the global comparison property holds corresponds to an Isaacs-Bellman equation for an appropriate family of Markov processes.

1.1. **Statement of The Main Results.**

Definition 1.1. *Given a set X and functions $u, v : X \to \mathbb{R}$, it is said that u touches v from below at $x_0 \in X$ if*

$$u(x) \le v(x), \quad \forall\, x \in X,$$
$$u(x_0) = v(x_0).$$

If the inequality is reversed, it is said that u touches v from above at x_0.

Definition 1.2. *Consider a set X and let $\mathcal{F} \subset \mathbb{R}^X$ be a class of real valued functions defined over X. Given a (possibly nonlinear) operator*

$$I : \mathcal{F} \subset \mathbb{R}^X \to \mathbb{R}^X,$$

*I is said to satisfy the **global comparison property (GCP)** if whenever $u \in \mathcal{F}$ touches $v \in \mathcal{F}$ from below at x_0 we have the inequality*

$$I(u, x_0) \le I(v, x_0).$$

Remark 1.3. *It is clear that the set of maps having the global comparison property is convex and closed with respect to (u, x)-pointwise limits, i.e. for limits $I_n \to I$ in the sense that*

$$\lim_{n \to 0} I_n(u, x) = I(u, x) \ \forall\, u \in \mathcal{F}, \ and \ \forall x \in X.$$

Our goal is to prove a representation theorem for nonlinear operators with the GCP. In order to include examples such as the nonlinear Dirichlet-to-Neumann mapping mentioned above, the main result necessarily deals the case that X is a Riemannian manifold (in that example, $X = \partial\Omega$).

Definition 1.4. *Let (M, g) be a d-dimensional C^3 Riemannian manifold with injectivity radius $r_0 > 0$, let $\exp_x$ denote the exponential map based at $x \in M$, and fix some $x \in M$. A linear functional, $L_x \in (C_b^\beta(M))^*$, is said to be a functional of Lévy type based at $x \in M$ if $L_x(u)$ has the following form*

$$L_x(u) = \operatorname{tr}(A\nabla^2 u(x)) + (B, \nabla u(x))_{g_x} + C u(x)$$
$$+ \int_{M \setminus \{x\}} u(y) - u(x) - \mathbb{1}_{B_{r_0}(x)}(y)(\nabla u(x), \exp_x^{-1}(y))_{g_x}\, \mu(dy), \qquad (1.2)$$

where $A : (TM)_x \to (TM)_x$ is a linear self-adjoint map such that $A \ge 0$, $B \in (TM)_x$, $C \in \mathbb{R}$ and μ is a Borel measure in $M \setminus \{x\}$ such that

$$\int_{M \setminus \{x\}} \min\{1, d(x, y)^2\}\, \mu(dy).$$

Remark 1.5. *When M is given by the Euclidean space $\mathbb{R}^d$, the exponential at x mapping simply becomes $y \to x + y$, and so the last term in (1.2) takes the more commonly seen form of*

$$\int_{\mathbb{R}^d \setminus \{x\}} u(y) - u(x) - (\nabla u(x), y - x)\, \mathbb{1}_{B_1(x)}(y)\, \mu(dy),$$

where μ is a Borel measure in $\mathbb{R}^d \setminus \{x\}$ such that

$$\int_{\mathbb{R}^d \setminus \{0\}} \min\{1, |x - y|^2\}\, \mu(dy).$$

We are now ready to state our main results.

Theorem 1.6. *Let (M, g) be as in Definition 1.4, and let $\beta \in [0, 2]$. Let $C_b^\beta(M)$ be one of the Banach spaces $C_b^{0,\beta}(M)$ or $C^{0,1}(M)$ if $\beta \in (0, 1)$; $C_b^1(M)$ if $\beta = 1$; $C_b^{1,\beta-1}(M)$ if $\beta \in (1, 2)$; $C^{1,1}(M)$ or $C_b^2(M)$ if $\beta = 2$. Let I*

$$I : C_b^\beta(M) \to C_b(M)$$

be a Lipschitz map having the global comparison property, and that satisfies the additional assumption that there is a modulus, ω with $\omega(r) \to 0$ as $r \to \infty$, such that for all r large enough,

$$\forall u, v \in C_b^\beta, \quad \|I(u) - I(v)\|_{L^\infty(B_r)} \le C\|u - v\|_{C^\beta(\overline{B_{2r}})} + C\omega(r)\|u - v\|_{L^\infty(M)}. \tag{1.3}$$

Then, I has the following min-max representation (proved in Section 4)

$$I(u, x) = \min_{v \in C_b^\beta} \max_{L_x \in \mathcal{K}_{Levy}(I)} \{I(v, x) + L_x(u - v)\}, \tag{1.4}$$

where $\mathcal{K}_{Levy}(I)$ is a collection of Lévy type linear functionals on $C_b^\beta(M)$, as in Definition 1.4 and (1.2). Moreover, the norm of each L_x is bounded by the Lipschitz norm of I. The formula (1.4) holds for u in different spaces, depending upon the domain of I. The cases are for respectively the domain of I and the type of u for which (1.4) holds are: domain is C_b^2, $u \in C_b^2$; domain is $C^{1,1}$, $u \in C_b^2$; domain is $C_b^{1,\gamma}$, $u \in C_b^{1,\gamma+\varepsilon}$ for any $0 < \varepsilon < 1 - \gamma$; domain is C_b^1, $u \in C_b^{1,\varepsilon}$ for any $0 < \varepsilon < 1$; domain is C_b^γ, $u \in C^{\gamma+\varepsilon}$ for any $0 < \varepsilon < 1 - \gamma$.

Proposition 1.7. *In the min-max formula of (1.4), not only do the functionals L_x have the Lévy-type form of (1.2), but they also reduce to simpler cases on β as follows:*

(1) *if $\beta = 2$ or $C^\beta = C^{1,1}(M)$, then all terms in (1.2) may be present;*
(2) *if $\beta \in [0, 2)$, excluding the case $C^{1,1}(M)$, but including the cases of C^1 and $C^{0,1}$, then $A^{ab} \equiv 0$ for all $x \in M$;*
(3) *if $\beta \in [0, 1)$ excluding the case $C^{0,1}$, then both $A^{ab} \equiv 0$ and $B^{ab} \equiv 0$ for all $x \in M$.*

A stronger version of the min-max holds if one imposes a further assumption on I,

$$\forall \, \mathcal{K} \subset\subset C_b^\beta(M), \text{ the family } \left\{ x \to \frac{I(v + u, x) - I(v, x)}{\|u\|_{C_b^\beta(M)}} \right\}_{u,v \in \mathcal{K}} \text{ is equicontinuous} \tag{1.5}$$

This assumption is satisfied if one assumes that I is a Lipschitz map from C_b^β to the Hölder space C_b^α (for any $\alpha > 0$), or even a space C_b^ω, where ω is some modulus of continuity.

Theorem 1.8. *Suppose that, in addition to the assumptions of Theorem 1.6, the operator I satisfies (1.5). Then, there is a family $\mathcal{L}$ of linear operators from C_b^β to C_b, such that*

$$I(u, x) = \min_{v \in C_b^\beta} \max_{L \in \mathcal{L}} \{I(v, x) + L(u - v, x)\}, \tag{1.6}$$

Furthermore, for each $x \in M$, the functional defined by $L(\cdot, x)$ belongs to the same class of functionals $\mathcal{K}_{Levy}(I)$ above.

Remark 1.9. *In the case that I is linear, Theorem 1.6 was shown by Courrége, for $M = \mathbb{R}^d$ [15, Theorem 1.5], and by Bony-Courrège-Priouret for an arbitrary d-dimensional manifold [6]. In fact, those works showed the result holds simply when L is a continuous linear operator from C^2 to C, endowed with the non-Banach space topology of local uniform convergence on compact sets.*

Remark 1.10. *If the operator I in Theorem 1.6 is convex, then the min-max formula simplifies to a max formula, see Lemma 4.34, and if I is linear, then there is no min-max.*

Remark 1.11. *As suggested by the result of Theorem 1.6, the GCP imposes significant structure on I. A good example of this, is that in fact I must depend on the C_b^β-norm in a very particular way. For example, one possible estimate that can be shown (not exactly the one we use, but illustrative enough) is for a fixed x,*

$$|I(u,x) - I(v,x)| \le C(R)\|I\|_{Lip(C_b^\beta, C_b)} \left(\|u - v\|_{C^\beta(B_R(x))} + \|u - v\|_{L^\infty(M)} \right).$$

A similar type of splitting of the estimate on the right hand side between C_b^β and L^∞ turns out to be fundamental to our method, and we explain it in detail in Section 4.2. We note for the reader familiar with the integro-differential theory that if I were already known to be of the Lévy form (1.2), then this decomposition is immediate for $\beta = 2$ (also for operators that are a min-max of (1.2) with uniform bounds on the ingredients).

1.2. **Background.** There are several precedents for this result. It was shown by Courrège [15] that a bounded linear operator $C^2(\mathbb{R}^d) \to C(\mathbb{R}^d)$ has the global comparison property if and only if it is of *Lévy type*, in (1.2), which was later extended to linear operators on functions in a manifold M in work of Bony, Courrège, and Priouret [6]. A related result by Hsu [28] provides a representation for the Dirichlet to Neumann map for the Laplacian in a smooth domain Ω, and this corresponds to studying the boundary process for a reflected Brownian motion. After a time rescaling, the boundary process is a pure-jump Lévy process on the boundary, and it's generator is of the form

$$L(u,x) = b(x) \cdot \nabla_\tau u(x) + \int_{\partial\Omega\setminus\{x\}} \left(u(y) - u(x) - \mathbb{1}_{B_1(x)}(y)\nabla_\tau u(x) \cdot (y - x) \right) k(x,y)\, d\sigma(y),$$

where ∇_τ denotes the tangential gradient, $b(x)$ is a tangent vector field to $\partial\Omega$, σ is the surface measure, and k is comparable to $|x - y|^{-d-1}$ for $|x - y|$ small. An interesting family of nonlocal operators on Riemannian manifolds are the fractional Paneitz operators, which are also conformally invariant; recently, such operators have been studied in relation to Dirichlet to Neumann maps by Chang and Gonzalez [11] and Case and Chang [10]; these linear operators satisfy the GCP, under certain curvature conditions. A related (nonlinear) Dirichlet to Neumann operator arising in conformal geometry is the boundary operator for the fully nonlinear Yamabe problem on manifolds with boundary [39].

If I is not necessarily linear but happens to satisfy the stronger *local* comparison principle, there are min-max results by many authors, e.g. Evans [19], Souganidis [47], Evans-Souganidis [21] and Katsoulakis [32]. In this case, the operator takes the form,

$$I(u,x) = F(x, u(x), \nabla u(x), D^2 u(x)),$$

where $F : \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^d \times \mathrm{Sym}(\mathbb{R}^d) \to \mathbb{R}$ can be expressed as

$$F(x, u, p, M) = \min_a \max_b \{\mathrm{tr}(A^{ab}(x)M) + B^{ab}(x) \cdot p + C^{ab}(x)u + f^{ab}(x)\}.$$

This was extended to even include the possibility of weak solutions acting as a *local* semi-group on $BUC(\mathbb{R}^d)$, related to image processing, in Alvarez-Guichard-Lions-Morel [1], and to weak solutions of sets satisfying an order preserving set flow by Barles-Souganidis in [3]. In [1] it was shown under quite general assumptions that certain nonlinear semigroups must be represented as the unique viscosity solution to a degenerate parabolic equation. Recent work of Gilboa and Osher [26] has explored the practical advantages of image processing algorithms that are not local. Thus, the family of nonlinear *local* elliptic operators has a simple description, and hence the representation of Lipschitz operators in the local case is more or less complete. So far, very little has been said about operators that don't necessarily have the local comparison principle, but only the weaker version that is the GCP (i.e. operators containing a nonlocal part).

1.3. **Some examples of the advantage of a min-max.** Using an equation that involves a min-max of linear operators of course goes back to studying differential games, where the equation gives information about the value and strategies of the game. However, here we briefly list some results where the flow of information is reversed: beginning with a nonlinear PDE, some results are more easily (or only) attainable after the solutions (sub or super solutions) are represented as value functions for certain differential games– via the dynamic programming principle. Some very early results on existence for solutions to nonlinear first order equations utilized the properties of the value function in a stochastic differential game and the vanishing viscosity method in Fleming [23] and Friedman [25]. Also, solving some similar nonlinear equations, the accretive operator method of Evans [18] utilized a convenient min-max structure. More refined properties of Hamilton-Jacobi equations, such as "blow-up" limits appear in Evans-Ishii [17] and inequalities for directional derivatives of solutions in Lions-Souganidis [41]. Applications to the structure of level sets, geometric motions, "generalized" characteristics, and finite domain/cone of dependence appear in Evans-Souganidis [21]. Some constructions of finite difference schemes in e.g. Kuo-Trudinger [37] utilized the fact that second order uniformly elliptic equations are necessarily a min max of linear operators in order to choose appropriate stencil sizes; and a min-max was used by Krylov [36] to produce a rate of convergence for some approximation schemes. The Lions-Papanicolaou-Varadhan preprint for homogenization of Hamilton-Jacobi equations [40] used the fact that any semigroup with the properties inherited by the homogenized limit must be a translation invariant Hamilton-Jacobi semigroup of viscosity solutions– a result very close in spirit to the one we show for nonlocal equations (see [40, Section 1.2] and work of Lions cited therein). Katsoulakis [32] used a min-max to leverage the value function of a stochastic differential game to show existence of a viscosity solution and its Lipschitz/Hölder regularity properties. More recently, Kohn-Serfaty exploited a min-max structure to make a link between solutions of fully nonlinear second order parabolic equations and a class of deterministic two-player games in the papers [33] and [34] (as opposed to the already known link with *stochastic* differential games). All of these results mentioned above are solely in the context of local equations.

1.4. **Nonlocal results that assume a min-max.** One of the reasons why there is such a strong link between nonlinear elliptic PDE and min-max formulas associated with differential games is that it turns out the property of being a unique viscosity solution of such an equation is more or less equivalent to satisfying a dynamic programming principle/equation. Thus, it is natural that even though in the nonlocal setting, no min-max formula for general operators was known to exist, many results assume their operators to have a min-max structure. Some of these examples are as follows. Some uniqueness theorems for viscosity solutions (weak solutions) to somewhat general nonlinear and nonlocal equations assume the operator to have a min-max structure in both Jakobsen-Karlsen [30] and Barles-Imbert [2]. Caffarelli-Silvestre [7, Sections 3 and 4] assume a min-max structure of their equations in proving some properties of viscosity solutions– but the main result of the paper, [7] does not make a min-max assumption. Silvestre [46] assumes the min-max in proving regularity results for critical nonlocal equations, where the nonlocal term is of order 1, the same as the drift. One of the authors in [43] and [44] assumes the nonlocal operators to have a min-max so as to be able to set-up a corrector equation in homogenization for some nonlocal problems. Furthermore, in [43] and [44] a homogenized limit equation is proved to exist, but it is only known as an abstract nonlinear nonlocal operator of a certain ellipticity class, and its precise structure is left as an unresolved question. Also, in connection to the known results for local Hamilton-Jacobi equations, Koike-Święch [35] showed that the value function for some stochastic differential games driven by Lévy noise is indeed the unique viscosity solution of the related nonlocal Isaacs equation. Thus, Theorem 1.6 in our current work can be seen as a sort of a posteriori justification for the existing min-max assumptions in the nonlocal literature.

1.5. **Notation.** Here we collect a table of notation that is used throughout the work.

Notation	Definition
M	Complete Riemannian manifold
d	dimension of M
$d(x, y)$	Geodesic distance on M
$TM, (TM)_x$	The tangent bundle to M and the tangent space at $x \in M$
$\exp_x$	The exponential map of the manifold M
r_0	a lower bound for the injectivity radius of M
$Q, Q', \dots$	cubes in some tangent space $(TM)_x$
Q^*	cube concentric with Q whose common length is increased by a factor of $9/8$
$\nabla^2 u(x)$	the Hessian of u over M
∇_a, ∇_{ab}	components of covariant derivatives on M w.r.t. a chart (e.g [27])
$\nabla_n^1 u(x)$	a discrete gradient over the finite set $\tilde{G}_n$
$\nabla_n^2 u(x)$	a discrete Hessian over the finite set $\tilde{G}_n$
$C_b(M)$	functions which are continuous and bounded in M, with the sup-norm
$C_b^2(M)$	functions for which $\nabla^2 u$ is continuous and bounded in M, with the sup-norm
$C_b^\beta(M)$	Any of: $C_b^2(M)$, $C^{1,1}(M)$, $C_b^{1,\beta-1}(M)$ if $1 \le \beta < 2$, $C^{0,1}(M)$, or $C_b^{0,\beta}(M)$ if $\beta < 1$
$C_c^\beta(M)$	functions in $C_b^\beta(M)$ that have compact support
$X_n^\beta(M)$	finite dimensional subspace of $C_b^\beta(M)$ given by a Whitney extension
$\mathcal{L}(X, Y)$	space of linear operators
$\mathrm{hull}(E)$	convex hull of the set E
$l(p, x; y)$	a "linear" function with gradient p, centered at x (Def 3.13)
$q(D, x; y)$	a "quadratic" function with Hessian D, centered at x (Def 3.13)
$p_{u,k}^\beta$	a "polynomial" approximation to u using l and q (eq (3.14))
ρ	a smooth approximation to $\min(t, 1)$, can be fixed for the entire work (Def 4.14)
$\eta_x^\delta, \tilde{\eta}_x^\delta$	smooth approximations to $\mathbb{1}_{B_{r_0}(x)}$ and $\mathbb{1}_{\{x\}}$ (Def 4.22).

1.6. **Outline of the rest of the paper.** In Section 2 we prove a "finite dimensional" version of Theorem 1.6 for operators acting on functions defined on a finite graph. In Section 3 and Section 4 we use finite dimensional approximations to extend the min-max formula to the case of a Riemannian manifold, proving Theorem 1.6. Finally, in Section 5 we mention several reasonable questions that could be addressed and which are directly related to our main result.

2. The min-max formula in the finite dimensional case

A cornerstone of our proof relies on the fact that in the finite dimensional setting, min-max representations for Lipschitz functions are known. Later we will produce a finite dimensional approximation to the original operator, I, and we will then invoke the tools from the finite dimensional setting. Here we collect the necessary theorems we need, and present them in a context that is consistent with our subsequent application.

Consider a finite set G, let $C(G) = \mathbb{R}^G$ denote the space of real valued functions defined on G. In this finite dimensional setting the characterization of linear maps satisfying the global comparison property is elementary. Thus, the importance of this section is not to establish a new result for Lipschitz maps, but rather to present all of the results in a way that will match our needs for extending the min-max to the infinite dimensional case.

Lemma 2.1. *Any bounded linear map* $L : C(G) \to C(G)$ *can be expressed as follows*

$$Lu(x) = c(x)u(x) + \sum_{y \in G, \ y \neq x} (u(y) - u(x))K(x, y) \quad \forall \, x \in G,$$

where $K(x, y) : G \times G \to \mathbb{R}$, $c : G \to \mathbb{R}$. *If it happens that* L *also satisfies the GCP, then* $K(x, y) \geq 0$ *for all* $x, y \in G$.

Proof. Consider the "canonical basis" of $C(G)$, $\{e_x\}_{x \in G}$, where for each $x \in G$,

$$e_x(y) = \begin{cases} 1 & \text{if } x = y, \\ 0 & \text{otherwise.} \end{cases}$$

This means that we can write u as

$$u(x) = \sum_{y \in G} u(y)e_y(x).$$

Then, for a generic $u \in C(G)$, we can use the linearity of L to write

$$(Lu)(x) = \sum_{y \in G} u(y)(Le_y)(x),$$

$$= u(x)(Le_x)(x) + \sum_{y \neq x} u(y)(Le_y)(x).$$

This can be rewritten as follows,

$$(Lu)(x) = u(x)(Le_x)(x) + u(x)\left(\sum_{y \neq x}(Le_y)(x)\right) - u(x)\left(\sum_{y \neq x}(Le_y)(x)\right) + \sum_{y \neq x} u(y)(Le_y)(x),$$

$$= u(x)\left((Le_x)(x) + \sum_{y \neq x}(Le_y)(x)\right) + \sum_{y \neq x}(u(y) - u(x))(Le_y)(x).$$

Let us define then

$$K(x, y) := (Le_y)(x), \quad \forall \, x, y \in G,$$

$$c(x) := (Le_x)(x) + \sum_{y \neq x}(Le_y)(x), \quad \forall \, x \in G.$$

Now, suppose we are in the special case that L has the GCP. Observe that $e_x(y) \geq 0$ with $e_x(y) = 0$ whenever $x \neq y$, with this in mind, and recalling that L satisfies the global comparison property, it is clear that

$$K(x, y) = (Le_x)(y) \geq 0, \quad \forall \, y \neq x,$$

and the lemma is proved. □

If I is nonlinear but Lipschitz, the above characterization can be extended as a min-max formula. We will use some machinery from nonsmooth analysis (see Clarke's book [14]). In particular, we will be making extensive use of the generalized Jacobian and some of its properties. Note we give it a slightly different name than the one used in [14].

Definition 2.2 ([14] Def 2.6.1). *For* $I : C(G) \to C(G)$, *the **Clarke differential of** I **at** u is defined as the set*

$$\mathcal{D}I(u) = \text{hull}\left\{\lim_{k \to \infty} DI(u_k) \ : \ \text{where } u_k \to u \text{ and } DI(u_k) \text{ exists for each } k\right\}.$$

Here "lim" is simply interpreted as the limit of a sequence of matrices (since this takes place in a finite dimensional vector space), and $DI(u_n)$ is the (Fréchet) derivative of I at u_n. Given a set E in a normed vector space, "hull(E)" denotes the smallest closed convex containing it.

It will also be convenient to have notation for the collection of all differentials:

Definition 2.3 (Full differential of I). *For $I : C(G) \to C(G)$, the **full differential of** I is the set*

$$\mathcal{D}I = \text{hull} \left(\bigcup_{u \in C(G)} DI(u) \right).$$

The main result of this section is the observation (which is more or less well known) that Lipschitz maps have a min-max structure. We record it here in a format that is useful to our subsequent approximations to I.

Lemma 2.4. *Let $I : C(G) \to C(G)$ be a Lipschitz map. Then, for any $u \in C(G)$ and $x \in G$,*

$$I(u, x) = \min_{v \in C(G)} \max_{L \in \mathcal{D}I} \{I(v, x) + L(u - v, x)\}, \tag{2.1}$$

where $\mathcal{D}I$ is as in Definition 2.3. This can equivalently be written as

$$I(u, x) = \min_{a} \max_{b} \left\{ f^a(x) + u(x)c^{ab}(x) + \sum_{y \in G, \; y \neq x} (u(y) - u(x))K^{ab}(x, y) \right\}. \tag{2.2}$$

If I happens to have the GCP, then it also holds that $K^{ab}(x, y) \geq 0$.

We first list some key properties of I before we prove Lemma 2.4.

Proposition 2.5. *The GCP is inherited under differentiation. Namely, if $I : C(G) \to C(G)$ is a Lipschitz mapping that has the GCP, then the same is true of any $L : C(G) \to C(G)$ in $\mathcal{D}I$.*

Proof. Assume first that I is differentiable at u and let L_u denote the derivative of I at u. Then,

$$\frac{d}{dt}\big|_{t=0} (I(u + t\phi, x) - I(u, x)) = L_u(\phi, x), \quad \forall \, \phi \in C(G), x \in G.$$

If $\phi(x) \leq 0$ for all x and $\phi(x_0) = 0$ for some x_0, it follows that (for every $t > 0$) $u + t\phi$ touches u from below at x_0, therefore (since I has the GCP)

$$I(u + t\phi, x_0) \leq I(u, x_0), \quad \forall \, t > 0$$
$$\Rightarrow L_u(\phi, x_0) \leq 0.$$

It follows L_u has the GCP. By definition, any $L \in \mathcal{D}I$ is a convex combination of limits of such L_u. Then, by Remark 1.3 we conclude that any $L \in \mathcal{D}I$ also has the GCP, and the proposition is proved.

$\square$

The following result is a very useful fact of the Clarke differential, and it shows that the differential set enjoys the mean value property.

Proposition 2.6. *Let $I : C(G) \to C(G)$ be a Lipschitz function. Then, for any $u, v \in C(G)$ there exists some $L \in \mathcal{D}F$ such that*

$$I(u) - I(v) = L(u - v).$$

Proof. See [14, Chapter 2, Proposition 2.6.5] for the proof.

$\square$

With these previous results in hand, we can now prove the main Lemma of this section.

Proof of Lemma 2.4. For any $v \in C(G)$, define an operator $K_v : C(G) \to C(G)$ as follows

$$K_v(u, x) = \max_{L \in \mathcal{DI}} \{I(v, x) + L(u - v, x)\}.$$

First, let us show that

$$I(u, x) = \min_{v \in C(G)} K_v(u, x). \tag{2.3}$$

Since $K_u(u, x) = I(u, x)$ for every u and x it holds that $I(u, x) \geq \min_{v \in C(G)} K_v(u, x)$.

Next, by Proposition 2.6 it follows that for any $u, v \in C(G)$ and any $x \in G$ there exists some $L \in \mathcal{DI}$ such that

$$I(u, x) = I(v, x) + L(u - v, x).$$

In particular,

$$K_v(u, x) = \max_{L \in \mathcal{DI}} \{I(v, x) + L(u - v, x)\} \geq I(u, x),$$

which proves (2.3) and hence (2.1). We note that (2.2) follows by applying Lemma 2.1 to each of the operators $L \in \mathcal{DI}$.

$\square$

3. A Whitney Extension For $C_b^\beta(M)$

In this section, we develop some tools necessary to build finite dimensional approximations to I. This will involve taking a sequence of finite sets $G_n \subset M$ "converging" to M, all while constructing an embedding map $C(G_n) \mapsto C^\beta(M)$ to approximate $C^\beta(M)$ by a finite dimensional subspace. Because we are concerned with approximations that will not corrupt too badly the Lipschitz norm of I, we had a natural choice to use the Whitney extension. If we were working in $M = \mathbb{R}^d$, then all of the results we would need are standard, and can be found e.g. in Stein's book [48, Chapter 6]. Unfortunately, we could find no references for these theorems for the Whitney extension on $M \neq \mathbb{R}^d$, and so for completeness, we provide the details here. We emphasize that nearly all of the theorems and proofs in the section are adaptations that mirror those of Stein's book [48], but are modified for the additional technical difficulties arising due to the Riemannian nature of M. A key fact is how the extension operator preserves regularity (Theorem 3.23). Along the way, we will also prove a few important lemmas: one regarding the behavior of the extension operators as $n \to \infty$ (Lemma 3.25), and a "corrector lemma" that says the extensions are in general order preserving up to a small error (Lemma 3.29).

For all of this section, (M, g) is a d-dimensional complete Riemannian manifold with injectivity radius bounded below by a constant $r_0 > 0$. We remind the reader that the choice to work on (M, g) rather than $\mathbb{R}^d$ is not just for mathematical generality– rather, since we intend to apply the min-max theory to the Dirichlet-to-Neumann operators of fully nonlinear equations, we must understand those operators acting on functions on $M = \partial\Omega$.

3.1. **Finite approximations to M, coverings, and partitions of unity.** The following basic lemma will be needed. It simply states that on a (uniform) small neighborhood of $0 \in (TM)_x$, the map $\exp_x$ is nearly an isometry.

Lemma 3.1. *Let M be a complete d-dimensional manifold with injectivity radius $r_0 > 0$ and bounded curvature. Then for any $\varepsilon \in (0, 1)$ there exists a $\delta > 0$ such that for any $w \in M$ we have*

$$(1 + \varepsilon)^{-1} |\exp_w^{-1}(x) - \exp_w^{-1}(y)|_{g_w} \leq d(x, y) \leq (1 + \varepsilon) |\exp_w^{-1}(x) - \exp_w^{-1}(y)|_{g_w}$$

for every $x, y \in B_{4\delta\sqrt{d}}(w)$.

Remark 3.2. *We note that the operation $\exp_w^{-1}(x) - \exp_w^{-1}(y)$ reduces simply to $x - y$ when M happens to be Euclidean space. The same can be said of $\exp_y^{-1}(x)$ –which will also appear later in a expression that involves $x - y$ in the case M is flat.*

Proof of Lemma 3.1. This is just, for example, the result in Lee's book [38, Prop 5.11] restated in our setting. We leave the proof to [38]. □
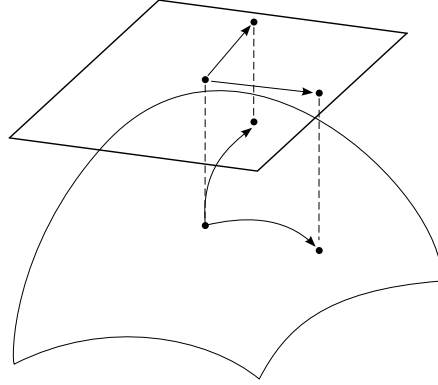

FIGURE 1. The Exponential Map

The above lemma says that we can control the amount by which the exponential map fails to be an isometry from $(TM)_w$ to M by restricting to a small enough neighborhood of the origin in $(TM)_w$. We fill fix a "distortion" factor ε, and cover M with sufficiently small balls where the above holds. We record this observation as a remark.

Remark 3.3. *Choose $\delta \in (0,1)$ sufficiently small so that conclusion of Lemma 3.1 holds with $\varepsilon = 1/100$. We fix an auxiliary sequence of points $\{w_i\}_i$ having the property*

$$M = \bigcup_i B_\delta(w_i). \tag{3.1}$$

Moreover, we select these points making sure the covering has the following property: there is a number $N_0 > 0$ such that any $x \in M$ lies in at most N_0 of the balls $\{B_{4\delta\sqrt{d}}(w_i)\}_i$.

From here on, we shall fix an infinite sequence of finite subsets M which, informally speaking, approximate the entire manifold (let us emphasize these points are different from the centers of the cover in Remark 3.3).

It will be useful to construct a sequence of discrete, but not necessarily finite, approximations to M (which will contain the finite ones). This sequence shall be denoted $\{\tilde{G}_n\}_n$, and it is assumed to have the following properties:

(1) The sequence is monotone increasing, $\tilde{G}_n \subset \tilde{G}_{n+1}, \ \forall\, n \in \mathbb{N}$.
(2) For every n, we have

$$\tilde{h}_n := \sup_{x \in M} d(x, \tilde{G}_n), \ \sup_n \tilde{h}_n \leq \delta/500, \ \lim_n \tilde{h}_n = 0. \tag{3.2}$$

(3) There exist a constant $\lambda > 0$ independent of n, such that

$$\inf_{\substack{x,y \in \tilde{G}_n \\ x \neq y}} d(x,y) \geq \lambda \tilde{h}_n. \tag{3.3}$$

Remark 3.4. *The existence of such a sequence of sets is not too difficulty to verify. For the sake of brevity, we only sketch its construction: take an orthogonal grid at each of the points w_i, and push them down via the respective exponential map, throw away points as needed.*

Remark 3.5. *The fact that $\tilde{h}_n$ is much smaller than δ is used at several points in the proof. In particular, the explicit factor of 500 in (3.2) is chosen to guarantee there are sufficiently many points of $\tilde{G}_n$ in any ball of radius δ, a fact that is not used until the Appendix (Proposition A.2), where we prove several important facts about the discretization of the gradient and the Hessian.*

Then, the sequence of finite sets $\{G_n\}_n$ is constructed as follows: we fix an auxiliary point $x_* \in M$ and let

$$M_n := B_{2^n}(x_*), \tag{3.4}$$

and define

$$G_n := \tilde{G}_n \cap M_{n+1}. \tag{3.5}$$

It is not surprising that the sequence $\{G_n\}_n$ has similar properties as $\{\tilde{G}_n\}$. As these properties will be used successively throughout the paper, we record them all in a single proposition.

Proposition 3.6. *The following properties are satisfied by $\{G_n\}_n$*

    (1) *If M is compact, then $G_n = \tilde{G}_n$ for all large enough n.*
    (2) *For every n we have $G_n \subset G_{n+1}$.*
    (3) *Each G_n is finite.*
    (4) *We have, with M_n as defined in (3.4), that $h_n = \tilde{h}_n$, in particular*

$$h_n := \sup_{x \in M_n} d(x, G_n), \text{ satisfies } \sup_n h_n \leq \delta/500, \quad \lim_n h_n = 0. \tag{3.6}$$

    (5) *Let h_n be as in (3.6) and λ as in (3.3), then for all sufficiently large n we have*

$$\inf_{\substack{x,y \in G_n \\ x \neq y}} d(x, y) \geq \lambda h_n. \tag{3.7}$$

Proof. Properties (1) and (2) are obvious. Next, from (3.3) it follows in particular that $\tilde{G}_n$ has no accumulation points, and thus Property (3) follows from the fact that M_n is bounded.

By the assumptions on $\tilde{G}_n$, for any $x \in M_n$ there is some $\hat{x} \in \tilde{G}_n$ such that

$$d(x, \hat{x}) \leq \tilde{h}_n.$$

Since $d(x, x_*) \leq 2^n$, it follows that $d(\hat{x}, x^*) \leq 2^n + h_n \leq 2^{n+1}$ since $\tilde{h}_n \leq 1$ for all n by (3.2). This means that $\hat{x} \in B_{2^{n+1}}(x_*) = M_{n+1}$, and that $\hat{x} \in G_n$. This shows that

$$\tilde{h}_n = \sup_{x \in M} d(x, \tilde{G}_n) = \sup_{x \in M_n} d(x, G_n) = h_n.$$

and Property (4) is proved. On the other hand, we have the trivial inequality

$$\inf_{\substack{x,y \in G_n \\ x \neq y}} d(x, y) \geq \inf_{\substack{x,y \in \tilde{G}_n \\ x \neq y}} d(x, y),$$

then (3.3) says this last term is at least $\lambda \tilde{h}_n$, which is equal to λh_n, which proves Property (5). $\square$

Remark 3.7. *If $M = \mathbb{R}^d$, for each $n \in \mathbb{N}$, we consider the Cartesian grid*

$$\tilde{G}_n := (2^{-2-n})\mathbb{Z}^d.$$

It is straightforward to see that $\{\tilde{G}_n\}_n$ has all the desired properties.

Remark 3.8. *Although the finite sets G_n will be the ones actually used in the proof of the main theorem, that will not happen until Section 4, for the rest of this section, we will be mostly concerned with $\tilde{G}_n$.*

We now start the construction. For each n we shall construct open covers $\{P_{n,k}\}_{k\in\mathbb{N}}$ and $\{P^*_{n,k}\}_{k\in\mathbb{N}}$ of $M \setminus \tilde{G}_n$, comprised of subsets of $M \setminus \tilde{G}_n$ (that is, the sets $P_{n,k}$ and $P^*_{n,k}$ will be disjoint from $\tilde{G}_n$). The sets in these covers will obtained by applying the exponential map to families of cubes lying in the tangent spaces $\{TM_{w_i}\}_i$. The cubes themselves are chosen following the classical Whitney cube decomposition, see [48, Chp 6, Thm 1].

Lemma 3.9. *For every n there exists two families of open sets $\{P_{n,k}\}_k, \{P_{n,k}*\}_k$ such that*

   (1) *For every k, there is some w_{i_k} $-\{w_i\}$ being the points fixed in Remark 3.3– such that*
$$P_{n,k} = \exp_{w_{i_k}}(Q_{n,k}), \quad P^*_{n,k} = \exp_{w_{i_k}}(Q^*_{n,k}),$$
   where $Q_{n,k}$ is a cube in $(TM)_{w_{i_k}}$, and $Q^*_{n,k}$ its concentric cube with length increased by a factor of $\frac{9}{8}$.

   (2) *For every k, we have*
$$\tfrac{1}{5}d(P_{n,k}, \tilde{G}_n) \le \operatorname{diam}(P_{n,k}) \le 5d(P_{n,k}, \tilde{G}_n),$$
$$\tfrac{1}{5}d(P^*_{n,k}, \tilde{G}_n) \le \operatorname{diam}(P^*_{n,k}) \le 7d(P^*_{n,k}, \tilde{G}_n).$$

   (3) *There is a universal $N > 0$, which in particular, is independent of n, such that if*
$$K_x := \{k \mid x \in P^*_{n,k}\}, \quad x \in M \setminus \tilde{G}_n, \tag{3.8}$$
   then
$$\#\{k \mid x \in P^*_{n,k}\} \le N \ \ \forall\, x \in M \setminus \tilde{G}_n. \tag{3.9}$$

   (4) *The sets $\{P_{n,k}\}_k$ cover the complement of $\tilde{G}_n$,*
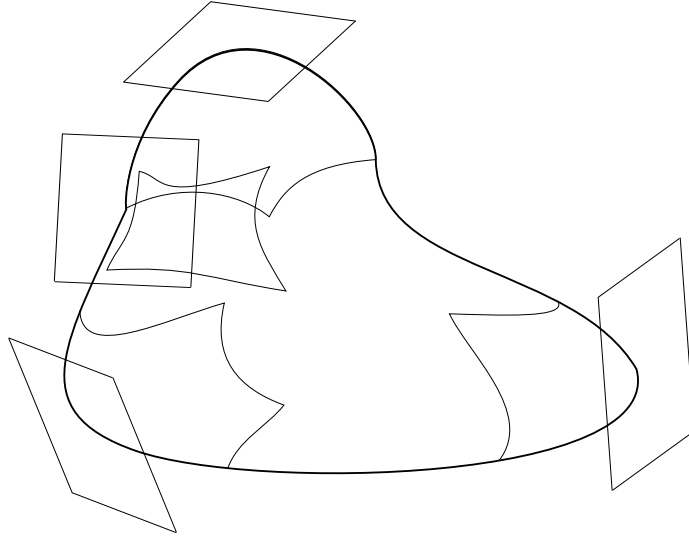$$\bigcup_k P_{n,k} = M \setminus \tilde{G}_n.$$

FIGURE 2. Example cubes, $Q_{n,k}$, in $(TM)_x$ projected to M, as $P_{n,k}$

Proof. Let δ be the constant from Remark 3.3. In what follows, we will lift a portion of $\tilde{G}_n$ to the vector space TM_{w_i}, for some nearby w_i. Then, we apply the Whitney cube decomposition to the resulting set [48, Chp 6], producing cubes in TM_{w_i} that will have the desired properties. These cubes are then mapped to M via exp_{w_i}.

For each w_i, we select

$$[e_{i,1}, \ldots, e_{i,d}], \quad \text{an orthonormal basis of } (TM)_{w_i},$$

the purpose of these bases is to allow us to set a rectangular grid in each of the tangent spaces. Which particular basis we choose each w_i will be immaterial. For each $n \in \mathbb{N}$ and w_i, define

$$F_{n,i} := \exp_{w_i}^{-1}(\tilde{G}_n \cap B_{3\delta\sqrt{d}}(w_i)), \quad \Omega_{n,i} := B_\delta(0) \setminus F_{n,i}.$$

For each i we construct a family of cubes in $(TM)_{w_i}$, denoted by $\mathcal{Q}_{n,i}$. The family is obtained by applying Whitney's cube decomposition in $(TM)_{w_i}$. Ultimately, this cube decomposition will be pushed down to $M \setminus \tilde{G}_n$ via the exponential map at w_i (see Figure 3).

Let us go over the cube decomposition. As we are working on a manifold, it will be convenient to consider cubes inside a small enough cube centered at the origin of $(TM)_{w_i}$. Keeping this in mind –and recalling that δ was chosen in Remark 3.3– we let $m_0 \in \mathbb{N}$ be the universal constant determined by

$$2\delta \leq 2^{-m_0} < 4\delta.$$

In other words, m_0 is the largest number such that $B_\delta(0) \subset (TM)_{w_i}$ is contained inside the cube centered at 0 with common side length equal to 2^{-m_0}, that is

$$Q_{2^{-m_0-1}}(0) = \{q \in (TM)_{w_i} : |(q, e_{i,l})_{g_x}| \leq 2^{-m_0-1}, \; l = 1, \ldots, d\}.$$

Then, considering only those cubes obtained by repeatedly bisecting the sides of $Q_{2^{-m_0-1}}(0)$, we define $\hat{\mathcal{Q}}_{n,i}$ to be the subfamily formed by those cubes Q for which we also have

$$Q \cap \{q \in (TM)_{w_i} \mid 2\mathrm{diam}(Q) \leq d(q, F_{n,i}) \leq 4\mathrm{diam}(Q)\} = \emptyset.$$

Then, let us say that a cube Q in $\hat{\mathcal{Q}}_{n,i}$ is maximal if there is no other cube Q' in the family such that $Q' \subset Q$. The family $\mathcal{Q}_{n,i}$ is then defined to be the subfamily of maximal cubes of $\hat{\mathcal{Q}}_{n,i}$.
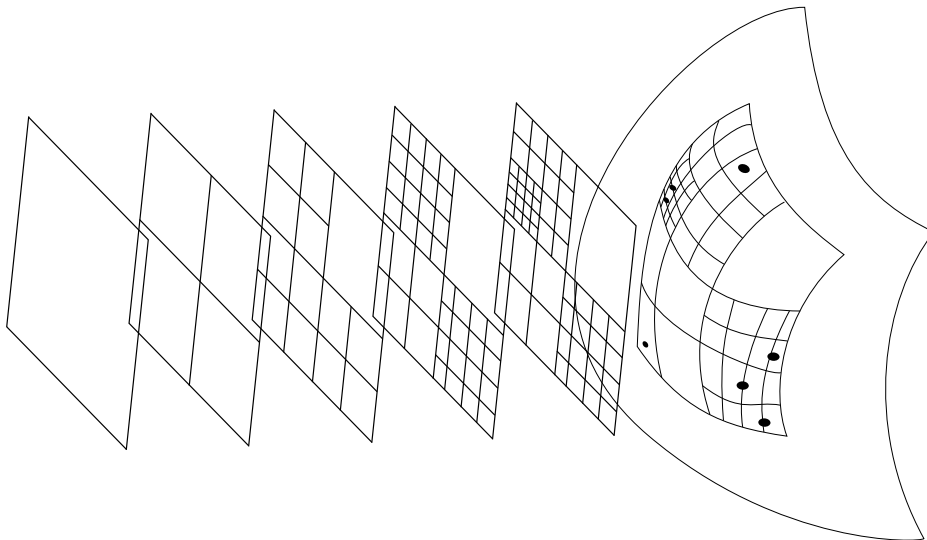

FIGURE 3. Decomposing Cubes In $(TM)_x$

The family $\mathcal{Q}_{n,i}$ has the following properties
(1) Any two distinct elements of $\mathcal{Q}_{n,i}$ have disjoint interiors.
(2) Every $q \in \Omega_{n,i}$ lies in the interior of a cube belonging to $\mathcal{Q}_{n,i}$.
(3) If $Q \in \mathcal{Q}_{n,i}$, then the common side length of Q is no larger than $2^{-m_0} \leq 4\delta$. In particular, Q lies inside $B_{2\delta\sqrt{d}}$, and Q^* lies inside $B_{3\delta\sqrt{d}}(0)$.

(4) There is a number N_1, independent of n and i, such that any $q \in \Omega_{n,i}$ lies in at most N_1 of the sets $\{Q^*\}_{Q \in \mathcal{Q}_{n,i}}$.

(5) The cubes in the family have a diameter comparable to their distance to $F_{n,i}$. Concretely,

$$\mathrm{diam}(Q) \leq d(Q, F_{n,i}) \leq 4\mathrm{diam}(Q), \quad \forall\, Q \in \mathcal{Q}_{n,i}. \tag{3.10}$$

We omit the verification of these properties, as it is standard. We refer the interested reader to [48, Chap. 6, Sec. 1] for details.

Let us immediately note that bounds akin to (3.10) extend to the respective "stretched" cubes Q^*. Indeed, fix some $Q \in \mathcal{Q}_{n,i}$. From $Q \subset Q^*$ we have $d(Q^*, F_{n,i}) \leq d(Q, F_{n,i})$, while from $d(Q, (Q^*)^c) = (1/8)\mathrm{diam}(Q)$ we have $d(Q^*, F_{n,i}) \geq d(Q, F_{n,i}) - (1/8)\mathrm{diam}(Q)$. From these observations and (3.10) it follows that

$$\tfrac{7}{9}\mathrm{diam}(Q^*) = \tfrac{7}{8}\mathrm{diam}(Q) \leq d(Q^*, F_{n,i}) \leq 4\mathrm{diam}(Q^*), \quad \forall\, Q \in \mathcal{Q}_{n,i}. \tag{3.11}$$

Having the families $\mathcal{Q}_{n,i}$ (for each i for which $F_{n,i} \neq \emptyset$), let us combine them into a single one, which will also be countable. Let $\{Q_{n,k}\}_k$ denote an enumeration of the elements of this larger family. Each $Q_{n,k}$ is a cube belonging to some tangent space $(TM)_{w_{i_k}}$ for some w_{i_k}.

Let $q_{n,k}$ denote the the center of $Q_{n,k}$, and $l_{n,k}$ its common side length. Then, we define

$$P_{n,k} := \exp_{w_i}(Q_{n,k}), \quad P^*_{n,k} := \exp_{w_i}(Q^*_{n,k}), \quad y_{n,k} = \exp_{w_{i_k}}(q_{n,k}),$$

This produces a family of sets for which Property (1) holds. Let us verify these families satisfy the other three Properties. Let us prove Property (2). Fix $P_{n,k} = \exp_{w_{i_k}}(Q_{n,k})$. Then,

$$\begin{aligned}
d(P_{n,k}, \tilde{G}_n) &\leq d(P_{n,k}, \tilde{G}_n \cap B_{3\delta\sqrt{d}}(w_{i_k})) \\
&\leq \tfrac{101}{100} d(Q_{n,k}, F_{n,i}) \\
&\leq 4\tfrac{101}{100}\mathrm{diam}(Q_{n,k}) \leq 4(\tfrac{101}{100})^2 \mathrm{diam}(P_{n,k}) \leq 5\mathrm{diam}(P_{n,k}).
\end{aligned}$$

The exact same argument yields

$$d(P^*_{n,k}, \tilde{G}_n) \leq 5\mathrm{diam}(P^*_{n,k})$$

This yields one side of the bounds in Property (2). Next, note that $Q_{n,k} \subset B_{2\delta\sqrt{d}}(0)$, which means that $\mathrm{diam}(Q_{n,k}) \leq 4\delta\sqrt{d}$ and

$$\begin{aligned}
d(P_{n,k}, \tilde{G}_n \setminus B_{3\delta\sqrt{d}}(w_{i_k})) &\geq \tfrac{100}{101} d(Q_{n,k}, \partial B_{3\delta\sqrt{d}}(0)) \\
&\geq \tfrac{100}{101}\delta\sqrt{d} \\
&\geq \tfrac{1}{4}\tfrac{100}{101}\mathrm{diam}(Q_{n,k}) \geq \tfrac{1}{4}(\tfrac{100}{101})^2\mathrm{diam}(P_{n,k}) \geq \tfrac{1}{5}\mathrm{diam}(P_{n,k}).
\end{aligned}$$

At the same time,

$$\begin{aligned}
d(P_{n,k}, \tilde{G}_n \cap B_{3\delta\sqrt{d}}(w_{i_k})) &\geq \tfrac{100}{101} d(Q_{n,k}, F_{n,i_k}) \geq \tfrac{100}{101}\mathrm{diam}(Q_{n,k}) \\
&\geq (\tfrac{100}{101})^2\mathrm{diam}(P_{n,k}) \\
&\geq \tfrac{1}{5}\mathrm{diam}(P_{n,k}).
\end{aligned}$$

Therefore,

$$d(P_{n,k}, \tilde{G}_n) \geq \tfrac{1}{5}\mathrm{diam}(P_{n,k}).$$

With the same argument, one can check that

$$d(P^*_{n,k}, \tilde{G}_n) \geq \tfrac{7}{36}(\tfrac{100}{101})^2\mathrm{diam}(P^*_{n,k}) \geq \tfrac{1}{7}\mathrm{diam}(P^*_{n,k}),$$

and Property (2) is proved. Next, recall the sequence $\{w_i\}$ is such that given $x \in M$, then

$$\#\{i \mid x \in B_{3\delta\sqrt{d}}(w_i)\} \leq N_0.$$

It follows that each x lies in at most N_0 of the sets $\{\Omega_{n,i}\}_i$, and Property (3) follows immediately by taking $N := N_0 N_1$. Finally, from (3.11), we have

$$P_{n,k} \subset P_{n,k}^* \subset M \setminus \tilde{G}_n \ \forall \, n, k.$$

Furthermore, since the balls $\{B_\delta(w_i)\}_i$ cover M, and each $B_\delta(w_i)$ is covered by $\{P_{n,k}\}$, we have

$$\bigcup_k P_{n,k} \supset \bigcup_i \{B_\delta(w_i) \setminus \tilde{G}_n\} = M \setminus \tilde{G}_n.$$

Thus we obtain Property (4), and the lemma is proved. $\square$

From this point onward, the sets $\tilde{G}_n, G_n$, and the associated family of open sets $\{P_{n,k}\}_k$ and $\{P_{n,k}^*\}_k$ will be fixed. For every k, by the "center" of $P_{n,k}$ we will mean the point $y_{n,k} = \exp_{w_{i_k}}(q_{n,k})$. Furthermore, $\hat{y}_{n,k}$ will denote a point in $\tilde{G}_n$ which realizes the distance from $y_{n,k}$ to $\tilde{G}_n$. Let us record these definitions for further reference:

$$y_{n,k} := \exp_{w_{i_k}}(q_{n,k}), \ \text{and} \ \hat{y}_{n,k} \in \tilde{G}_n \ \text{such that} \ d(y_{n,k}, \hat{y}_{n,k}) = d(y_{n,k}, \tilde{G}_n). \tag{3.12}$$

The following elementary fact will be used repeatedly in this section, we record it as a remark.

Remark 3.10. *Let $x \in P_{n,k}^*$. Then we have the inequalities*

$$\tfrac{1}{7}\mathrm{diam}(P_{n,k}^*) \leq d(x, \tilde{G}_n) \leq 6\mathrm{diam}(P_{n,k}^*).$$

Let us prove this. By the triangle inequality $d(x, \tilde{G}_n) \leq d(P_{n,k}^*, \tilde{G}_n) + \mathrm{diam}(P_{n,k}^*)$. Then, (2) from Lemma 3.9 says that

$$d(x, \tilde{G}_n) \leq 5\mathrm{diam}(P_{n,k}^*) + \mathrm{diam}(P_{n,k}^*) \leq 6\mathrm{diam}(P_{n,k}^*).$$

On the other hand, since $d(P_{n,k}^*, \tilde{G}_n)$ is just the infimum of $d(\cdot, \tilde{G}_n)$ over $P_{n,k}^*$,

$$d(x, \tilde{G}_n) \geq d(P_{n,k}^*, \tilde{G}_n) \geq \tfrac{1}{7}\mathrm{diam}(P_{n,k}^*),$$

the second inequality being again thanks to (2) from Lemma 3.9.

Continuing in parallel with the classical approach to the extension problem [48, Chapter 6], we construct a partition of unity for $M \setminus \tilde{G}_n$ associated to the family $\{P_{n,k}\}_{n,k}$. Since we work on a Riemannian manifold, we will need to compute covariant derivatives for scalar functions, up to third order (since the highest regularity we will be concerned with is $C^{2,\alpha}$, this will suffice for all our purposes). For a review of the definition of $\nabla^i \phi$ and its basic properties, see the end of Section 1.1 in [27, Chapter 1].

Lemma 3.11 (Partition of unity). *For every n, there is a family of smooth functions $\{\phi_{n,k}\}_k$ such that*

(1) $\sum_k \phi_{n,k}(x) = 1$ *for all $x \in M \setminus \tilde{G}_n$.*

(2) $0 \leq \phi_{n,k} \leq 1$ *in $M \setminus \tilde{G}_n$ and $\phi_{n,k} \equiv 0$ outside $P_{n,k}^*$.*

(3) *There is a constant C such that for every $x \in M \setminus \tilde{G}_n$, every n, k and $i = 1, 2, 3$ we have*

$$|\nabla^i \phi_{n,k}(x)|_{g_x} \leq \frac{C}{(\mathrm{diam}(P_{n,k}^*))^i}.$$

Proof. As is standard for a construction of a partition of unity, we will begin with auxiliary functions, $\tilde{\phi}_{n,k}$ that are basically smooth bumps localized at the centers of the sets $P_{n,k}$, and then we normalize their sum to obtain the desired family, $\{\phi_{n,k}\}$.

Let us fix an auxiliary function $\psi_0 : \mathbb{R}^d \to \mathbb{R}$ with the following properties

$$\psi_0 \in C^\infty(\mathbb{R}^d), \quad \psi_0 \equiv 1 \text{ in } [-1,1]^d, \quad \psi_0 \equiv 0 \text{ outside } [-\tfrac{9}{8}, \tfrac{9}{8}]^d.$$

Using the basis $[e_{i,1}, \ldots, e_{i,d}]$ for each i, we can "push" the above function to smooth functions

$$\psi_i : (TM)_{w_i} \to \mathbb{R}.$$

Then, for each k we let $\tilde{\phi}_{n,k}$ be defined by

$$\tilde{\phi}_{n,k}(x) = \begin{cases} \psi_{i_k}\left((l_{n,k}/2)^{-1}(\exp_{w_{i_k}}^{-1}(x) - q_{n,k}) \right) & \text{inside } P_{n,k}^* \\ 0 & \text{outside } P_{n,k}^*. \end{cases}$$

Here $l_{n,k} = (\sqrt{d})^{-1}\mathrm{diam}(Q_{n,k})$ is the common length of the sides of $Q_{n,k}$. Since $P_{n,k}^*$ lies uniformly in a normal neighborhood, and $\psi_0 \in C^\infty$, it follows for each k that $\tilde{\phi}_{n,k}$ is a smooth function. Moreover, using the definition of $\tilde{\phi}_{n,k}$ above it is straightforward to check that

$$\tilde{\phi}_{n,k}(x) \equiv 1 \text{ in } P_{n,k}, \quad \tilde{\phi}_{n,k}(x) \equiv 0 \text{ outside } P_{n,k}^*.$$

Furthermore, from the chain rule it follows easily there is a universal C such that for $i = 1, 2, 3$,

$$\sup_{x \in M} |\nabla^i \tilde{\phi}_{n,k}(x)|_{g_x} \leq \frac{C}{(l_{n,k})^i} \leq \frac{C}{(\mathrm{diam}(P_{n,k}^*))^i}.$$

Next, we consider the function

$$\phi_n(x) := \sum_k \tilde{\phi}_{n,k}(x).$$

Note that at most N of the sets $P_{n,k}^*$ contain x (Lemma 3.9), and therefore, only at most N of the functions $\tilde{\phi}_{n,k}$ are non-zero. Thus the sum defining ϕ_n is locally the sum of at most N non-zero smooth functions. In particular, we may differentiate to obtain

$$\nabla^i \phi_n(x) = \sum_k \nabla^i \tilde{\phi}_{n,k}(x), \quad i = 1, 2, 3.$$

Let us estimate the derivatives of $\phi_n(x)$, for each $i = 1, 2, 3$ we have

$$|\nabla^i \phi_n(x)|_{g_x} \leq \sum_k |\nabla \tilde{\phi}_{n,k}(x)|_{g_x} \leq \sum_{k \in K_x} \frac{C}{(\mathrm{diam}(P_{n,k}^*))^i}$$

Then, by Remark 3.10

$$k \in K_x \Rightarrow x \in P_{n,k}^* \Rightarrow d(x, \tilde{G}_n) \leq 6\mathrm{diam}(P_{n,k}^*).$$

Using again that $\#K_x \leq N$, we conclude for $i = 1, 2, 3$, that

$$|\nabla^i \phi_n(x)|_{g_x} \leq N \frac{C}{(d(x, \tilde{G}_n))^i} \leq \frac{C}{(d(x, \tilde{G}_n))^i}.$$

On the other hand, for every x there is at least one k such that $x \in P_{n,k}$, thus

$$1 \leq \phi_n(x) \leq N \quad \forall\, x \in M \setminus \tilde{G}_n.$$

We may now define the actual family of functions $\{\phi_{n,k}\}_{n,k}$. For each k, let

$$\phi_{n,k}(x) := \frac{\tilde{\phi}_{n,k}(x)}{\phi_n(x)}.$$

It is simple to check that this family of functions has Property (1). Indeed, using that the sum is locally finite, we have

$$\sum_k \phi_k(x) = \sum_k \frac{\tilde{\phi}_{n,k}(x)}{\phi_n(x)} = \frac{1}{\phi_n(x)} \sum_k \tilde{\phi}_{n,k}(x) = \frac{\phi_n(x)}{\phi_n(x)} = 1.$$

On the other hand, Property (2) follows as each $\tilde{\phi}_{n,k}$ is non-negative and supported in $P_{n,k}^*$ from the definition of $\tilde{\phi}_{n,k}$. As for Property (3), we compute

$$\nabla \phi_{n,k}(x) = \frac{\nabla \tilde{\phi}_{n,k}(x)}{\phi_n(x)} - \frac{\tilde{\phi}_{n,k}(x)}{\phi_n(x)^2} \nabla \phi_n(x)$$

$$\nabla^2 \phi_{n,k}(x) = \frac{\nabla^2 \tilde{\phi}_{n,k}(x)}{\phi_n(x)} - \frac{\tilde{\phi}_{n,k}(x)}{\phi_n(x)} \nabla^2 \phi_n(x).$$

Combining the estimates for the derivatives of $\tilde{\phi}_{n,k}$ and ϕ_n yields, for each $x \in M \setminus \tilde{G}_n$,

$$|\nabla \phi_{n,k}(x)|_{g_x} \leq |\nabla \tilde{\phi}_{n,k}(x)|_{g_x} + |\nabla \phi_n(x)|_{g_x} \leq \frac{C}{\operatorname{diam}(P_{n,k}^*)} \,,$$

$$|\nabla^2 \phi_{n,k}(x)|_{g_x} \leq |\nabla^2 \tilde{\phi}_{n,k}(x)|_{g_x} + |\nabla^2 \phi_n(x)|_{g_x} \leq \frac{C}{(\operatorname{diam}(P_{n,k}^*))^2} \,.$$

Where we have used Remark 3.10 (once again) to obtain the second bound in each case. The respective bound for $\nabla^3 \phi_{n,k}(x)$ follows similarly, and we omit the details.

$\square$

Remark 3.12. *As stated in Lemma 3.11, we have $\sum_k \phi_{n,k} \equiv 1$ on $M \setminus \tilde{G}_n$. After repeatedly differentiating this identity we obtain another identity that will be of use later on,*

$$\sum_k \nabla^i \phi_{n,k}(x) \equiv 0 \text{ in } M \setminus \tilde{G}_n, \quad i = 1, 2, 3. \tag{3.13}$$

3.2. **Local interpolators.** We have constructed a "cube" covering of $M \setminus \tilde{G}_n$ (since $P_{n,k}$ are only cubes when seen in the right exponential chart), and a corresponding partition of unity. Next, we need to fix a choice for "local" interpolating functions. Specifically, we need to define what will take the place of the local linear and quadratic functions in the usual Whitney extensions.

Recall that $\delta \in (0,1)$ was chosen in Remark 3.3 so the exponential map was roughly an isometry in balls of radius $4\delta\sqrt{d}$. In particular this means that for $y \in M$, the inverse exponential map $\exp_y^{-1}$ is a well defined, uniformly smooth map from $B_{4\delta\sqrt{d}}(y)$ to a neighborhood of zero in $(TM)_y$. This smooth map defines local charts on M having useful properties (they are normal systems of coordinates), and using such charts we shall introduce (locally defined) functions that will play the role of "linear" and "quadratic" functions near a given point $y \in M$.

Definition 3.13. *Given $y \in M$ and a vector $p \in (TM)_y$, define $l(p, y; \cdot) : B_{4\delta\sqrt{d}}(y) \to \mathbb{R}$ by*

$$l(p, y; x) := (\exp_y^{-1}(x), p)_{g_y}, \quad \forall \, x \in B_{4\delta\sqrt{d}}(y).$$

Given a self-adjoint linear transformation $D \in \mathcal{L}((TM)_y)$, define $q(D, y; \cdot) : B_{4\delta\sqrt{d}}(y) \to \mathbb{R}$ by

$$q(D, y; x) := \tfrac{1}{2}(D \exp_y^{-1}(x), \exp_y^{-1}(x))_{g_y}, \quad \forall \, x \in B_{4\delta\sqrt{d}}(y).$$

Remark 3.14. *An equivalent formulation of the above is the following. In $B_{4\delta\sqrt{d}}(y)$ one obtains coordinate functions $\xi^1, \ldots, \xi^d$ by choosing an orthonormal basis $\{e_i\}$ at $(TM)_y$ and setting*

$$\xi^i(x) := (e_i, (\exp_y)^{-1}(x))_{g_y}.$$

Then, the functions l and q seen in these coordinates are simply linear and quadratic polynomials,

$$l(p, y; x) = p_i \xi^i, \quad q(D, y; x) = \tfrac{1}{2} D_{ij} \xi^i \xi^j.$$

Where p_i and D_{ij} are the components of p and D in the basis $\{e_i\}$.

Moreover, these coordinates are normal, *meaning that the Christoffel symbols vanish at the origin of the system of coordinates $\xi^1 = \ldots = \xi^d = 0$, that is, at the point corresponding to y itself. In particular, it follows that*

$$\nabla l(p, y; y) = p, \quad \nabla^2 l(p, y; y) = 0,$$
$$\nabla q(D, y; y) = 0, \quad \nabla^2 q(D, y; y) = D.$$

Which confirms the idea that l and q play the role of linear and quadratic functions near a point.

The next remark explains an important technical fact. Namely, for each k the set $P_{n,k}^*$ lies in a sufficiently small neighborhood of $\hat{y}_{n,k}$ so that, given p or D, the functions $l(p, \hat{y}_{n,k}; \cdot)$ and $q(D, \hat{y}_{n,k}; \cdot)$ are well defined and smooth in $P_{n,k}^*$.

Remark 3.15. *Let $Q_{n,k}^*$, $q_{n,k}$, be as in the proof of Lemma 3.9, and let $y_{n,k}$, $\hat{y}_{n,k}$ be as introduced in (3.12). As noted in the proof of Lemma 3.9, the common side length of each of the cubes $Q_{n,k}$ is at most 4δ. Since $q_{n,k}$ is the center of $Q_{n,k}$, it follows that $Q_{n,k}^*$ lies inside the ball of radius $(\tfrac{9}{8})2\delta\sqrt{d}$ centered at $q_{n,k}$. In this case, Remark 3.3 says that*

$$P_{n,k}^* \subset B_{(\frac{101}{100})(\frac{9}{4})\delta\sqrt{d}}(y_{n,k}).$$

At the same time, $d(\hat{y}_{n,k}, y_{n,k}) = d(y_{n,k}, \tilde{G}_n) \le \tilde{h}_n$, and $\tilde{h}_n \le \delta/500$ by (3.2). Then, the triangle inequality yields $d(x, \hat{y}_{n,k}) \le d(x, y_{n,k}) + d(\hat{y}_{n,k}, y_{n,k}) \le \operatorname{diam}(P_{n,k}^*) + \tilde{h}_n \le (\tfrac{101}{100})(\tfrac{9}{4})\delta\sqrt{d} + \tfrac{1}{500}\delta$, for $x \in P_{n,k}^*$. This shows that,

$$P_{n,k}^* \subset B_{3\delta\sqrt{d}}(\hat{y}_{n,k}).$$

In light of the discussion at the beginning of this section, we know that $\exp_{\hat{y}_{n,k}}^{-1}$ is well defined and smooth in the larger ball $B_{4\delta\sqrt{d}}(\hat{y}_{n,k})$. Therefore, we conclude that given p or D the functions $l(p, \hat{y}_{n,k}; \cdot)$ and $q(D, \hat{y}_{n,k}; \cdot)$ are well defined functions in $P_{n,k}^*$ which are also smooth.

We refer the reader to the Appendix (Definition A.3, A.8) for the definition of the discrete gradient and discrete Hessian,

$$\nabla_n^1 u(x) \in (TM)_x, \quad \nabla_n^2 u(x) \in \mathcal{L}((TM)_x)$$

defined for every $x \in \tilde{G}_n$. With these, we introduce the local interpolation operators $p_{u,k}^\beta(x)$. These are real valued functions defined as follows, recall $\hat{y}_{n,k}$ from (3.12), then

$$p_{u,k}^\beta : P_{n,k}^* \to \mathbb{R},$$

is defined as follows

$$p_{(u,k)}^\beta(x) := \begin{cases} u(\hat{y}_{n,k}) & \text{if } \beta \in (0,1) \\ u(\hat{y}_{n,k}) + l(\nabla_n^1 u(\hat{y}_{n,k}), \hat{y}_{n,k}; x) & \text{if } \beta \in [1,2) \\ u(\hat{y}_{n,k}) + l(\nabla_n^1 u(\hat{y}_{n,k}), \hat{y}_{n,k}; x) + q(\nabla_n^2 u(\hat{y}_{n,k}), \hat{y}_{n,k}; x) & \text{if } \beta = 2. \end{cases} \quad (3.14)$$

Thus, $p_{(u,k)}^\beta$ yields respectively a constant/first order/second order approximation to u in $P_{n,k}^*$.

Using the chain rule, and the smoothness of $\exp_{\hat{y}_{n,k}}^{-1}$ in $P_{n,k}^*$ (as explained in Remark 3.15), we have the following proposition.

Proposition 3.16. *The following estimate holds with a constant independent of n:*

$$\|p_{(u,k)}^\beta\|_{C^\beta(P_{n,k}^*)} \leq \begin{cases} C|u(\hat{y}_{n,k})| & \text{if } \beta \in (0,1) \\ C\left(|\nabla_n^1 u(\hat{y}_{n,k})|_{g_{\hat{y}_{n,k}}} + |u(\hat{y}_{n,k})|\right) & \text{if } \beta \in [1,2) \\ C\left(|\nabla_n^2 u(\hat{y}_{n,k})|_{g_{\hat{y}_{n,k}}} + |\nabla_n^1 u(\hat{y}_{n,k})|_{g_{\hat{y}_{n,k}}} + |u(\hat{y}_{n,k})|\right) & \text{if } \beta = 2. \end{cases}$$

3.3. **The Whitney extension.** With the partition of unity $\{\phi_{n,k}\}_k$ and the local interpolation operators at hand, we are ready to introduce the Whitney extension operators E_n^β.

Definition 3.17. *For each n, we define*

(1) *The restriction operator $\tilde{T}_n : C_b^\beta(M) \to C(\tilde{G}_n)$, defined in the usual manner*

$$\tilde{T}_n(u,x) := u(x) \ \forall \ x \in \tilde{G}_n.$$

(2) *The extension operator of order β, $\tilde{E}_n^\beta : C(\tilde{G}_n) \to C_b^\beta(M)$ defined by*

$$\tilde{E}_n^\beta(u,x) := \begin{cases} u(x) & \text{if } x \in \tilde{G}_n \\ \sum_k p_{(u,k)}^\beta(x)\phi_{n,k}(x) & \text{if } x \notin \tilde{G}_n. \end{cases}$$

(3) *The "projection" map $\tilde{\pi}_n^\beta : C_b^\beta(M) \mapsto C_b^\beta(M)$ defined by*

$$\tilde{\pi}_n^\beta := \tilde{E}_n^\beta \circ \tilde{T}_n.$$

Remark 3.18. *The fact that $\tilde{E}_n^\beta$ and $\tilde{\pi}_n^\beta$ map to C_b^β is not at all trivial, and it will be proved below in Theorem 3.23.*

Remark 3.19. *On the other hand, it is not difficult to see that if $u \in C(\tilde{G}_n)$ vanishes in $\tilde{G}_n \setminus G_n$ then $\tilde{E}_n^\beta(u)$ vanishes outside M_{n+2} and in particular has compact support. Indeed, by recalling (3.4), (3.5), and the definition of $P_{n,k}$, one can show in this case that for $x \notin M_{n+2}$ and $k \in K_x$ one has that $u(\hat{y}_{n,k}), \nabla_n^1 u(\hat{y}_{n,k})$, and $\nabla_n^2 u(\hat{y}_{n,k})$ all vanish, and thus $\tilde{E}_n^\beta(u,x) = 0$ for $x \notin M_n$.*

Accordingly, if $u \in C_b^\beta(M)$ is a function with compact support, then for n large enough $\tilde{T}_n \circ u$ vanishes in $\tilde{G}_n \setminus G_n$, and it follows $\tilde{\pi}_n^\beta(u)$ is compactly supported inside M_n.

Our immediate goal is controlling the regularity of $\tilde{\pi}_n^\beta u$ in terms of u. For the sake of notation, we shall write for the rest of this section

$$f(x) := \tilde{\pi}_n^\beta(u,x). \tag{3.15}$$

The following propositions, leading to Theorem 3.23, intend show that the maps $\tilde{\pi}_n^\beta$ are well behaved with respect to the C_b^β norm in a manner which is independent of the sets $\tilde{G}_n$. The validity of these bounds in a manner that does not depend on the set $\tilde{G}_n$ is a crucial feature of the Whitney extension.

Among these propositions, we highlight two. First, we have Proposition 3.20, which says $\tilde{\pi}_n^\beta u(x)$ (and its respective derivatives) approach $u(x)$ as x approaches $\tilde{G}_n$. Meanwhile, Proposition 3.22 states that away from $\tilde{G}_n$ the functions $\tilde{\pi}_n^\beta u(x)$ have the correct regularity. Once again, we remind the reader that these estimates are standard for the Whitney extension when $M = \mathbb{R}^d$, and refer to [48, Chap. 6, Section 2]). Here we review their straightforward adaptation to more general M for the sake of completeness.

Proposition 3.20. *Let $x \in M \setminus \tilde{G}_n$ and $u \in C_b^\beta(M)$. There is a universal constant C such that, if $\beta \in [0,3)$ and $f(x) := \tilde{\pi}_n^\beta(u,x)$, we have*

$$|f(x) - f(\hat{x})| \leq C\|u\|_{C^\beta(M)} d(x,\tilde{G}_n)^{\min\{1,\beta\}}$$

Furthermore,

$$|\nabla_a f(x) - \nabla_a f(\hat{x})| \le C\|u\|_{C^\beta(M)} d(x, \tilde{G}_n)^{\min\{1, \beta-1\}}, \quad \text{if } \beta \ge 1,$$

$$|\nabla_{ab}^2 f(x) - \nabla_{ab}^2 f(\hat{x})| \le C\|u\|_{C^\beta(M)} d(x, \tilde{G}_n)^{\min\{1, \beta-2\}}, \quad \text{if } \beta \ge 2.$$

Here, ∇_a and ∇_{ab}^2 are respectively the components of the first and second covariant derivatives of f with respect to an orthonormal frame.

Proof. For the sake of explaining the key ideas of the proof without getting distracted with technicalities, we postpone the proof of the higher derivatives estimates to Section 3.5.

Let $x \in M \setminus \tilde{G}_n$ and $\hat{x} \in \tilde{G}_n$ such that $d(x, \tilde{G}_n) = d(x, \hat{x})$. Recalling that $f = u$ on $\tilde{G}_n$, and using the first property of $\{\phi_{n,k}\}_k$ from Lemma 3.11, we have that $f(\hat{x}) - f(x)$ is equal to

$$u(\hat{x}) - \sum_k u(\hat{y}_{n,k})\phi_{n,k}(x) \quad \text{if } \beta \in [0, 1),$$

$$u(\hat{x}) - \sum_k (u(\hat{y}_{n,k}) + l(\nabla_n^1 u(\hat{y}_{n,k}), \hat{y}_{n,k}; x))\phi_{n,k}(x) \quad \text{if } \beta \in [1, 2),$$

$$u(\hat{x}) - \sum_k (u(\hat{y}_{n,k}) + l(\nabla_n^1 u(\hat{y}_{n,k}), \hat{y}_{n,k}; x) + q(\nabla_n^2 u(\hat{y}_{n,k}), \hat{y}_{n,k}; x))\phi_{n,k}(x) \quad \text{if } \beta \in [2, 3).$$

Let us consider each case individually. If $\beta \in [0, 1)$, the identity $\sum_k \phi_{n,k}(x) = 1$ allows us to write

$$f(\hat{x}) - f(x) = \sum_k (u(\hat{x}) - u(\hat{y}_{n,k}))\phi_{n,k}(x)$$

$$= \sum_{k \in K_x} (u(\hat{x}) - u(\hat{y}_{n,k}))\phi_{n,k}(x).$$

The set K_x being the one defined in (3.8). Then, the triangle inequality and $|\phi_{n,k}| \le 1$ yields

$$|f(\hat{x}) - f(x)| \le \|u\|_{C^\beta} \sum_{k \in K_x} d(\hat{x}, \hat{y}_{n,k})^\beta.$$

The triangle inequality says that

$$d(\hat{x}, \hat{y}_{n,k}) \le d(\hat{x}, x) + d(x, y_{n,k}) + d(y_{n,k}, \hat{y}_{n,k}),$$

where, according to (3.12), $d(y_{n,k}, \hat{y}_{n,k}) = d(y_{n,k}, \tilde{G}_n)$. In this case, we see that $d(y_{n,k}, \hat{y}_{n,k}) \le d(x, \tilde{G}_n) + d(x, y_{n,k})$, and we conclude that

$$d(\hat{x}, \hat{y}_{n,k}) \le 2d(x, \tilde{G}_n) + 2\text{diam}(P_{n,k}^*).$$

Then, using Remark 3.10, it follows that

$$d(\hat{x}, \hat{y}_{n,k}) \le 16d(x, \tilde{G}_n) \ \forall \ k \in K_x. \tag{3.16}$$

Furthermore, recall (3.9) which says that $\#K_x \le N$. All of this leads to the estimate

$$|f(\hat{x}) - f(x)| \le C\|u\|_{C^\beta} d(x, \tilde{G}_n)^\beta.$$

Instead, if $\beta \in [1, 2)$, we have

$$f(\hat{x}) - f(x) = \sum_k (u(\hat{x}) - u(\hat{y}_{n,k}))\phi_{n,k}(x) + \sum_k l(\nabla_n^1 u(\hat{y}_{n,k}), \hat{y}_{n,k}; x))\phi_{n,k}(x)$$

$$= \sum_{k \in K_x} (u(\hat{x}) - u(\hat{y}_{n,k}))\phi_{n,k}(x) + \sum_{k \in K_x} l(\nabla_n^1 u(\hat{y}_{n,k}), \hat{y}_{n,k}; x))\phi_{n,k}(x).$$

Just as before, using the triangle inequality and the fact that $|\phi_n, k| \leq 1$, it follows that

$$|f(\hat{x}) - f(x)| \leq \sum_{k \in K_x} |u(\hat{x}) - u(\hat{y}_{n,k})| + \sum_{k \in K_x} |l(\nabla_n^1 u(\hat{y}_{n,k}), \hat{y}_{n,k}; x)|$$

$$\leq \|u\|_{C^1} \sum_{k \in K_x} d(\hat{x}, \hat{y}_{n,k}) + \sum_{k \in K_x} |\nabla_n^1 u(\hat{y}_{n,k})|_{g_x} d(\hat{y}_{n,k}, x).$$

Proposition A.15 in the Appendix guarantees that $|\nabla_n^1 u(\hat{y}_{n,k})|_{g_x} \leq C\|u\|_{C^1}$, for some universal C. Using this bound, the fact that $\|u\|_{C^1} \leq \|u\|_{C^\beta}$ for $\beta \geq 1$, and the last inequality above, it follows that

$$|f(\hat{x}) - f(x)| \leq C\|u\|_{C^\beta} \sum_{k \in K_x} d(\hat{x}, \hat{y}_{n,k}).$$

From this point one argues exactly as done for $\beta \in [0, 1)$ to conclude that

$$|f(\hat{x}) - f(x)| \leq C\|u\|_{C^\beta} d(x, \tilde{G}_n).$$

The proof for $\beta \in [2, 3)$ is entirely analogous, and we leave the details to the reader. This proves the first estimate.

As mentioned above, we refer to Section 3.5 for the proofs for $\nabla_a f$ and $\nabla_{ab}^2 f$.

$\square$

We delay the technical proof of the following auxiliary proposition until the Appendix B.

Proposition 3.21. *Let $x \in M \setminus \tilde{G}_n$ and $u \in C^\beta$. There is a universal constant C such that the following bounds hold. First, if $0 \leq \beta < 1$,*

$$|\nabla(\tilde{E}_n^\beta \circ \tilde{T}_n)u(x)| \leq C\|u\|_{C^\beta} d(x, \tilde{G}_n)^{\beta-1}.$$

If $1 \leq \beta < 2$, we have

$$|\nabla^2(\tilde{E}_n^\beta \circ \tilde{T}_n)u(x)| \leq C\|u\|_{C^\beta} d(x, \tilde{G}_n)^{\beta-2}.$$

Finally, if $2 \leq \beta < 3$, we have

$$|\nabla^3(\tilde{E}_n^\beta \circ \tilde{T}_n)u(x)| \leq C\|u\|_{C^\beta} d(x, \tilde{G}_n)^{\beta-3}.$$

Using Proposition 3.21 it is easy to show that $\tilde{\pi}_n^\beta(u, x)$ is regular away from $\tilde{G}_n$.

Proposition 3.22. *Let x_0 and r be fixed such that $B_{4r}(x_0) \subset M \setminus \tilde{G}_n$. Then, given $u \in C_b^\beta(M)$, for $f = \tilde{\pi}_n^\beta u$ we have the estimate*

$$[\nabla^i f]_{C^{\beta-i}(B_r(x_0))} \leq C\|u\|_{C^\beta(M)}, \quad \text{for } \beta \in [i, i+1), \ \ i = 0, 1, 2.$$

Proof. Here, we only prove the statement for $\beta \in (0, 1)$, and we defer the remaining two cases until later, in Section 3.5.

The case $\beta \in (0, 1)$. Let $x_1, x_2 \in B_r(x_0)$, and $x(t) : [0, L] \to M$ a minimal geodesic between them, parametrized with respect to arc length, so $L = d(x_1, x_2)$. Then, by the triangle inequality

$$d(x(t), x_0) \leq d(x(t), x_1) + d(x_1, x_0)$$
$$\leq d(x_1, x_2) + d(x_1, x_0)$$
$$\leq d(x_1, x_0) + d(x_2, x_0) + d(x_1, x_0) \leq 3r.$$

In particular, it follows that $d(x(t), \tilde{G}_n) \geq r$ for all $t \in [0, L]$. Then,

$$
\begin{aligned}
|f(x_1) - f(x_2)| &= \int_0^L \frac{d}{dt} f(x(t))\, dt \\
&= \int_0^L (\nabla f(x(t)), \dot{x}(t))\, dt \leq C\|u\|_{C^\beta(M)} r^{\beta-1} L,
\end{aligned}
$$

the last inequality being thanks to Proposition 3.21 and the fact that $d(x(t), \tilde{G}_n) \geq r$ for all t. Since $d(x_1, x_2) \leq 2r$ and $\beta - 1 < 0$, we conclude that

$$
|f(x_1) - f(x_2)| \leq C\|u\|_{C^\beta(M)} r^{\beta-1} d(x_1, x_2) \leq C\|u\|_{C^\beta(M)} d(x_1, x_2)^\beta.
$$

The remaining cases (those with $\beta \geq 1$) are proved in Section 3.5. $\square$

For readers with a background in elliptic PDE, and in particular, those not familiar with the Whitney extension, it may be useful to make a naïve but possibly illustrative analogy with the derivation of global regularity estimates for solutions of elliptic equations. Proposition 3.22 is a kind of interior estimate, where in order to bound the solution in a ball, one needs the "equation" (here, being the extension) to take place in a bigger ball. Likewise, Proposition 3.20 is analogous to estimates at the boundary. In this sense, $\tilde{G}_n$ is the kind of boundary and u provides the boundary values. Furthermore, the way these two estimates are "glued" in the next proof bears a great resemblance to the proof of global regularity estimates for elliptic equations from interior and boundary estimates.

With the previous two estimates in hand, we are ready to prove that $\tilde{\pi}_n^\beta$ is a bounded map from C_b^β to C_b^β.

Theorem 3.23. *If $u \in C_b^\beta(M)$, then $\tilde{\pi}_n^\beta \in C_b^\beta(M)$ and, for some universal C,*

$$
\|\tilde{\pi}_n^\beta u\|_{C^\beta(M)} \leq C\|u\|_{C^\beta(M)}.
$$

Proof. As before we write $f = \tilde{\pi}_n^\beta u$. Let us first show

$$
\|f\|_{L^\infty} \leq C\|u\|_{C^\beta},
$$

for all $\beta \in [0, 2]$. If $x \in M \setminus \tilde{G}_n$, then

$$
f(x) = \sum_k p_{(u,k)}^\beta(x) \phi_{n,k}(x).
$$

Proposition 3.16 implies that

$$
\sup_{x \in P_{n,k}^*} |p_{(u,k)}^\beta(x)| \leq C\|u\|_{C^\beta}.
$$

Then,

$$
\sup_{x \in M} |f(x)| \leq C\|u\|_{C^\beta}.
$$

Let us now prove $f(x)$ has the right regularity. The argument is separated in cases depending on β, in each case the proof will consist in "gluing" the interior and boundary estimates proved for $\tilde{\pi}_n^\beta$ in Propositions 3.22 and 3.20.

The case $\beta \in [0, 1)$. Let $x_1, x_2 \in M \setminus \tilde{G}_n$. If $4d(x_1, x_2) < \max\{d(x_1, \tilde{G}_n), d(x_2, \tilde{G}_n)\}$, then we can apply Proposition 3.22 and conclude that

$$
|f(x_1) - f(x_2)| \leq C\|u\|_{C^\beta} d(x_1, x_2)^\beta.
$$

Consider on the other hand the case $4d(x_1, x_2) \geq \max\{d(x_1, \tilde{G}_n), d(x_2, \tilde{G}_n)\}$, then, for $\hat{x}_i \in \tilde{G}_n$ such that $d(x_i, \hat{x}_i) = d(x_i, \tilde{G}_n)$ we have

$$|f(x_1) - f(x_2)| \leq |f(x_1) - f(\hat{x}_1)| + |f(\hat{x}_1) - f(\hat{x}_2)|$$
$$+ |f(\hat{x}_2) - f(x_2)|.$$

Applying Proposition 3.20 to the first and third terms, and recalling that $f(\hat{x}_i) = u(\hat{x}_i)$,

$$|f(x_1) - f(x_2)| \leq C\|u\|_{C^\beta}(d(x_1, \hat{x}_1)^\beta + d(x_2, \hat{x}_2)^\beta) + \|u\|_{C^\beta} d(\hat{x}_1, \hat{x}_2)^\beta.$$

Given that in this case we have $d(x_1, \hat{x}_1) + d(x_2, \hat{x}_2) \leq 8d(x_1, x_2)$, we can use the triangle inequality to conclude that $d(\hat{x}_1, \hat{x}_2) \leq 10d(x_1, x_2)$, therefore

$$|f(x_1) - f(x_2)| \leq C\|u\|_{C^\beta} d(x_1, x_2)^\beta.$$

Combining the above estimates we obtain the desired bound for $\beta \in [0, 1)$.

The case $\beta \in [1, 2)$. Let us show first that if $u \in C_b^1$, then $f \in C_b^1$, and $\nabla f(x) = \nabla_n^1 u(x)$ for every $x \in \tilde{G}_n$. In order to do this, we shall show that $\nabla_a f(x)$ is continuous in x for every index a. Note that

$$\nabla_a f(x) = \sum_k \nabla_a \left(l(\nabla_n^1 u(\hat{y}_{n,k}), \hat{y}_{n,k}; x) \right) \phi_{n,k}(x)$$
$$+ \sum_k \left(u(\hat{y}_{n,k}) + l(\nabla_n^1 u(\hat{y}_{n,k}), \hat{y}_{n,k}; x) \right) \nabla_a \left(\phi_{n,k}(x) \right).$$

Recall that any point $x_0 \in M \setminus \tilde{G}_n$ has a neighborhood where at most N of the terms in the above sums are non-zero. Since each term is continuous in x, it follows that $\nabla_a f(x)$ is continuous in $M \setminus \tilde{G}_n$. It remains to show the continuity for a point $x_0 \in \tilde{G}_n$. Let us also recall Remark 3.10, which says that for any $x \in M \setminus \tilde{G}_n$ we have

$$\text{diam}(P_{n,k}^*) \leq 7d(x, \tilde{G}_n), \quad \forall k \in K_x,$$

where K_x was defined in (3.8). Then, since $d(x_0, \tilde{G}_n \setminus \{x_0\}) > 0$, it follows that if x is sufficiently close to $x_0 \in \tilde{G}_n$, then $x \in P_{n,k}^*$ implies that there is a unique closest point in $\tilde{G}_n$ to $y_{n,k}$, x_0 itself. In other words (recall $\hat{y}_{n,k}$ was defined in (3.12)),

$$\hat{y}_{n,k} = x_0 \quad \forall x \in K_x.$$

This means that if x is sufficiently close to x_0, $\nabla_a f(x)$ has the form

$$\nabla_a f(x) = \sum_k \nabla_a \left(l(\nabla_n^1 u(x_0), x_0; x) \right) \phi_{n,k}(x)$$
$$+ \sum_k \left(u(x_0) + l(\nabla_n^1 u(x_0), x_0; x) \right) \nabla_a \left(\phi_{n,k}(x) \right)$$
$$= \nabla_a \left(l(\nabla_n^1 u(x_0), x_0; x) \right).$$

Where we used that $\phi_{n,k}$ is a partition of unity: (1) in Lemma 3.11 and the identity (3.13) to obtain the last identity. From the last inequality we see that as $x \to x_0$ we have $\nabla_a f(x) \to \nabla_a f(x_0) = \nabla_n^1 u(x_0)$ and thus $u \in C_b^1$.

Next, let us show the Hölder bound for $\beta \in (1, 2)$. Let $x_1, x_2 \in M_n$. If $4d(x_1, x_2) < \max\{d(x_1, \tilde{G}_n), d(x_2, \tilde{G}_n)\}$, then we can apply Proposition 3.22 and conclude that

$$|\nabla_a f(x_1) - \nabla_a f(x_2)| \leq C\|u\|_{C^\beta} d(x_1, x_2)^{\beta-1}.$$

If instead we have $4d(x_1, x_2) \geq \max\{d(x_1, \tilde{G}_n), d(x_2, \tilde{G}_n)\}$, then the triangle inequality yields

$$|\nabla_a f(x_1) - \nabla_a f(x_2)| \leq |\nabla_a f(x_1) - \nabla_a f(\hat{x}_1)| + |\nabla_a f(\hat{x}_1) - \nabla_a f(\hat{x}_2)| + |\nabla_a f(\hat{x}_2) - \nabla_a f(x_2)|.$$

Therefore, by Proposition 3.20

$$|\nabla_a f(x_1) - \nabla_a f(x_2)| \leq C\|u\|_{C^\beta} d(x_1, \hat{x}_1)^{\beta-1} + |\nabla_a f(\hat{x}_1) - \nabla_a f(\hat{x}_2)|.$$

On the other hand, since in a neighborhood of $\hat{x}_i$ we have $\nabla \tilde{\pi}_n^\beta u(x) = \nabla_n^1 u(\hat{x}_i)$ (see Def 3.17 and A.3), the first half of Proposition A.16 says that

$$|\nabla_a f(\hat{x}_1) - \nabla_a f(\hat{x}_2)| \leq C\|u\|_{C^\beta} d(\hat{x}_1, \hat{x}_2)^{\beta-1}.$$

Hence, it follows that

$$|\nabla_a f(x_1) - \nabla_a f(x_2)| \leq C\|u\|_{C^\beta} d(x_1, x_2)^{\beta-1}, \ \ \forall\, x_1, x_2 \in M_n.$$

The case $\beta \in [2, 3)$**.** An argument entirely analogous to that used for $\beta \in [1, 2)$ shows that if $u \in C_b^2$, then $f \in C_b^2$, with $\nabla^i f(x) = \nabla_n^i u(x)$ for $i = 1, 2$ and every $x \in \tilde{G}_n$.

Then, let us prove that $\nabla_{ab}^2 f(x)$ are Hölder continuous for $\beta > 2$. As in the previous cases, suppose first that $4d(x_1, x_2)$ is no larger than $\max\{d(x_1, \tilde{G}_n), d(x_2, \tilde{G}_n)\}$. In this case, Proposition 3.22 yields

$$|\nabla_{ab}^2 f(x_1) - \nabla_{ab}^2 f(x_2)| \leq C\|u\|_{C^\beta} d(x_1, x_2)^{\beta-2}.$$

Consider now the case where $4d(x_1, x_2) \geq \max\{d(x_1, \tilde{G}_n), d(x_2, \tilde{G}_n)\}$. We shall argue in a parallel manner to the case $\beta \in [1, 2)$. First off, we have

$$|\nabla_{ab}^2 f(x_1) - \nabla_{ab}^2 f(x_2)| \leq |\nabla_{ab}^2 f(x_1) - \nabla_{ab}^2 f(\hat{x}_1)| + |\nabla_{ab}^2 f(\hat{x}_1) - \nabla_{ab}^2 f(\hat{x}_2)|$$
$$+ |\nabla_{ab}^2 f(\hat{x}_2) - \nabla_{ab}^2 f(x_2)|.$$

Next, since in a neighborhood of $\hat{x}_i$ we have $\nabla^2 \tilde{\pi}_n^\beta u(x) = \nabla_n^2 u(\hat{x}_i)$ (see Def 3.17 and A.3), the second part of Proposition A.16 says that

$$|\nabla_{ab}^2 f(\hat{x}_1) - \nabla_{ab}^2 f(\hat{x}_2)| \leq C\|u\|_{C^\beta} d(\hat{x}_1, \hat{x}_2)^{\beta-2}.$$

Since $d(\hat{x}_1, \hat{x}_2) \leq 10 d(x_1, x_2)$ and $4d(x_1, x_2)$ is larger than $d(x_1, \tilde{G}_n)$ and $d(x_2, \tilde{G}_n)$,

$$|\nabla_{ab}^2 f(x_1) - \nabla_{ab}^2 f(x_2)| \leq C\|u\|_{C^\beta} d(x_1, x_2)^{\beta-2}, \ \ \forall\, x_1, x_2 \in M_n.$$

This concludes the proof of the theorem. $\qquad\square$

The operators $\tilde{\pi}_n^\beta$ also enjoy the useful property of having a finite range of dependence, a property that will play a role in some arguments of Section 4.

Lemma 3.24. *Assume that* $\beta \in [0, 3)$*. There is a universal constant* C*, such that if* $K, K' \subset M$ *are open sets such that* $d(K', M \setminus K) \geq r + 10^3 h_n$*, then*

$$\|\tilde{\pi}_n^\beta u - \tilde{\pi}_n^\beta v\|_{C^\beta(K)} \leq C(1 + r^{-\beta})\|u - v\|_{C^\beta(K')}, \ \ \forall\, u, v \in C_b^\beta(M).$$

Proof. Given K and K' with $d(K', M \setminus K) \geq r + 10^3 h_n$. It will be convenient to introduce an "intermediate" set,

$$\tilde{K} := \{x \in M \mid d(x, K) \leq 400 \tilde{h}_n\}.$$

In other words, $\tilde{K}$ is the closure of the $400\tilde{h}_n$-neighborhood of K. Thanks to the triangle inequality and the assumption on K and K' we have $d(\tilde{K}, M \setminus K') \geq r$.

Next, we construct a function $\eta = \eta_{\tilde{K}, K'}$ such that

$$0 \leq \eta \leq 1, \ \ \eta \equiv 1 \text{ in } \tilde{K}, \ \ \eta \equiv 0 \text{ in } M \setminus K'.$$

It is not difficult to see that η can be chosen so that (for some universal C)

$$\|\eta\|_{C^\beta(M)} \leq C\left(1 + \frac{1}{d(K', M \setminus K'')^\beta}\right) \leq C\left(1 + r^{-\beta}\right).$$

In fact, this can be done using the Whitney decomposition itself, see the "regularized distance" construction in [48, Chapter 6, Section 2.1] .

Let $u, v \in C_b^\beta$. As proved later in Proposition A.11 (see Appendix), if $x \in M$ and $w \in C_b^\beta$ is identically zero in $B_{400\tilde{h}_n}(x)$, then

$$\tilde{\pi}_n^\beta w \equiv 0 \text{ in } B_{100\tilde{h}_n}(x).$$

We apply this to the function $w = (u - v) - \eta(u - v) \in C_b^\beta$, and to any $x \in K$, making use of the fact that w vanishes in $\tilde{K}$. It follows that for every x in a small neighborhood of K we have

$$\tilde{\pi}_n^\beta(u, x) = \tilde{\pi}_n^\beta(\eta u, x).$$

In this case, it is clear that $\|\tilde{\pi}_n^\beta u\|_{C^\beta(K)} \leq \|\tilde{\pi}_n^\beta(\eta u)\|_{C^\beta(M)}$. Then, by Theorem 3.23,

$$\|\tilde{\pi}_n^\beta u\|_{C^\beta(K)} \leq C\|\eta u\|_{C^\beta(M)}.$$

Using that $\eta \equiv 0$ in $M \setminus K'$, the Leibniz rule, and the bound on $\|\eta\|_{C_b^\beta}$, we conclude that

$$\|\tilde{\pi}_n^\beta u\|_{C^\beta(K)} \leq C(1 + r^{-\beta})\|u\|_{C^\beta(K')}.$$

$$\square$$

Lemma 3.25. *There is universal constant C such that for any $u \in C_b^3(M)$ we have*

$$\|\tilde{\pi}_n^\beta u - u\|_{C^\beta(M)} \leq Ch_n^\gamma\|u\|_{C^3(M)}.$$

Here, $\gamma = i - \beta$ if $\beta \in [i - 1, i)$, for $i = 1, 2, 3$.

Remark 3.26. *It will be evident from the proof, that the C_b^3 norm on the right hand side can be weakened when $\beta < 2$. Here we simply state the lemma with C_b^3 for the sake of brevity.*

Proof. Recall we are writing f for $\tilde{\pi}_n^\beta u$, and that given $x \in M$ we write $\hat{x}$ for an element of $\tilde{G}_n$ for which $d(x, \hat{x}) = d(x, \tilde{G}_n)$.

We begin by estimating $\|f - u\|_{L^\infty(M)}$. Since $f \equiv u$ in $\tilde{G}_n$, we have

$$\begin{aligned}
|f(x) - u(x)| &\leq |f(x) - f(\hat{x})| + |u(\hat{x}) - u(x)| \\
&\leq \|f\|_{C^1}d(x, \hat{x}) + \|u\|_{C^1}d(x, \hat{x}) \\
&\leq C\|u\|_{C^3}d(x, \hat{x}).
\end{aligned}$$

Then, regardless of β we have,

$$\sup_{x \in M} |f(x) - u(x)| \leq \sup_{x \in M} C\|u\|_{C^3}d(x, \tilde{G}_n) \leq C\|u\|_{C^3}\tilde{h}_n.$$

Note this already shows $\|f - u\|_{L^\infty(M)}$ goes to zero with a rate determined by $\tilde{h}_n$. To bound $\|f - u\|_{C^\beta(M)}$, it remains to control the Hölder seminorm of either u, ∇u, or $\nabla^2 u$, depending on the range where β lies. Let us treat the case $\beta \in (0, 1)$ first, which means we must estimate $[u]_{C^\beta}$. We defer the proof of the remaining two cases ($\beta \in [1, 2)$ and $\beta \in [2, 3)$ until later, in Section 3.5.

The case $\beta \in [0, 1)$. Let $x_1, x_2 \in K$. We shall bound

$$\frac{|f(x_1) - u(x_1) - (f(x_2) - u(x_2))|}{d(x_1, x_2)^\beta}.$$

In what follows, it will be useful to fix $\hat{x}_i \in \tilde{G}_n$ such that $d(x_i, \hat{x}_i) = d(x_i, \tilde{G}_n)$ for $i = 1, 2$. First, suppose that $d(x_1, x_2) \leq \max\{d(x_1, \hat{x}_1), d(x_2, \hat{x}_2)\}$, then

$$|f(x_1) - u(x_1) - (f(x_2) - u(x_2))| \leq \|f - u\|_{C^1(M)}d(x_1, x_2),$$

$$\frac{|f(x_1) - u(x_1) - (f(x_2) - u(x_2))|}{d(x_1, x_2)^\beta} \le \frac{\|f - u\|_{C^1} d(x_1, x_2)}{d(x_1, x_2)^\beta}.$$

Since $\|f - u\|_{C^1} \le \|f\|_{C^1} + \|u\|_{C^1}$, Theorem 3.23 yields $\|f - u\|_{C^1} \le C\|u\|_{C^1}$.

Using that $\beta < 1$, we have $d(x_1, x_2)^{1-\beta} \le \max\{d(x_1, \hat{x}_1)^{1-\beta}, d(x_2, \hat{x}_2)^{1-\beta}\} \le \tilde{h}_n^{1-\beta}$. Then, for this case we have

$$\frac{|f(x_1) - u(x_1) - (f(x_2) - u(x_2))|}{d(x_1, x_2)} \le C\|u\|_{C^1(M)} \tilde{h}_n^{1-\beta}.$$

Second, let us consider the case where $d(x_1, x_2) > \max\{d(x_1, \hat{x}_1), d(x_2, \hat{x}_2)\}$. Then, we proceed by writing

$$|f(x_1) - u(x_1) - (f(x_2) - u(x_2))| \le |f(x_1) - u(x_1)| + |f(x_2) - u(x_2)|.$$

Next, due to $f = u$ in $\tilde{G}_n$, for $i = 1, 2$ we have

$$|f(x_i) - u(x_i)| = |f(x_i) - u(x_i) - (f(\hat{x}_i) - u(\hat{x}_i))| \le (\|f\|_{C^1} + \|u\|_{C^1})d(x_i, \hat{x}_i),$$

and since $\|f\|_{C^1} \le C\|u\|_{C^1}$ (Theorem 3.23), we conclude that

$$|f(x_i) - u(x_i)| \le C\|u\|_{C^1} d(x_i, \hat{x}_i), \quad i = 1, 2.$$

The assumption $d(x_1, x_2) > \max\{d(x_1, \hat{x}_1), d(x_2, \hat{x}_2)\}$ yields that $d(x_i, \hat{x}_i) \le d(x_i, \hat{x}_i)^{1-\beta} d(x_1, x_2)^\beta$ and furthermore $d(x_i, \hat{x}_i) \le \tilde{h}_n^{1-\beta} d(x_1, x_2)^\beta$ both for $i = 1, 2$ (this uses again that $\beta < 1$, since it means that $t \to t^{1-\beta}$ is nondecreasing). Therefore,

$$|f(x_i) - u(x_i)| \le C\|u\|_{C^1} d(x_i, \hat{x}_i)^{1-\beta} d(x_1, x_2)^\beta \le C\|u\|_{C^1} \tilde{h}_n^{1-\beta} d(x_1, x_2)^\beta.$$

Then, in this case we also conclude that

$$\frac{|f(x_1) - u(x_1) - (f(x_2) - u(x_2))|}{d(x_1, x_2)^\beta} \le C\|u\|_{C^1} \tilde{h}_n^{1-\beta}.$$

Combining the estimates for either case, we conclude that

$$[f]_{C^\beta(M)} = \sup_{x_1 \neq x_2} \frac{|f(x_1) - u(x_1) - (f(x_2) - u(x_2))|}{d(x_1, x_2)^\beta} \le C\|u\|_{C^1} \tilde{h}_n^{1-\beta}.$$

Now, since $\tilde{h}_n \le 1$ always, we have $\tilde{h}_n \le \tilde{h}_n^{1-\beta}$ for all n, therefore, we have proved that

$$\begin{aligned}
\|f - u\|_{C^\beta(M)} &:= \|f - u\|_{L^\infty(M)} + [f]_{C^\beta(M)} \\
&\le C\|u\|_{C^1(M)} \tilde{h}_n + C\|u\|_{C^1(M)} \tilde{h}_n^{1-\beta}, \\
&\le C\|u\|_{C^1(M)} \tilde{h}_n^{1-\beta}.
\end{aligned}$$

For the proofs for $\beta \ge 1$, see Section 3.5.

$\square$

3.4. The Whitney extension is almost order preserving.
When $\beta \in [0, 1)$ it turns out that $\tilde{E}_n^\beta$ preserves the ordering of functions.

Remark 3.27. *Suppose $\beta < 1$. If $u, v \in C(\tilde{G}_n)$ and $u(x) \le v(x) \ \forall \, x \in \tilde{G}_n$, then*

$$\tilde{E}_n^\beta(u, x) \le \tilde{E}_n^\beta(v, x) \ \ \forall \, x \in M.$$

Indeed, take $u \le v$ in $\tilde{G}_n$ and $x \in M \setminus \tilde{G}_n$. Then, from the definition of E_n^β when $\beta < 1$, we have

$$\begin{aligned}
u(x) &= \sum_k u(\hat{y}_{n,k}) \phi_{n,k}(x) \\
&\le \sum_k v(\hat{y}_{n,k}) \phi_{n,k}(x) = v(x),
\end{aligned}$$

where we used that $\phi_{n,k} \geq 0$ and $u(\hat{y}_{n,k}) \leq v(\hat{y}_{n,k})$ for every k.

It is unclear –or rather unlikely– that the operators continue to be order preserving for β larger than 1. However, when considering the extension among functions in $\tilde{G}_n$ that are sufficiently regular (in the sense that they are the restriction of smooth functions) then E_n^β preserves the ordering up to a small correcting function whose C^β norm vanishes as n goes to infinity. It is worthwhile to point out to a recent preprint of Fefferman, Israel, and Luli [22], where a closely related question, the interpolation of functions with a positivity constraint, is studied.

The next proposition -which is chiefly needed for Lemma 3.29 below- quantifies the intuitive fact that if $u \in C_b^3$ vanishes at a point $x_0 \in \tilde{G}_n$, and $u \geq 0$ everywhere in $\tilde{G}_n$, then the gradient and the negative eigenvalues of the Hessian of $\tilde{\pi}_n^\beta u$ at x_0 must be small when n is large.

We will need a cutoff function in the next few proofs. We fix one, and call it ϕ_0 such that

$$\phi_0 : \mathbb{R} \to \mathbb{R}, \ 0 \leq \phi_0 \leq 1, \ \phi_0 \text{ smooth}, \ \phi_0 \equiv 1 \text{ in } [-1,1], \text{ and } \phi_0 \equiv 0 \text{ in } \mathbb{R} \setminus [2,2]. \qquad (3.17)$$

Proposition 3.28. *Let $w \in C_b^3(M)$ be nonnegative in $\tilde{G}_n$ and such that $w(x_0) = 0$ at some $x_0 \in \tilde{G}_n$. Then, with some universal C we have*

$$|\nabla \tilde{\pi}_n^\beta w(x_0)|_{g_{x_0}} \leq C\|w\|_{C^3} h_n \text{ if } \beta \geq 1.$$
$$|(\nabla^2 \tilde{\pi}_n^\beta w(x_0))_-|_{g_{x_0}} \leq C\|w\|_{C^3} h_n \text{ if } \beta \geq 2.$$

Here, we recall that $\tilde{h}_n$ is as defined in (3.2), and note that for a given matrix D, D_- denotes its negative part.

Proof of Proposition 3.28. According to 2) in Proposition A.12 if $d(x, x_0) \leq 4\delta\sqrt{d}$ (recall δ was defined in Remark 3.3), then

$$|w(x) - w(x_0) - l(\nabla \tilde{\pi}_n^\beta w(x_0), x_0; x)| \leq \|u\|_{C^2} d(x, x_0)^2.$$

Then, using that $w(x_0) = 0$ and $w(x) \geq 0$ for all $x \in \tilde{G}_n$, we conclude that

$$0 \leq l(\nabla \tilde{\pi}_n^\beta w(x_0), x_0; x) + \|\tilde{\pi}_n^\beta w\|_{C^2} d(x, x_0)^2, \ \ \forall \, x \in \tilde{G}_n \cap B_{4\delta\sqrt{d}}(x_0). \qquad (3.18)$$

Then, using that $d(x_0, \tilde{G}_n) \leq \tilde{h}_n$, it is not hard to see there is some $x_1 \in \tilde{G}_n$ with $d(x_1, x_0) \leq 4\delta\sqrt{d}$ such that (for some universal C),

$$l(\nabla \tilde{\pi}_n^\beta w(x_0), x_0; x_1) \leq -C^{-1}|\nabla \tilde{\pi}_n^\beta w(x_0)|_{g_{x_0}}|\exp_{x_0}^{-1}(x_1)|_{g_{x_0}}.$$

Then, using (3.18) with this x_1, and, Theorem 3.23, we see that

$$|\nabla \tilde{\pi}_n^\beta w(x_0)|_{g_{x_0}} \leq C\|\tilde{\pi}_n^\beta w\|_{C^2} d(x, x_0) \leq C\|w\|_{C^2} h_n,$$

proving the first estimate.

Next, we prove the second estimate. Let $\beta \geq 2$. Assume first that $\nabla \tilde{\pi}_n^\beta w(x_0) = 0$. Then, we may use Proposition A.12 as before to obtain,

$$0 \leq q(\nabla^2 \tilde{\pi}_n^\beta w(x_0), x_0; x) + \|\tilde{\pi}_n^\beta w\|_{C^3} d(x, x_0)^3, \ \ \forall \, x \in \tilde{G}_n \cap B_{4\delta\sqrt{d}}(x_0). \qquad (3.19)$$

Then, as in the previous case, one can see there is some x_1 with $d(x_1, x_0) \leq 4\delta\sqrt{d}$ such that

$$q(\nabla^2 \tilde{\pi}_n^\beta w(x_0), x_0; x_1) \leq -C^{-1}|(\nabla^2 \tilde{\pi}_n^\beta w(x_0))_-|_{g_x}|\exp_{x_0}^{-1}(x_1)|_{g_x}$$
$$= -C^{-1}|(\nabla^2 \tilde{\pi}_n^\beta w(x_0))_-|_{g_x} d(x_1, x_0)^2.$$

Using (3.19) with this x_1, we conclude that

$$|(\nabla^2 \tilde{\pi}_n^\beta w(x_0))_-|_{g_x} \leq C\|\tilde{\pi}_n^\beta w\|_{C^3} d(x_1, x_0) \leq C\|w\|_{C^3} \tilde{h}_n.$$

If $\nabla \tilde{\pi}_n^\beta w(x_0) \neq 0$, we apply the above argument to the function

$$\tilde{w} = \tilde{\pi}_n^\beta(w, x) - l(\nabla \tilde{\pi}_n^\beta(w)(x_0), x_0; x), \text{ defined in } B_{4\delta\sqrt{d}}(x_0).$$

As explained at the end of Remark 3.14, we always have $\nabla^2 l(\nabla \tilde{\pi}_n^\beta(w)(x_0), x_0; x_0) = 0$, thus the Hessian at x_0 is not perturbed by this change. Moreover, it is clear that $\|\tilde{w}\|_{C^\beta} \leq C\|w\|_{C^\beta}$, with a universal C, and the proof follows. $\qquad\square$

Using Proposition 3.28, we now show the existence of a kind of "corrector" to the Whitney extension, in the sense that $\tilde{\pi}_n^\beta w$ plus this corrector is non-negative in M whenever w is non-negative in $\tilde{G}_n$, the corrector having a C^β norm which vanishes as n goes to infinity.

Lemma 3.29. *Fix $\beta \in [0, 3)$. Let $w \in C_b^3(M)$ and suppose that there is some $x_0 \in \tilde{G}_n$ such that*

$$w(x) \geq 0, \ \forall \, x \in \tilde{G}_n,$$

$$w(x_0) = 0, \ x_0 \in \tilde{G}_n.$$

Then, there is a function R_{β,n,w,x_0} with $R_{\beta,n,w,x_0}(x_0) = 0$ and such that

$$(\tilde{\pi}_n^\beta w)(x) + R_{\beta,n,w,x_0}(x) \geq 0 \ \ \forall \, x \in M,$$

$$\|R_{\beta,n,w,x_0}\|_{C^\beta(M)} \leq C h_n^\gamma \|w\|_{C^3(M)}.$$

Here, $\gamma := i - \beta$ for $\beta \in [i-1, i)$, $i = 2, 3$, while $\tilde{h}_n$ is as in (3.2).

Proof. As in previous proofs, for any $x \in M$, let $\hat{x} \in \tilde{G}_n$ be a point such that $d(x, \tilde{G}_n) = d(x, \hat{x})$. Further, by Remark 3.27 the lemma is trivial with $R_{\beta,n,w,x_0} \equiv 0$ in the case $\beta \in [0, 1)$.

The case $\beta \in [1, 2)$. First, we must take care of the first order part of w near x_0, by writing

$$\tilde{w}_n(x) = \tilde{\pi}_n^\beta w(x) - l(\nabla \tilde{\pi}_n^\beta w(x_0), x_0; x)\phi_0(d(x, x_0)^2)$$

Where $\phi_0(t)$ is a smooth function which is identically equal to 1 for $t \leq (\delta/4)^2$ and vanishes for $t > (\delta/2)^2$. Let us gather a few properties of $\tilde{w}_n$. First, thanks to Proposition 3.16 we have

$$\|\tilde{w}_n\|_{C^2} \leq C\|\tilde{\pi}_n^\beta w\|_{C^2}$$

Moreover, $\tilde{w}_n$ has a vanishing gradient at x_0

$$\nabla \tilde{w}_n(x_0) = 0.$$

Given $x \in M$, let $\hat{x} \in \tilde{G}_n$ denote some point such that $d(x, \tilde{G}_n) = d(x, \tilde{x})$. Then, from the positivity assumption on w, we have $\tilde{w}_n(\hat{x}) \geq -l(\nabla \tilde{\pi}_n^\beta w(x_0), x_0; \hat{x})\phi_0(d(x, x_0)^2)$ for any $x \in \tilde{G}_n$. Then, given $x \in B_{\delta/2}(x_0)$, we have

$$\tilde{w}_n(x) \geq \tilde{w}_n(\hat{x}) - C\|\tilde{w}_n\|_{C^1} d(x, \hat{x})$$

$$\geq -l(\nabla \tilde{\pi}_n^\beta w(x_0), x_0; \hat{x})\phi_0(d(x, x_0)^2) - C\|w\|_{C^3} d(x, \hat{x}),$$

$$\geq -C|\nabla \pi_n^\beta w(x_0)|_{g_{x_0}} d(\hat{x}, x_0)\phi_0(d(x, x_0)^2) - C\|w\|_{C^3} h_n$$

For such x, we have that $d(\hat{x}, x_0) \leq d(x, x_0) + d(\hat{x}, x) \leq \delta + h_n$, therefore

$$\tilde{w}_n(x) \geq -C|\nabla \pi_n^\beta w(x_0)|_{g_{x_0}}(\delta + \tilde{h}_n) - C\|w\|_{C^3} h_n.$$

Using Proposition 3.28, we conclude that,

$$\tilde{w}_n(x) \geq -C\|w\|_{C^3} h_n, \ \ \forall \, x \in M. \tag{3.20}$$

Next, we use that $\tilde{w}_n(x_0) = 0$ and $\nabla \tilde{w}_n(x_0) = 0$, together with Proposition A.12, to obtain the bound

$$\tilde{w}_n(x) \geq -C\|w\|_{C^3} d(x, x_0)^2, \ \ \forall \, x \in M. \tag{3.21}$$

The idea is to combine these two estimates to construct the desired function, using (3.20) away from x_0, and (3.21) near x_0. We define a preliminary function $\tilde{R}_{\beta,n,w,x_0}$ as follows,

$$\tilde{R}_{\beta,n,w,x_0}(x) := C\|w\|_{C^3} h_n \eta\left(\frac{d(x,x_0)^2}{h_n}\right).$$

Here, $\eta : \mathbb{R}_+ \to \mathbb{R}$ is an auxiliary smooth, nondecreasing function such that

$$\eta(t) = t \text{ in } [0,1/2], \quad \eta(t) \equiv 1 \text{ in } [1,\infty).$$

Then, if $d(x,x_0)^2 \geq Ch_n$, from (3.20) we have

$$\tilde{w}_n + \tilde{R}_{\beta,n,w,x_0} \geq 0.$$

On the other hand, if $d(x,x_0)^2 \leq Ch_n$ we use (3.21) to obtain

$$\begin{aligned}
\tilde{w}_n(x) + \tilde{R}_{\beta,n,w,x_0}(x) &\geq -C\|w\|_{C^3}d(x,x_0)^2 + R_{\beta,n,w,x_0}(x) \\
&\geq -C\|w\|_{C^3}d(x,x_0)^2 + C\|w\|_{C^3}d(x,x_0)^2 \\
&\geq 0.
\end{aligned}$$

Moreover,

$$\nabla_a \tilde{R}_{\beta,n,w,x_0}(x) = C\|w\|_{C^3}\eta'\left(\frac{d(x,x_0)^2}{h_n}\right) 2d(x,x_0)\nabla_a d(x,x_0).$$

Thus, if $d(x,x_0)^2 \geq h_n$, $\nabla_a R_{\beta,n,w,x_0}(x) = 0$. If $d(x,x_0)^2 \leq h_n$ then

$$|\nabla_a \tilde{R}_{\beta,n,w,x_0}(x) - \nabla_a \tilde{R}_{\beta,nw,x_0}(x')| \leq C\|w\|_{C^3}d(x,x').$$

This may be rewritten as,

$$\frac{|\nabla_a \tilde{R}_{\beta,n,w,x_0}(x) - \nabla_a \tilde{R}_{\beta,n,w,x_0}(x')|}{d(x,x')^{\beta-1}} \leq C\|w\|_{C^3}h_n^{2-\beta}.$$

In conclusion, letting $R_{\beta,n,w,x_0} := \tilde{R}_{\beta,n,w,x_0}(x) - l(\nabla\tilde{\pi}_n^\beta w(x_0), x_0; x)\phi_0(d(x,x_0)^2)$ it follows that $\tilde{\pi}_n^\beta w + R_{\beta,n,w,x_0} \geq 0$ everywhere and

$$\|R_{\beta,n,w,x_0}\|_{C^\beta} \leq Ch_n^{2-\beta}\|w\|_{C^3}.$$

Thus R_{β,n,w,x_0} as constructed has the desired properties.

The case $\beta \in [2,3)$. This time, we must get rid of the first and second order parts of $\tilde{\pi}_n^\beta w$ near x_0. Therefore, we write

$$\tilde{w}_n(x) := \tilde{\pi}_n^\beta w(x) - \left(l(\nabla\tilde{\pi}_n^\beta w(x_0), x_0; x) + q(\nabla^2\tilde{\pi}_n^\beta w(x_0), x_0; x)\right)\phi_0(d(x,x_0)^2).$$

Where ϕ_0 is the same function from the case $\beta \in [1,2)$. Then, as in the previous case we have two inequalities,

$$\tilde{w}_n(x) \geq -C\|w\|_{C^3}h_n,$$

and

$$\tilde{w}_n(x) \geq -C\|w\|_{C^3}d(x,x_0)^3$$

Then, we introduce the function

$$\tilde{R}_{\beta,n,w,x_0}(x) := C\|w\|_{C^3} h_n \eta\left(\frac{d(x,x_0)^3}{h_n}\right).$$

where η is the same function from the previous case. If $d(x,x_0)^3 \geq h_n$ it follows that

$$\tilde{w}_n(x) + \tilde{R}_{\beta,n,w,x_0}(x) = \tilde{w}_n(x) + C\|w\|_{C^3}h_n \geq 0.$$

On the other hand, if $d(x, x_0)^3 \leq h_n$, then

$$\tilde{w}_n(x) + \tilde{R}_{\beta,n,w,x_0}(x) \geq \tilde{w}_n(x) + C\|w\|_{C^3} d(x, x_0)^3 \geq 0.$$

Letting $R_{\beta,n,w,x_0} := \tilde{R}_{\beta,n,w,x_0}(x) - \left(l(\nabla \tilde{\pi}_n^\beta w(x_0), x_0; x) + q(\nabla^2 \tilde{\pi}_n^\beta w(x_0), x_0; x) \right) \phi_0(d(x, x_0)^2)$ we conclude that $\tilde{\pi}_n^\beta w + R_{\beta,n,w,x_0} \geq 0$ in M. $\qquad\square$

3.5. Remaining proofs for the case where $\beta \geq 1$.
Here we present the proof of the more technical cases in Proposition 3.20, Proposition 3.22, and Lemma 3.25.

Proof of Proposition 3.20 for $\beta \geq 1$. **The case $\beta \in [1, 2)$.** In this case, f has the form

$$f(x) = \sum_k \left(u(\hat{y}_{n,k}) + l(\nabla_n^1 u(\hat{y}_{n,k}), \hat{y}_{n,k}; x) \right) \phi_{n,k}(x).$$

Let $x \in M \setminus \tilde{G}_n$ and $\hat{x} \in \tilde{G}_n$ be such that $d(x, \hat{x}) = d(x, \tilde{G}_n)$.

$$\nabla_a f(x) = \sum_k \nabla_a \left(l(\nabla_n^1 u(\hat{y}_{n,k}), \hat{y}_{n,k}; x) \right) \phi_{n,k}(x)$$

$$+ \sum_k \left(u(\hat{y}_{n,k}) + l(\nabla_n^1 u(\hat{y}_{n,k}), \hat{y}_{n,k}; x) \right) \nabla_a \left(\phi_{n,k}(x) \right)$$

$$= \sum_k \nabla_a \left(l(\nabla_n^1 u(\hat{y}_{n,k}), \hat{y}_{n,k}; x) \right) \phi_{n,k}(x) + \sum_k \left(u(\hat{y}_{n,k}) + l(\nabla_n^1 u(\hat{y}_{n,k}), \hat{y}_{n,k}; x) \right) \nabla_a \left(\phi_{n,k}(x) \right).$$

For $\hat{x} \in \tilde{G}_n$, we have

$$\nabla_a f(\hat{x}) = \nabla_n^1 u(\hat{x}),$$

which is not too difficult to show. Since the proof of this fact essentially follows the same argument used later on in the proof of Theorem 3.23 –in the case $\beta \in [1, 2)$–, we omit the proof.

Then, using (3.13) in the above expression for $\nabla_a f(x)$, we see that

$$\nabla_a f(x) - \nabla_a f(\hat{x}) = \sum_k \left(\nabla_a \left(l(\nabla_n^1 u(\hat{y}_{n,k}), \hat{y}_{n,k}; x) \right) - \nabla_a f(\hat{x}) \right) \phi_{n,k}(x)$$

$$+ \sum_k \left(u(\hat{y}_{n,k}) + l(\nabla_n^1 u(\hat{y}_{n,k}), \hat{y}_{n,k}; x) - u(\hat{x}) \right) \nabla_a \left(\phi_{n,k}(x) \right).$$

Recall that the only non-zero terms above are those with $k \in K_x$ (defined in Lemma 3.9). For such k, thanks to Definition 3.13 and Proposition A.15 we have

$$|\nabla_a \left(l(\nabla_n^1 u(\hat{y}_{n,k}), \hat{y}_{n,k}; x) \right) - \nabla_a f(\hat{x})| \leq |\nabla_a \left(l(\nabla_n^1 u(\hat{y}_{n,k}), \hat{y}_{n,k}; x) \right) - \nabla_a \left(l(\nabla_n^1 u(\hat{y}_{n,k}), \hat{y}_{n,k}; \hat{x}) \right)|$$

$$+ |\nabla_a \left(l(\nabla_n^1 u(\hat{y}_{n,k}), \hat{y}_{n,k}; \hat{x}) \right) - \nabla_a f(\hat{x})|$$

$$\leq C\|u\|_{C^\beta} d(x, \hat{x})^{\beta-1}.$$

Adding these for every $k \in K_x$, and using that $\#K_x \leq N$,

$$\left| \sum_k \left(\nabla_a \left(l(\nabla_n^1 u(\hat{y}_{n,k}), \hat{y}_{n,k}; x) \right) - \nabla_a f(\hat{x}) \right) \phi_{n,k}(x) \right| \leq C\|u\|_{C^\beta} d(x, \hat{x})^{\beta-1}. \qquad (3.22)$$

Let us bound the remaining terms (compare with [48, Chp 6, Sec 2.3.2]). Let $k \in K_x$, we seek a bound for the quantity

$$|u(\hat{y}_{n,k}) + l(\nabla_n^1 u(\hat{y}_{n,k}), \hat{y}_{n,k}; x) - u(\hat{x})|.$$

Assume that $\hat{y}_{n,k} \neq \hat{x}$ (otherwise the quantity is zero and there is nothing to prove). By the triangle inequality, to bound this quantity it suffices to bound the sum

$$|u(\hat{y}_{n,k}) + l(\nabla u(\hat{y}_{n,k}), \hat{y}_{n,k}; \hat{x}) - u(\hat{x})| + |l(\nabla^1_n u(\hat{y}_{n,k}), \hat{y}_{n,k}; x) - l(\nabla u(\hat{y}_{n,k}), \hat{y}_{n,k}; \hat{x})|.$$

Thanks to Proposition A.12 in the Appendix, for each $k \in K_x$, we have the bound

$$|u(\hat{y}_{n,k}) + l(\nabla u(\hat{y}_{n,k}), \hat{y}_{n,k}; \hat{x}) - u(\hat{x})| \leq \|u\|_{C^\beta} d(\hat{x}, \hat{y}_{n,k})^\beta. \tag{3.23}$$

At the same time, Lemma A.14 yields $|\nabla^1_n u(\hat{y}_{n,k}) - \nabla u(\hat{y}_{n,k})|_{g_x} \leq C\|u\|_{C^\beta} h_n^{\beta-1}$ (see Appendix). Then, from the definition of the operators l (see also Remark 3.14), it follows that

$$|l(\nabla^1_n u(\hat{y}_{n,k}), \hat{y}_{n,k}; x) - l(\nabla u(\hat{y}_{n,k}), \hat{y}_{n,k}; \hat{x})| \leq C\|u\|_{C^\beta} h_n^{\beta-1} d(x, \hat{y}_{n,k}).$$

Since $\hat{y}_{n,k} \neq \hat{x}$, we have $d(\hat{x}, \hat{y}_{n,k}) \geq \lambda h_n$, thanks to (3.3). Thus $h_n^{\beta-1} \leq \lambda^{1-\beta} d(x, \hat{y}_{n,k})^{\beta-1}$ and we conclude there is some universal constant C such that

$$|l(\nabla^1_n u(\hat{y}_{n,k}), \hat{y}_{n,k}; x) - l(\nabla u(\hat{y}_{n,k}), \hat{y}_{n,k}; \hat{x})| \leq C\|u\|_{C^\beta} d(x, \hat{y}_{n,k})^\beta. \tag{3.24}$$

Then, as argued earlier to obtain (3.16) (using Remark 3.10 once again) we have

$$d(\hat{x}, \hat{y}_{n,k}) \leq 16 d(x, \tilde{G}_n), \ \ \forall \, k \in K_x,$$

which trivially implies the bound $d(x, \hat{y}_{n,k}) \leq 17 d(x, \tilde{G}_n)$ for every $k \in K_x$. Combining this with (3.23) and (3.24), we obtain the bound

$$|u(\hat{y}_{n,k}) + l(\nabla^1_n u(\hat{y}_{n,k}), \hat{y}_{n,k}; x) - u(\hat{x})| \leq C\|u\|_{C^\beta} d(x, \hat{x})^\beta \ \ \forall \, k \in K_x.$$

Given that $|\nabla \phi_{n,k}|_{g_x} \leq C \mathrm{diam}(P^*_{n,k})^{-1}$ (Lemma 3.11), the last inequality above, and the fact that $\#K_x \leq N$ (Lemma 3.9), it follows that

$$\left| \sum_k \left(l(\nabla^1_n u(\hat{y}_{n,k}), \hat{y}_{n,k}; x) - l(\nabla^1_n u(\hat{x}), \hat{x}; x) \right) \nabla_a \phi_{n,k}(x) \right| \leq C\|u\|_{C^\beta} d(x, \hat{x})^{\beta-1}. \tag{3.25}$$

Combining these we conclude that

$$|\nabla_a f(x) - \nabla_a f(\hat{x})| \leq C\|u\|_{C^\beta(M)} d(x, \hat{x})^{\beta-1}, \ \ \forall \, x \in M \setminus \tilde{G}_n,$$

as we wanted.

The case $\beta \in [2, 3)$. Finally, in this case we have

$$\nabla^2_{ab} f(x) = \sum_k \nabla^2_{ab} \left(l(\nabla^1_n u(\hat{y}_{n,k}), \hat{y}_{n,k}; x) \right) \phi_{n,k}(x)$$

$$+ \sum_k \nabla_a \left(l(\nabla^1_n u(\hat{y}_{n,k}), \hat{y}_{n,k}; x) \right) \nabla_b \phi_{n,k}(x) + \nabla_b \left(l(\nabla^1_n u(\hat{y}_{n,k}), \hat{y}_{n,k}; x) \right) \nabla_a \phi_{n,k}(x)$$

$$+ \sum_k \left(l(\nabla^1_n u(\hat{y}_{n,k}), \hat{y}_{n,k}; x) \right) \nabla^2_{ab} \phi_{n,k}(x).$$

The argument from this point on is entirely analogous to the one for $\beta \in [1, 2)$. We only sketch the details. One uses (3.13) and the above identity to write an expression $\nabla^2_{ab} f(x) - \nabla^2_{ab} f(\hat{x})$. This expression is itself separated into various sums grouped according to whether a term has a factor of $\phi_{n,k}$, $\nabla_a \phi_{n,k}$, or $\nabla_{ab} \phi_{n,k}$. Then, one proceeds to use Proposition A.12 and Lemma A.14 to obtain bounds for the various terms, in a manner analogous to the case $\beta \in [1, 2)$. In conclusion, one arrives at the desired bound,

$$|\nabla^2_{ab} f(x) - \nabla^2_{ab} f(\hat{x})| \leq C\|u\|_{C^\beta} d(x, \hat{x})^{\beta-2}.$$

$\square$

Proof of Proposition 3.22 for $\beta \geq 1$. We recall some of the setup, already used in the case $\beta < 1$. We let $x_1, x_2 \in B_r(x)$, where $B_{4r}(x) \subset M \setminus \tilde{G}_n$, so that $d(x_i, \tilde{G}_n) \geq r$ for $i = 1, 2$. Let $x(t)$ denote again the geodesic going from x_1 to x_2, parametrized with arc length, so that $x(0) = x_1, x(L) = x_2$ where $L = d(x_1, x_2)$. Under these circumstances, we have

$$d(x(t), \tilde{G}_n) \geq r, \ \ \forall\, t \in [0, L].$$

We now consider each of the remaining cases.

The case $\beta \in [1, 2)$. Invoking the chain rule, and Proposition 3.21 as done for $\beta < 1$, we have

$$\left| \frac{d}{dt} \nabla_a f(x(t)) \right| \leq C \|u\|_{C^\beta} r^{\beta-2}.$$

In particular, integrating from $t = 0$ to $t = L$ we have

$$|\nabla_a f(x_1) - \nabla_a f(x_2)| \leq \int_0^L \left| \frac{d}{dt} \nabla_a f(x(t)) \right| \, dt \leq C r^{\beta-2} \|u\|_{C^\beta(M)} d(x_1, x_2).$$

Since $\beta - 2 < 0$ and $d(x_1, x_2) \leq 2r$, it follows that $r^{\beta-2} \leq 2^{2-\beta} d(x_1, x_2)^{\beta-2}$. Then,

$$|\nabla_a f(x_1) - \nabla_a f(x_2)| \leq C \|u\|_{C^\beta(M)} d(x_1, x_2)^{\beta-1}.$$

The case $\beta \in [2, 3)$. This time we use the third derivative estimate from Proposition 3.21, which yields

$$\left| \frac{d}{dt} \nabla^2_{ab} f(x(t)) \right| \leq C \|u\|_{C^\beta} r^{\beta-3}.$$

Then,

$$|\nabla^2_{ab} f(x_1) - \nabla^2_{ab} f(x_2)| \leq \int_0^L \left| \frac{d}{dt} \nabla^2_{ab} f(x(t)) \right| \, dt \leq C r^{\beta-3} \|u\|_{C^\beta(M)} d(x_1, x_2)$$

This time, since $\beta - 3 < 0$ and $d(x_1, x_2) \leq 2r$, we have $r^{\beta-3} \leq 2^{3-\beta} d(x_1, x_2)^{\beta-3}$ and therefore

$$|\nabla^2_{ab} f(x_1) - \nabla^2_{ab} f(x_2)| \leq C \|u\|_{C^\beta(M)} d(x_1, x_2)^{\beta-2}.$$

$$\square$$

Proof of Lemma 3.25 for $\beta \geq 1$. **The case $\beta \in [1, 2)$.** In this case we need to go further and bound the Hölder seminorm of $\nabla_a f$, for every index a. Observe that

$$|\nabla_a f(x) - \nabla_a u(x)| \leq |\nabla_a f(x) - \nabla_a f(\hat{x})| + |\nabla_a f(\hat{x}) - \nabla_a u(\hat{x})| + |\nabla_a u(\hat{x}) - \nabla_a u(x)|.$$

Evidently,

$$|\nabla_a f(x) - \nabla_a f(\hat{x})| \leq C \|u\|_{C^2} d(x, \hat{x}),$$
$$|\nabla_a u(\hat{x}) - \nabla_a u(x)| \leq \|u\|_{C^2} d(x, \hat{x}).$$

Where we have used that $\|f\|_{C^2} \leq C \|u\|_{C^2}$ in the first inequality. According to Lemma A.14, $|\nabla f(\hat{x}) - \nabla u(\hat{x})|_{g_x}$, is bounded from above by $C \|u\|_{C^2} h_n$ (recall that $\nabla \tilde{\pi}^\beta_n w$ and $\nabla^1_n u$ agree at points in $\tilde{G}_n$). Since $d(x, \hat{x}_n) \leq h_n$, we conclude that

$$\sup_{x \in M} |\nabla f(x) - \nabla u(x)|_{g_x} \leq C \|u\|_{C^2} h_n \leq C \|u\|_{C^3} h_n.$$

The Hölder seminorm of $\nabla f(x) - \nabla u(x)$ is estimated using an argument analogous to the one used in the case $\beta \in [0, 1)$. Let $x_1, x_2 \in M$, and let ∇_a be as usual. Suppose first that $d(x_1, x_2) \leq h_n$. Then, using that $\|f - u\|_{C^2} \leq \|f\|_{C^2} + \|u\|_{C^2} \leq C \|u\|_{C^2}$ (by Theorem 3.23),

$$|\nabla_a f(x_1) - \nabla_a u(x_1) - (\nabla_a f(x_2) - \nabla_a u(x_2))| \leq C \|u\|_{C^2} d(x_1, x_2).$$

Using that $2 - \beta < 0$ and $d(x_1, x_2) \leq h_n$, it follows that

$$\frac{|\nabla_a f(x_1) - \nabla_a u(x_1) - (\nabla_a f(x_2) - \nabla_a u(x_2))|}{d(x_1, x_2)^{\beta - 1}} \leq C\|u\|_{C^2} d(x_1, x_2)^{2-\beta}$$

$$\leq C\|u\|_{C^2} h_n^{2-\beta}.$$

Next, let us consider what happens if x_1, x_2 are such that $d(x_1, x_2) > h_n$. First, we note that

$$|\nabla_a f(x_1) - \nabla_a u(x_1) - (\nabla_a f(x_2) - \nabla_a u(x_2))|$$
$$\leq |\nabla_a f(x_1) - \nabla_a u(x_1)| + |\nabla_a f(x_2) - \nabla_a u(x_2)|.$$

To estimate these two terms, we decompose each of them again. We have, for $i = 1, 2$

$$|\nabla_a f(x_i) - \nabla_a u(x_i)| \leq |\nabla_a f(x_i) - \nabla_a f(\hat{x}_i)| + |\nabla_a f(\hat{x}_i) - \nabla_a u(\hat{x}_i)| + |\nabla_a u(\hat{x}_i) - \nabla_a u(x_i)|.$$

Now, on one hand we have the estimates

$$|\nabla_a f(x_i) - \nabla_a f(\hat{x}_i)| \leq C\|u\|_{C^2} d(x_i, \hat{x}_i),$$
$$|\nabla_a u(x_i) - \nabla_a u(\hat{x}_i)| \leq \|u\|_{C^2} d(x_i, \hat{x}_i),$$

while on the other hand Lemma A.14 says that $|\nabla_a f(\hat{x}_i) - \nabla_a u(\hat{x}_i)| \leq C\|u\|_{C^2} d(x_i, \hat{x}_i)$. Gathering these bounds and using that $d(x_1, x_2) > h_n \geq d(x_i, \hat{x}_i)$, we conclude that

$$|\nabla_a f(x_1) - \nabla_a u(x_1) - (\nabla_a f(x_2) - \nabla_a u(x_2))| \leq C\|u\|_{C^2} h_n.$$

Then, since $\beta \in [1, 2)$,

$$\frac{|\nabla_a f(x_1) - \nabla_a u(x_1) - (\nabla_a f(x_2) - \nabla_a u(x_2))|}{d(x_1, x_2)^{\beta - 1}} \leq C\|u\|_{C^2} h_n^{2-\beta}.$$

In conclusion, for $x_1, x_2 \in M$ with $x_1 \neq x_2$ we have

$$\frac{|\nabla_a f(x_1) - \nabla_a u(x_1) - (\nabla_a f(x_2) - \nabla_a f(x_2))|}{d(x_1, x_2)^{\beta - 1}} \leq C\|u\|_{C^2} h_n^{2-\beta}.$$

Therefore, as in the case $\beta \in [0, 1)$, we conclude that

$$\|f - u\|_{C^\beta} = \|f - u\|_{L^\infty} + \|\nabla f - \nabla u\|_{L^\infty} + [\nabla f - \nabla u]_{C^{\beta-1}} \leq C\|u\|_{C^2} h_n^{2-\beta},$$

proving the estimate in this case.

The case $\beta \in [2, 3)$. In this case we must also take into account the values of $\nabla^2 f$. Similarly as in the previous cases, we use a triangle inequality to estimate the difference $\nabla^2 f(x) - \nabla^2 u$. Let a, b be indices in one of the usual exponential system of coordinates, then

$$|\nabla_{ab}^2 f(x) - \nabla_{ab}^2 u(x)| \leq |\nabla_{ab}^2 f(x) - \nabla_{ab}^2 f(\hat{x})| + |\nabla_{ab}^2 f(\hat{x}) - \nabla_{ab}^2 u(\hat{x})| + |\nabla_{ab}^2 u(x) - \nabla_{ab}^2 u(\hat{x})|.$$

On the other hand, we have, from Theorem 3.23

$$|\nabla_{ab}^2 f(x) - \nabla_{ab}^2 f(\hat{x})| \leq C\|u\|_{C^3} d(x, \hat{x}),$$
$$|\nabla_{ab}^2 u(\hat{x}) - \nabla_{ab}^2 u(x)| \leq \|u\|_{C^3} d(x, \hat{x}),$$

and, again from Lemma A.14 in the Appendix,

$$|\nabla_{ab}^2 f(\hat{x}) - \nabla_{ab}^2 u(\hat{x})| \leq C h_n \|u\|_{C^3}.$$

Combining these inequalities, we conclude that

$$\sup |\nabla_{ab}^2 f(x) - \nabla_{ab}^2 u(x)| \leq C\|u\|_{C^3} h_n.$$

Now we consider the Hölder seminorm. Fix $x_1, x_2 \in M$. As before, consider first the case where $d(x_1, x_2) \leq h_n$, in which case it is clear that

$$|\nabla_{ab} f(x_1) - \nabla_{ab} u(x_1) - (\nabla_{ab} f(x_2) - \nabla_{ab} u(x_2))| \leq \|u\|_{C^3} d(x_1, x_2).$$

Then, using that $3 - \beta < 0$, and that $d(x_1, x_2) \le h_n$, it follows that

$$\frac{|\nabla_{ab} f(x_1) - \nabla_{ab} u(x_1) - (\nabla_{ab} f(x_2) - \nabla_{ab} u(x_2))|}{d(x_1, x_2)^{\beta - 2}} \le C\|u\|_{C^3} d(x_1, x_2)^{3-\beta} \le C\|u\|_{C^3} h_n^{3-\beta}.$$

Let us now take the opposite case, that is when $d(x_1, x_2) > h_n$. Then

$$|\nabla_{ab} f(x_1) - \nabla_{ab} u(x_1) - (\nabla_{ab} f(x_2) - \nabla_{ab} u(x_2))|$$
$$\le |\nabla_{ab} f(x_1) - \nabla_{ab} u(x_1)| + |\nabla_{ab} f(x_2) - \nabla_{ab} u(x_2)|$$

As before,

$$|\nabla_{ab} f(x_i) - \nabla_{ab} u(x_i)| \le |\nabla_{ab} f(x_i) - \nabla_{ab} f(\hat{x}_i)| + |\nabla_{ab} f(\hat{x}_i) - \nabla_{ab} u(\hat{x}_i)| + |\nabla_{ab} u(x_i) - \nabla_{ab} u(\hat{x}_i)|$$

Next, we have

$$|\nabla_{ab} f(x_i) - \nabla_{ab} f(\hat{x}_i)| \le C\|u\|_{C^3} d(x_i, \hat{x}_i),$$
$$|\nabla_{ab} u(x_i) - \nabla_{ab} u(\hat{x}_i)| \le \|u\|_{C^3} d(x_i, \hat{x}_i).$$

These inequalities, together with the bound $|\nabla_{ab} f(\hat{x}_i) - \nabla_{ab} u(\hat{x}_i)| \le C\|u\|_{C^3} h_n$ from Lemma A.14 in the Appendix, yield

$$|\nabla_{ab} f(x_1) - \nabla_{ab} u(x_1) - (\nabla_{ab} f(x_2) - \nabla_{ab} u(x_2))| \le C\|u\|_{C^3} h_n.$$

Using that $d(x_1, x_2) > h_n$, we see that in this case

$$\frac{|\nabla_{ab} f(x_1) - \nabla_{ab} u(x_1) - (\nabla_{ab} f(x_2) - \nabla_{ab} u(x_2))|}{d(x_1, x_2)^{\beta - 2}} \le C\|u\|_{C^3} h_n^{3-\beta}.$$

The rest of the proof is entirely analogous to the previous case, and the Lemma is proved. $\qquad\square$

4. The Min-max formula in infinite dimensions: Functions on (M, g)

This section has two goals: defining a "finite dimensional" approximation to I; and showing that the approximation can be used, along with Section 2, to prove Theorem 1.6. First we develop the approximation, and second we establish Theorem 1.6.

4.1. **Approximations to I and their structure.** We are now ready to introduce the finite dimensional approximations to the Lipschitz map $I : C_b^\beta(M) \to C_b(M)$. Recall that in Definition 3.17 we introduced the restriction and extension operators $\tilde{T}_n$ and $\tilde{E}_n^\beta$, below we introduce slight modifications of these operators, which have the advantage that they depend only on the values of u over G_n, and not all of $\tilde{G}_n$.

Definition 4.1. *For each n, we define*

(1) *The restriction operator $T_n : C_b^\beta(M) \to C(G_n)$, defined by*

$$T_n(u, x) := u(x) \ \forall \ x \in G_n.$$

(2) *The extension operator of order β, $E_n^\beta : C(G_n) \to C_b^\beta(G_n)$, defined by*

$$E_n^\beta(u, x) := E_n^\beta(\tilde{u}, x)$$

where E_n^β is the extension operator from Definition 3.17, and $\tilde{u} \in C(G_n)$ denotes the function which agrees with u in G_n and is defined to be zero in $\tilde{G}_n \setminus G_n$.

(3) *Again, we have a projection map, which we denote π_n^β, and is defined by $\pi_n^\beta := E_n^\beta \circ T_n$.*

From Remark 3.19, it becomes clear that, if $u \in C_b^\beta(M)$ has compact support, then if n is large enough, then $E_n^\beta \circ T_n = \tilde{E}_n^\beta \circ \tilde{T}_n u$. Therefore

$$u \in C_c^\beta(M) \Rightarrow \tilde{\pi}_n^\beta(u, x) = \pi_n^\beta(u, x) \ \forall \ x \in M, \quad \text{for all large enough } n. \tag{4.1}$$

Using this, we can use the apply the results about $\tilde{\pi}_n^\beta$ from Section 3 to π_n^β when dealing with functions supported in some compact set K and n large enough (depending on K).

We will create two approximations, which we call i_n and I_n. The distinction is that i_n is legitimately defined on the finite dimensional space, $C(G_n)$, whereas I_n will be defined on $C_b^\beta(M)$, but is finite dimensional in the sense that it returns the same value for any two functions that agree on G_n. Introducing I_n will be important so that both I_n and I have the same domain and co-domain.

To this end, we let T_n, E_n^β, and π_n^β from Definition 4.1, and now we define

$$I_n : C_b^\beta(M) \to C_b(M), \quad I_n := \pi_n^0 \circ I \circ \pi_n^\beta,$$

that is to say

$$I_n(u, x) := E_n^0 T_n I(E_n^\beta T_n u, x). \tag{4.2}$$

The approximations I_n will be seen to well approximate I on a set that is dense with respect to local uniform convergence in $C_b^\beta(M)$ (as opposed to norm convergence).

Definition 4.2. *Define, for $\beta \in [0, 2]$, the finite dimensional subspace $X_n^\beta \subset C_b^\beta(M)$ by*

$$X_n^\beta := E_n^\beta(C(G_n)).$$

Proposition 4.3 (Convergence of I_n on $C_c^3(M)$). *With I_n defined in (4.2) and for every compact $K \subset M$ and any $R > 0$, we have*

$$\lim_{n \to \infty} \sup_{\|u\|_{C_c^3(K)} \le R} \|I_n u - I u\|_{L^\infty(M)} = 0.$$

Proof of Proposition 4.3. First of all, let us recall that that $\pi_n^\beta u = E_n^\beta T_n u$. Then, we have

$$\|I(u) - I(\pi_n^\beta u)\|_{L^\infty(M)} \le \|I\|_{\mathrm{Lip}(C_b^\beta, C)} \|u - \pi_n^\beta u\|_{C^\beta(M)}.$$

On the other hand, successive applications of Theorem 3.23 and the linearity of π_n^0 imply that

$$\|\pi_n^0 I(u) - \pi_n^0 I(\pi_n^\beta u)\|_{L^\infty(M)} \le C\|I(u) - I(\pi_n^\beta u)\|_{L^\infty(M)}$$
$$\le C\|I\|_{\mathrm{Lip}(C_b^\beta, C)} \|u - \pi_n^\beta u\|_{C^\beta(M)}.$$

It follows that

$$\|I_n u - I u\|_{L^\infty(M)} \le C\|I\|_{\mathrm{Lip}(C_b^\beta, C)} \|u - \pi_n^\beta u\|_{C^\beta(M)}.$$

At this point, we can apply Lemma 3.25 to the right hand side of the last inequality (using (4.1)), and we see that for sufficiently large n,

$$\|I_n u - I u\|_{L^\infty(M)} \le C\|I\|_{\mathrm{Lip}(C_b^\beta, C)} \tilde{h}_n^\gamma \|u\|_{C_b^3(M)}$$

where $\tilde{h}_n$ is as defined in (3.2) and γ is as in Lemma 3.25. It follows that for all large n, and all $u \in C_b^3(M)$ which are compactly supported in K, we have

$$\|I_n u - I_u\|_{L^\infty(M)} \le C\tilde{h}_n^\gamma \|I\|_{\mathrm{Lip}(C_b^\beta, C)} R,$$

and the Proposition is proved. $\qquad \square$

For each n, the map I_n may be thought of as a finite dimensional approximation to I in the following sense. We define the map

$$i_n : C(G_n) \to C(G_n), \quad i_n := T_n \circ I \circ E_n^\beta. \tag{4.3}$$

Thus, I_n and i_n are related by

$$I_n = E_n^0 \circ i_n \circ T_n.$$

In particular, this shows that although $I_n : C_b^\beta(M) \to C_b(M)$, I_n is uniquely determined by its values on functions in X_n^β, and functions in X_n^β are uniquely determined by their values on G_n.

Remark 4.4. *As suggested by the results in Section 3.4 in particular Lemma 3.29, except for when $\beta \in (0,1)$, it is not expected that i_n or I_n will enjoy the GCP. However, this is not a set-back because the GCP is recovered in the limit as $n \to \infty$. The potential failure of the GCP originates with the composition by E_n^β, and the latter operator may not be order preserving when $\beta \geq 1$.*

Lemma 4.5. *There is a universal constant, C, so that $\|i_n\|_{Lip(C(G_n),C(G_n))} \leq C\|I\|_{Lip(C_b^\beta,C_b)}$ and $\|I_n\|_{Lip(C_b^\beta,C_b)} \leq C\|I\|_{Lip(C_b^\beta,C_b)}$.*

Comments on Lemma 4.5. This is a straightforward consequence of the bound in Theorem 3.23, that $\|E_n^\beta T_n u\|_{C^\beta} \leq C\|u\|_{C^\beta}$, and the definitions of both i_n and I_n. $\qquad\square$

The advantage of this presentation is that we may now use the results from Section 2 to obtain a min-max formula for I_n, via the theory applied to i_n. First, we make an observation that relates the differentiability properties of i_n and I_n.

Lemma 4.6. *Assume that $u \in X_n^\beta(M)$ and $u_n = T_n u$. The map, i_n, is Frèchet differentiable at u_n if and only if I_n is Frèchet differentiable at $u = E_n^\beta u_n$. Furthermore,*

$$DI_n|_u = E_n^0 \circ Di_n|_{u_n} \circ T_n.$$

Comments on Lemma 4.6. This is a straightforward consequence of the uniqueness of DI_n and Di_n as well as the chain rule. We omit the details. $\qquad\square$

We define the analog of the Clarke differential for i_n in the context of I_n.

Definition 4.7. *The differential, $\mathcal{D}I_n$, is defined as*

$$\mathcal{D}I_n = \text{hull}\big\{ L \in \mathcal{L}(C_b^\beta, C_b) \ : \ \exists \{u_k^n\}_k \text{ s.t. } DI_n|_{u_k^n} \text{ exists } \forall\, k \text{ and}$$
$$\lim_{k\to\infty} DI_n|_{u_k^n}(f,x) = L(f,x) \ \forall f \in C_b^\beta, \forall\, x \in G_n \big\}.$$

An immediate corollary of this definition and Lemma 4.6 is

Corollary 4.8. *Composition by E_n^0 and T_n over $\mathcal{D}i_n$ gives $\mathcal{D}I_n$:*

$$\mathcal{D}I_n = \big\{ E_n^0 l T_n \ : \ l \in \mathcal{D}i_n \big\}.$$

Lemma 4.9. *For each n, the map $I_n : C_b^\beta(M) \to C_b(M)$ admits a min-max formula when evaluated over the set G_n; i.e.*

$$\forall u \in C_b^\beta(M), \ \forall\, x \in G_n, \quad I_n(u,x) = \min_{v \in C_b^\beta(M)} \max_{L \in \mathcal{D}I_n} \{I_n(v,x) + L(u-v,x)\}. \tag{4.4}$$

Here $\mathcal{D}I_n$ is as in Definition 4.7.

Proof of Lemma 4.9. To begin the proof, we make a few simple but useful observations about the range of i_n. First,

$$v_n \in C(G_n) \iff v_n = T_n v \text{ for some } v \in C_b^\beta(M),$$

and second

$$\left\{ T_n I E_n^\beta v_n \; : \; v_n \in C(G_n) \right\} = \left\{ T_n I E_n^\beta T_n v \; : \; v \in C_b^\beta(M) \right\}.$$

Applying Lemma 2.4 to i_n, we see that for all $u_n \in C(G_n)$ and all $x \in G_n$,

$$
\begin{aligned}
i_n(u_n, x) &= \min_{v_n \in C(G_n)} \max_{L_n \in \mathcal{D}i_n} \{ i_n(v_n, x) + L_n(u_n - v_n, x) \} \\
&= \min_{v_n \in C(G_n)} \max_{L_n \in \mathcal{D}i_n} \{ T_n I E_n^\beta(v_n, x) + L_n(u_n - v_n, x) \} \\
&= \min_{v \in C_b^\beta(M)} \max_{L_n \in \mathcal{D}i_n} \{ T_n I E_n^\beta T_n(v, x) + L_n(u_n - T_n v, x) \}.
\end{aligned}
$$

Thus, replacing u_n by $T_n u$, we see that for all $u \in C_b^\beta$ and $x \in G_n$,

$$i_n(T_n u, x) = \min_{v \in C_b^\beta(M)} \max_{L \in \mathcal{D}i_n} \{ T_n I E_n^\beta T_n(v, x) + L(T_n(u - v), x) \}.$$

This shows that for all $u, v \in C_b^\beta(M)$ and for all $x \in G_n$, the inequality:

$$i_n(T_n u, x) \le i_n(T_n v, x) + \max_{L_n \in \mathcal{D}i_n} \{ L_n(T_n(u - v), x) \};$$

and unraveling the notation for i_n, we see that

$$T_n I E_n^\beta T_n(u, x) \le T_n I E_n^\beta T_n(v, x) + \max_{L_n \in \mathcal{D}i_n} \{ L_n T_n((u - v), x) \}.$$

Thanks to the fact that E_n^0 is monotone and linear, as well as Corollary 4.8, we have

$$
\begin{aligned}
E_n^0 T_n I E_n^\beta T_n(u, x) &\le E_n^0 T_n I E_n^\beta T_n(v, x) + E_n^0 \max_{L_n \in \mathcal{D}i_n} \{ L_n T_n((u - v), x) \} \\
&\le E_n^0 T_n I E_n^\beta T_n(v, x) + \max_{L_n \in \mathcal{D}i_n} \{ E_n^0 L_n T_n((u - v), x) \} \\
&= E_n^0 T_n I E_n^\beta T_n(v, x) + \max_{\tilde{L}_n \in \mathcal{D}I_n} \{ \tilde{L}_n((u - v), x) \},
\end{aligned}
$$

Where we note in the middle inequality that if $L_n^{(x)}$ is a collection that point-by-point attains the max, then by Corollary 4.8, $E_n^0 L_n^{(x)}$ is an admissible family in $\mathcal{D}I_n$. Thus, by definition of I_n, we see that for all $u, v \in C_b^\beta(M)$ and all $x \in G_n$,

$$I_n(u, x) \le I_n(v, x) + \max_{\tilde{L}_n \in \mathcal{D}I_n} \{ \tilde{L}_n((u - v), x) \}.$$

Taking a min over $v \in C_b^\beta(M)$, we have achieved (4.4) for all $x \in G_n$. $\square$

Lemma 4.10. *If $L_n \in \mathcal{D}I_n$, then $\|L_n\|_{C_b^\beta \to C_b} \le C \|I\|_{C_b^\beta \to C_b}$.*

Proof of Lemma 4.10. First, assume that $u \in C_b^\beta(M)$ and that I_n is Fréchet differentiable at u. Let $\phi \in C_b^\beta(M)$ and $\psi \in C_b^\beta(M)$, and let $t > 0$.

$$
\begin{aligned}
\left\| \frac{I(u + t\phi) - I(u)}{t} - \frac{I(u + t\psi) - I(u)}{t} \right\|_{L^\infty(M)} &= \left\| \frac{I(u + t\phi) - I(u + t\psi)}{t} \right\|_{L^\infty(M)} \\
&\le \frac{1}{t} \|I_n\|_{Lip(C_b^\beta, C_b)} \|t(\phi - \psi)\|_{C^\beta(M)} \\
&\le C \|I\|_{Lip(C_b^\beta, C_b)} \|\phi - \psi\|_{C^\beta(M)}.
\end{aligned}
$$

Letting $t \to 0$ establishes the bound for $DI_n(u)$. We also note that in a Banach space, norm bounds are closed under convex combinations and weak limits, hence they also hold for $\mathcal{D}I_n$. $\square$

It will be useful to know that the assumption (1.3) is also obeyed by the operators I_n uniformly in n. This is indeed the case up to a slight enlargement factor, which is due to the result of the finite range of dependence of the operators $E_n^\beta T_n$, proved in Lemma 3.24.

Lemma 4.11. *There is a universal constant, C, such that for ω as in assumption (1.3), I_n inherits a slightly modified version of (1.3) in the form of*

$$\forall u, v \in C_b^\beta, \quad \|I_n(u) - I_n(v)\|_{L^\infty(B_r)} \leq C\|u - v\|_{C^\beta(\overline{B_{2r+3}})} + C\omega(r)\|u - v\|_{L^\infty(M)}, \qquad (4.5)$$

Proof of Lemma 4.11. This is immediate from two applications of Lemma 3.24, combined with the assumptions (1.3). $\square$

Lemma 4.12. *Up to a uniform constant, any $L_n \in \mathcal{D}I_n$ also inherits the properties of Lemma 4.11.*

Comments on Lemma 4.12. This follows in a similar way as the proof of Lemma 4.11, combined with the observations of the proof of Lemma 4.10. $\square$

Lemma 4.13. *Let $L \in \mathcal{D}I_n$. Suppose that $w \in C_b^3(M)$ is nonnegative and $w(x_0) = 0$, $x_0 \in G_n$. Then*

$$L(w, x_0) \geq -Ch_n^\gamma \|w\|_{C^3(M)}$$

where $\lim h_n = 0$ and h_n is defined in (3.6).

Proof of Lemma 4.13. Since the lower bound for L is preserved under convex combinations and limits, then, given the definition of $\mathcal{D}I_n$ it is clear that it suffices to prove the inequality when L is the classical derivative of I_n at points of differentiability for I_n. To this end, let us fix $u \in C_b^\beta$, an arbitrary point of differentiability of I_n, and let L_u denote the respective derivative.

We apply Lemma 3.29 to w, to obtain the remainder polynomial, R_{β,n,m,x_0} and conclude that for any $t > 0$ we have

$$u + tw + tR_{\beta,n,w,x_0} \geq u \quad \forall\, x \in M,$$

with equality at $x = x_0$ (recall that $R_{\beta,n,w,x_0}(x_0) = 0$). Since I has the global comparison property, it follows that

$$I(u + tw + tR_{\beta,n,w,x_0}, x_0) \geq I(u, x_0) \quad \forall\, t > 0.$$

Furthermore, since I is a Lipschitz map,

$$\begin{aligned} I(u + tw, x_0) &\geq I(u + tw + tR_{\beta,n,w,x_0}, x_0) - tC\|R_{\beta,n,w,x_0}\|_{C^\beta} \\ &\geq I(u, x_0) - tC\|R_{\beta,n,w,x_0}\|_{C^\beta}. \end{aligned}$$

It follows that

$$L_u(w, x_0) = \frac{d}{dt}\Big|_{t=0^+} I(u + tw, x_0) \geq -C\|R_{\beta,n,w,x_0}\|_{C^\beta}$$

Since $w \in C_c^3(M)$, Lemma 3.29 also says that

$$\|R_{\beta,n,w,x_0}\|_{C^\beta} \leq Ch_n^\gamma \|w\|_{C^3}$$

Thus,

$$L_u(w, x_0) \geq -Ch_n^\gamma \|w\|_{C^3}$$

This holds for every u where I_n is differentiable. Therefore, by the Definition 4.7 of $\mathcal{D}I_n$ in it also holds for any $L \in \mathcal{D}I_n$. $\square$

4.2. **Some nice properties of I, I_n, and π_n.** Here we will collect some useful observations about I, I_n, and π_n^β. They seem to be useful in their own right, and we hope they will appear elsewhere, but they are also essential for extracting limits of operators in $\mathcal{D}I_n$, and so we mention them here.

For the remainder of this section, we will use many times a function ρ, which is simply a smooth function that behaves like $t \mapsto \min\{t, 1\}$. We define it below.

Definition 4.14. *Let ρ be fixed from here until the end of this section as a function that satisfies*
$$\rho(s) = s \; \forall s \in [0, 1), \quad \rho(s) \equiv 3/2 \; \forall s \in [2, \infty), \; and \; |\rho'| + |\rho''| \leq 4.$$

Lemma 4.15. *Let $x \in M$, and let $\phi \in C_b^\beta(M)$ be any function such that $0 \leq \phi \leq 1$ and $\phi(x) = 0$. Then for any $u, v \in C_b^\beta(M)$,*
$$|I(\phi u, x) - I(\phi v, x)| \leq (\|I\|_{Lip(C_b^\beta, C_b)} \cdot \|\phi\|_{C^\beta(M)}) \cdot \|u - v\|_{L^\infty(\mathrm{spt}(\phi))}, \tag{4.6}$$

as well as
$$|I(u, x) - I(v, x)| \leq \|I\|_{Lip(C_b^\beta, C_b)} \left(\|(1 - \phi)(u - v)\|_{C^\beta(M)} + \|\phi\|_{C^\beta(M)} \cdot \|u - v\|_{L^\infty(\mathrm{spt}(\phi))} \right). \tag{4.7}$$

Proof of Lemma 4.15. First we establish (4.6). Note that for all $y \in \mathrm{spt}(\phi)$,
$$u(y) - v(y) \leq \|u - v\|_{L^\infty(\mathrm{spt}(\phi))},$$

and so for all $y \in M$, we also have
$$\phi(y)u(y) - \phi(y)v(y) \leq \phi(y)\|u - v\|_{L^\infty(\mathrm{spt}(\phi))}.$$

This says that the function $\phi v + \phi\|u - v\|_{L^\infty(\mathrm{spt}(\phi))}$ touches ϕu from above at any x such that $\phi(x) = 0$. By the GCP, we have
$$I(\phi u, x) \leq I(\phi v + \phi\|u - v\|_{L^\infty(\mathrm{spt}(\phi))}, x),$$

so that
$$I(\phi u, x) - I(\phi v, x) \leq I(\phi v + \phi\|u - v\|_{L^\infty(\mathrm{spt}(\phi))}, x) - I(\phi v, x)$$
$$\leq \|I\|_{Lip(C_b^\beta, C_b)} \cdot \|(\phi\|u - v\|_{L^\infty(\mathrm{spt}(\phi))})\|_{C^\beta(M)}$$
$$= \|I\|_{Lip(C_b^\beta, C_b)} \cdot \|\phi\|_{C^\beta(M)} \cdot \|u - v\|_{L^\infty(\mathrm{spt}(\phi))}.$$

The proof of (4.7) is similar, working with the inequality
$$(1 - \phi)u + \phi u - \phi v \leq (1 - \phi)u + \phi\|u - v\|_{L^\infty(\mathrm{spt}(\phi))},$$

which becomes an equality at any x such that $\phi(x) = 0$. Thus, the GCP gives
$$I((1 - \phi)u + \phi u, x) \leq I((1 - \phi)u + \phi v + \phi\|u - v\|_{L^\infty(\mathrm{spt}(\phi))}, x),$$

and after subtracting from both sides, we have
$$I((1 - \phi)u + \phi u, x) - I((1 - \phi)v + \phi v, x)$$
$$\leq I((1 - \phi)u + \phi v + \phi\|u - v\|_{L^\infty(\mathrm{spt}(\phi))}, x) - I((1 - \phi)v + \phi v, x)$$
$$\leq \|I\|_{Lip(C_b^\beta, C_b)} \cdot \|((1 - \phi)(u - v) + \phi\|u - v\|_{L^\infty(\mathrm{spt}(\phi))})\|_{C^\beta(M)}$$
$$\leq \|I\|_{Lip(C_b^\beta, C_b)} \cdot \left(\|(1 - \phi)(u - v)\|_{C^\beta(M)} + \|\phi\|_{C^\beta(M)} \cdot \|u - v\|_{L^\infty(\mathrm{spt}(\phi))} \right).$$
$$\square$$

In particular, using the results and proof of Lemma 4.15, after choosing an appropriate ϕ to approximate $B_R(x)$, we have as a corollary,

Corollary 4.16. *Given and $R > 0$, there exists a constant, $C(R)$, depending only on dimension such that for any x fixed, $r > 0$,*

$$\|I(u) - I(v)\|_{L^\infty(B_R)} \le C(R)\|I\|_{Lip(C_b^\beta, C_b)}\left(\|(u-v)\|_{C^\beta(B_{R+1})} + \|u-v\|_{L^\infty((B_R)^C)}\right),$$

as well as $C(R, r)$ which blows up as R, r are both small,

$$|I(u, x) - I(v, x)| \le \|I\|_{Lip(C_b^\beta, C_b)}\left(C(R)\|(u-v)\|_{C^\beta(B_{R+r}(x))} + C(R, r)\|u-v\|_{L^\infty((B_R(x))^C)}\right),$$

where $\omega(r) \to 0$ as $r \to \infty$ and comes from the limit in the extra assumption.

Sketch of the proof of Corollary . We just comment that this follows by making an appropriate choice of test functions in Lemma 4.15. $\qquad\square$

A very useful estimate, somewhat related to Corollary 4.16, involves the Whitney extension and touching a function from above. The proof of this uses Lemma 3.29 to a great degree. We record it as a proposition for later use.

Proposition 4.17. *Let $x_0 \in G_n$ be fixed, and let $f \in C_b(M)$ be such that $f(x_0) = 0$. Let $\beta \in [0, 3)$ and $\varepsilon \in [0, 1)$. Consider the function $w(x) := f(x)\rho(d(x, x_0)^{\beta+\varepsilon})$. There is a dimensional constant, C, and a function R_{n,x_0} such that $R_{n,x_0}(x_0) = 0$, $\|R_{n,x_0}\|_{C^\beta} \to 0$ as $n \to \infty$, and*

$$\pi_n^\beta(w, x) \le C\|f\|_{L^\infty}(\rho(d(x, x_0)^{\beta+\varepsilon}) + R_{n,x_0}(x)).$$

Here, ρ is the function introduced in Definition 4.14.

Proof. For the sake of brevity, we only provide the details for the case where $\beta > 2$, the other cases are simpler and the details are left to the reader. It will be convenient to introduce the following two functions

$$w_0(x) := \rho(d(x, x_0)^{\beta+\varepsilon}),$$
$$\tilde{w}(x) := \|f\|_{L^\infty}w_0(x) - f(x)w_0(x) = \|f\|_{L^\infty}w_0(x) - w(x).$$

Clearly, $w_0(x), \tilde{w}(x) \ge 0$ for all $x \in M$ and $w_0(x_0) = \tilde{w}(x_0) = 0$. Using the definition of π_n^β, and the positivity of $\|f\|_{L^\infty} - f(x)$, it is not difficult to show that

$$\pi_n^\beta(\tilde{w}, x) \ge -C\|f\|_{L^\infty}d(x, x_0)^{\beta+\varepsilon},$$
$$\pi_n^\beta(\tilde{w}, x) \ge -C\|f\|_{L^\infty}h_n.$$

Then, imitating the argument used in Lemma 3.29, we can construct a function function $\tilde{R}_{n,\beta+\varepsilon,x_0}$ such that $\tilde{R}_{n,\beta+\varepsilon,x_0}(x_0) = 0$, $\|\tilde{R}_{n,\beta+\varepsilon,x_0}\|_{C^\beta} \to 0$ as $n \to \infty$, and

$$\pi_n^\beta(\tilde{w}, x) + \|f\|_{L^\infty}\tilde{R}_{n,\beta+\varepsilon,x_0}(x) \ge 0 \Rightarrow \|f\|_{L^\infty(M)}(\pi_n^\beta(w_0, x) + \tilde{R}_{n,\beta+\varepsilon,x_0}(x)) \ge \pi_n^\beta(w, x).$$

On the other hand, from Proposition A.12, we have that if $d(x, x_0) \le 4\delta\sqrt{d}$, then

$$\pi_n^\beta(w_0, x) \le l(\nabla\pi_n^\beta(w_0), x_0; x) + q(\nabla^2\pi_n^\beta(w_0), x_0; x) + C\|w_0\|d(x, x_0)^{\beta+\varepsilon}.$$

Using that $\nabla w_0(x_0) = 0$ for $\beta > 1$, $\nabla^2 w_0(x_0) = 0$ for $\beta > 2$, together with Lemma A.14, it is easy to see that (cf. Proposition 3.16)

$$\|l(\nabla\pi_n^\beta(w_0), x_0; \cdot)\|_{C^\beta(B_\delta(x_0))} \le C\tilde{h}_n\|w_0\|_{C^\beta}, \text{ for } \beta > 1,$$
$$\|q(\nabla^2\pi_n^\beta(w_0), x_0; \cdot)\|_{C^\beta(B_\delta(x_0))} \le C\tilde{h}_n^{\beta-2}\|w_0\|_{C^\beta}, \text{ for } \beta > 2.$$

Combining this with the function $\tilde{R}_{n,\beta+\varepsilon,x_0}$, it is not hard to see there is a function R_{n,x_0} vanishing at x_0, such that

$$\pi_n^\beta(w,x) \le C\|f\|_{L^\infty(M)}\left(d(x,x_0)^{\beta+\varepsilon} + R_{n,x_0}(x)\right),$$
$$\lim_{n\to 0}\|R_{n,x_0}\|_{C^\beta} = 0,$$

and the proposition is proved. $\qquad\square$

The estimate in Proposition 4.17 lead , via the GCP, to a useful estimate for I and I_n.

Lemma 4.18. *Let $x \in G_n$, $\beta \in (0,2]$, $f \in C_b(M)$, and $u \in C_b^\beta(M)$ be fixed. Define the function, w, to be $w(y) = f(y)\rho(d(x,y)^\beta)$. Then for a universal C it holds that*

$$I(\pi_n^\beta u + \pi_n^\beta w, x) - I(\pi_n^\beta u, x) \le C\|f\|_{L^\infty(M)}\|\rho(d(x,\cdot)^\beta) + R_{n,x}(\cdot)\|_{C^\beta(M)},$$

and

$$E_n^0 T_n I(\pi_n^\beta u + \pi_n^\beta w, x) - E_n^0 T_n I(\pi_n^\beta u, x) \le C\|f\|_{L^\infty(M)}\|\rho(d(x,\cdot)^\beta) + R_{n,x}(\cdot)\|_{C^\beta(M)},$$

where $R_{n,x}$ is as in Proposition 4.17.

Proof of Lemma 4.18. We note that by Proposition 4.17 there is the touching of the two functions at x:

$$\pi_n^\beta u(y) + \pi_n^\beta w(y) \le \pi_n^\beta u(y) + C\|f\|_{L^\infty}(\rho(d(x,y)^\beta) + R_{n,x}(y)).$$

Thus, by the GCP,

$$I(\pi_n^\beta u + \pi_n^\beta w, x) \le I(\pi_n^\beta u + C\|f\|_{L^\infty}(\rho(d(x,\cdot)^\beta) + R_{n,x}(\cdot)), x).$$

Subtracting $I(\pi_n^\beta, x)$ from both sides, and using the Lipschitz assumption on I, we see that

$$I(\pi_n^\beta u + \pi_n^\beta w, x) - I(\pi_n^\beta u, x) \le I(\pi_n^\beta u + C\|f\|_{L^\infty}(\rho(d(x,\cdot)^\beta) + R_{n,x}(\cdot)), x) - I(\pi_n^\beta u, x)$$
$$\le C\|I\|_{Lip(C_b^\beta, C_b)} \cdot \|f\|_{L^\infty} \cdot \|\rho(d(x,\cdot)^\beta) + R_{n,x}(\cdot)\|_{C^\beta(M)}.$$

We remark that also, the operator

$$E_n^0 T_n I,$$

is an operator with the GCP if one only considers contact points belonging to G_n. Therefore, substituting $E_n^0 T_n I$ instead of I in the previous calculation preserves the result. $\qquad\square$

4.3. **The Structure of $\mathcal{D}I_n$, compactness, and weak limits.** In this section, we investigate in more detail the structure of the operators, $L_n \in \mathcal{D}I_n$. In particular, for $x \in G_n$, each $L_n(\cdot, x)$ is expressed as the sum of an (approximately) local part and a nonlocal part (see Lemma 4.25). The local and nonlocal parts are given in terms of a discrete measure associated to L_n and x, and using this we obtain compactness properties and other limiting properties for the L_n.

Lemma 4.19. *For all $L_n \in \mathcal{D}I_n$ and for all $x \in G_n$, there exist discrete signed Borel measures, μ_x^n, and functions $C^n(x)$ such that for all $u \in C_b^\beta(M)$,*

$$\forall x \in G_n, \quad L_n(u,x) = C^n(x)u(x) + \int_{M\setminus\{x\}} u(y) - u(x)\, \mu_x^n(dy).$$

Moreover,

$$C^n(x) = L_n(1,x), \tag{4.8}$$

and

$$\mu_x^n(dy) = \sum_{y \ne x} K^n(x,y)\delta_y(dy), \quad with\ K(x,y) = L_n(E_n^\beta e_y, x),$$

where $e_y \in C(G_n)$ are the "basis" functions introduced in Section 2.

Proof of Lemma 4.19. The proof is immediate from Section 2 for any $l_n \in \mathcal{D}i_n$, which are linear mappings from $C(G_n) \to C(G_n)$. However, thanks to Corollary 4.8, we know that any such L_n is of the form $E_n^0 \circ l_n \circ T_n$, for some $l_n \in \mathcal{D}i_n$. Since E_n^0 is, by definition, an extension operator, and the Lemma only uses information of u and l_n restricted to G_n, we see that indeed the formula from Lemma 2.1 and l_n is preserved for L_n as well. □

Recall that if $x \in G_{n'}$ for some n', then $x \in G_n$ all $n \geq n'$. This means that, for $x \in \cup G_n$, and for any sequence of operators L_n with $L_n \in \mathcal{D}I_n$, we have a respective sequence of Borel measures $\{\mu_x^n\}_{n \geq n'}$. Therefore, we are interested in obtaining bounds on these measures that allow us to obtain some kind of limit (at least along subsequences) as $n \to \infty$. These bounds are obtained in the following two lemmas.

Lemma 4.20. *Let $\beta \in (0,2]$. If $L_n \in \mathcal{D}I_n$, then L_n obeys the estimate of Corollary 4.16. Moreover, given $x \in G_{n'}$ fixed, , $\varepsilon \in [0,1)$, $f(x) = 0$, and $w(y) = f(y)\rho(d(x,y)^{\beta+\varepsilon})$, we have, for $n \geq n'$,*

$$L_n(w,x) \leq C\|f\|_{L^\infty(M)}\|\rho(d(x,\cdot)^{\beta+\varepsilon}) + R_{n,x}(\cdot)\|_{C^\beta(M)}.$$

Proof of Lemma 4.20. This follows by an argument entirely analogous to the one in Lemma 4.10. In this case, one invokes Lemma 4.18 to establish the estimates on any $L_n \in \mathcal{D}I_n$ that is an actual Fréchet derivative, $L_n = DI_n|_u$ at some $u \in C_b^\beta$, then pass the resulting estimate by density and convexity to all other elements of $\mathcal{D}I_n$. □

Lemma 4.21. *Let $L_n \in \mathcal{D}I_n$, and $\{\mu_x^n\}_{x \in G_n}$ the respective signed measures associated to L_n by Lemma 4.19. If m_x^n is the signed measure defined as*

$$m_n^x(dy) = \rho(d(x,y)^\beta)\mu_x^n(dy),$$

then, the total variation of m_x^n, denoted $|m_x^n|$, is bounded independently of n and x.

When dealing with $C^\beta = C_b^1$, we replace m_x^n by

$$m_n^x(dy) = \rho(d(x,y)^{1+\varepsilon})\mu_x^n(dy) \text{ for } \varepsilon \in (0,1).$$

Proof. We note that for x fixed and for any $f \in C_b^\beta(M)$, the function $f(y)\rho(d(x,y)^\beta) \in C_b^\beta(M)$. Furthermore, since $f(x)\rho(d(x,x)^\beta) = 0$, we obtain via Lemma 4.19 that

$$L_n(f\rho(d(x,\cdot)^\beta),x) = \int_M f(y)\rho(d(x,y)^\beta)\mu_x^n(dy).$$

Thus, the estimate of Lemma 4.20 immediately shows that

$$\int_M f(y)m_x^n(dy) \leq C\|f\|_{L^\infty}\|\rho(d(x,\cdot)^\beta) + R_{n,x}(\cdot)\|_{C^\beta(M)},$$

and we obtain the bound taking the supremum over f with $\|f\|_{L^\infty} \leq 1$, by duality.

□

Definition 4.22. *We use smooth approximations to the indicator and bump functions. Let x be fixed, with η_x^ϵ and $\tilde{\eta}_x^\epsilon$ be smooth functions satisfying*

$$0 \leq \eta_x^\epsilon(y) \leq 1, \quad \eta_x^\epsilon(y) \geq \mathbb{1}_{B_{r_0}}(y), \quad \eta_x^\epsilon(y) \searrow \mathbb{1}_{B_{r_0}(x)}(y), \text{ as } \epsilon \to 0$$
$$0 \leq \tilde{\eta}_x^\epsilon(y) \leq 1, \quad \tilde{\eta}_x^\epsilon(y) \geq \mathbb{1}_{B_\epsilon(x)}(y), \quad \tilde{\eta}_x^\epsilon(y) \searrow \mathbb{1}_{\{x\}}(y), \text{ as } \epsilon \to 0.$$

Definition 4.23. *For $\beta \in [0, 3)$ and $\epsilon \in (0, 1)$, the ϵ-Taylor "polynomial" of u centered at x is the function $T_x^{\epsilon,\beta}(u, y) \in C_b^\beta(M)$ given by*

$$T_x^{\epsilon,\beta}(u, y) = \begin{cases} u(x) & \text{if } \beta \in (0, 1) \\ u(x) + \eta_x^\epsilon(y)l(x, \nabla u(x); y) & \text{if } \beta \in [1, 2) \\ u(x) + \eta_x^\epsilon(y)l(x, \nabla u(x); y) + \tilde{\eta}_x^\epsilon(y)q(x, \nabla^2 u(x); y) & \text{if } \beta \in [2, 3). \end{cases}$$

Definition 4.24. *Fix $L_n \in \mathcal{DI}_n$ and $x \in G_n$, and let μ_x^n be the measure from Lemma 4.19 and let I denote the identity matrix $(TM)_x \to (TM)_x$.*
Then, we define $A^{\epsilon,n}(x) : (TM)_x \to (TM)_x$ by

$$A^{\epsilon,n}(x) = \int_M \tilde{\eta}_x^\epsilon(y)q(I, x; y)\mu_x^n(dy). \tag{4.9}$$

Furthermore, using duality, we define $B^{\epsilon,n}(x) \in (TM)_x$ as the unique vector in $(TM)_x$ such that

$$(B^{\epsilon,n}(x), p)_{g_x} = \int_M \eta_x^\epsilon(y)l(p, x; y)\mu_x^n(dy), \quad \forall\, p \in (TM)_x. \tag{4.10}$$

Lemma 4.25. *Let $L_n \in \mathcal{DI}_n, x \in M$ and $u \in C_b^3(M)$. Then, for some "remainder term", denoted $(\text{Error})_{L_n,u,x}$, we have the following representation for $L_n(u, x)$: If $\beta = 2$, then*

$$L_n(u, x) = \text{tr}(A^{\epsilon,n}(x)\nabla^2 u(x)) + (B^{\epsilon,n}(x), \nabla u(x))_{g_x} + C^n(x)u(x)$$

$$+ \int_M u(y) - T_x^{\epsilon,\beta}(u, y)\, \mu_x^n(dy) + (\text{Error})_{L_n,u,x};$$

if $\beta \in [1, 2)$, then

$$L_n(u, x) = (B^{\epsilon,n}(x), \nabla u(x))_{g_x} + C^n(x)u(x) + \int_M u(y) - T_x^{\epsilon,\beta}(u, y)\, \mu_x^n(dy) + (\text{Error})_{L_n,u,x};$$

and if $\beta \in (0, 1)$, then (note there is no remainder term in this case)

$$L_n(u, x) = C^n(x)u(x) + \int_M u(y) - u(x)\, \mu_x^n(dy).$$

Moreover, for every $\epsilon > 0$ fixed, the term $(\text{Error})_{L_n,u,x}$ satisfies the estimate

$$|(\text{Error})_{L_n,u,x}| \le Ch_n\|u\|_{C^3(M)}.$$

While $A^{\epsilon,n}(x)$, $B^{\epsilon,n}(x)$, and $C^n(x)$ satisfy the estimates

$$|A^{\epsilon,n}(x)|_{g_x} \le C, \quad |B^{\epsilon,n}(x)|_{g_x} \le C, \quad |C^n(x)| \le C.$$

In all cases C denoting a universal constant.

Proof of Lemma 4.25. When $\beta \in (0, 1)$, then we just apply Lemma 4.19 directly to L_n, and the Lemma in this case is trivial. For $\beta > 1$, the key observation is that we can write, for fixed $x \in G_{n'}$ and $n \ge n'$,

$$L_n(\cdot, x) = L_n(\cdot, x) \circ T_x^{\epsilon,\beta} + L_n(\cdot, x) \circ (\text{Id} - T_x^{\epsilon,\beta}).$$

Then, the first three terms in the desired expression for $L_n(u, x)$ arise from $L_n(\cdot, x) \circ T_x^{\epsilon,\beta}$, using Definition 4.23 to obtain $A^{\epsilon,n}(x)$ and $B^{\epsilon,n}(x)$. The term $(\text{Error})_{L_n,u,x}$ arises simply due to the perturbation of the gradient and Hessian made when applying π_n^β. However, Lemma A.14 guarantees the error made is bounded by $Ch_n\|u\|_{C^3(M)}$.

As for the term $L_n(\cdot, x) \circ (\text{Id} - T_x^{\epsilon,\beta})$, note that by definition

$$u(x) - T_x^{\epsilon,\beta}(u, x) = 0,$$

and so the terms $C^n(x)$ from Lemma 4.19 are not present in the representation of the second term. $\qquad\square$

The next lemma yields lower bounds for $A^{\epsilon,n}$ and μ_x^n. These bounds say that for large n, (and for fixed ϵ and $x \in \cup G_n$), $A^{\epsilon,n}$ is almost a positive semi-definite matrix, and μ_x^n is almost a positive measure.

Lemma 4.26. *There is a universal constant C, such that if $x \in G_{n'}$, and $n \geq n'$, then:*
With I denoting the identity map $(TM)_x \to (TM)_x$, we have

$$A^{\epsilon,n}(x) \geq -Ch_n^\gamma \epsilon^{-3} I.$$

Moreover, for all $f \in C_b^3(M)$ such that $f \geq 0$ and $f(x) = 0$, we have

$$\int_M f(y)\mu_x^n(dy) \geq -Ch_n^\gamma \|f\|_{C^3(M)}.$$

Here, h_n is as defined in (3.6), and γ is as in Lemma 3.25.

Proof of Lemma 4.26. Both of these results are immediate consequences of Lemma 4.13. Indeed, for the case of $A^{\epsilon,n}(x)$, consider a fixed unit vector, $v \in (TM)_x$, and the function

$$w(y) = \tilde{\eta}_x^\epsilon(y)q(v \otimes v, x; y),$$

where $\tilde{\eta}_x^\epsilon(y)$ is the function from Definition 4.22.

On the other hand, from the definition of q, we have that $\nabla^2 w(x) = v \otimes v$ and $\nabla w(x) = 0$, see Remark 3.14. It is also clear that $w(y) \geq 0$ for all y and that $w(x) = 0$. Then, applying Lemma 4.19 to w, it follows that

$$L_n(w, x) = \int_M \tilde{\eta}_x^\epsilon(y)q(v \otimes v, x; y)\mu_x^n(dy).$$

In light of the formula (4.9), we have that

$$L_n(w, x) = \text{tr}(A^{\epsilon,n}(x)v \otimes v).$$

Then, using Lemma 4.13 to bound $L_n(w, x)$, we conclude that

$$\text{tr}(A^{\epsilon,n}(x)v \otimes v) \geq -Ch_n^\gamma \|w\|_{C^3(M)}.$$

Using that $\tilde{\eta}_x^\epsilon$, $\|w\|_{C^3} \leq C\epsilon^{-3}$, as well as Proposition 3.16, we obtain the lower bound for $A^{\epsilon,n}(x)$.

It remains to prove the bound for μ_x^n. We use Lemma 4.19 once again, and apply to a function $f \in C^3(M)$ such that $f(x) = 0$, which yields

$$L_n(f, x) = \int_M f(y)\mu_x^n(dy).$$

Then, Lemma 4.13 applied to the left hand side yields the desired bound. $\qquad\square$

The next lemma is concerned with the "pointwise" limits for sequences $\{L_n\}$ where for each n we have $L_n \in \mathcal{DI}_n$ for each n. The lemma says essentially the following: given $x \in \bigcup G_{n'}$, the sequence $\{L_n(\cdot, x)\}_{n \geq n'}$, seen as a sequence of linear functionals $C_c^\beta(M) \to \mathbb{R}$, must converge along a subsequence to a functional of Levy type based at x (recall Definition 1.4).

Lemma 4.27. *Let $x \in G_{n'}$, and for every $n \geq n'$ let $L_n \in \mathcal{DI}_n$. There is a subsequence $n_k \to \infty$ such that $L_{n_k}(\cdot, x)$ converges weakly to some $L_x : C_b^\beta \to \mathbb{R}$, that is,*

$$\lim_{k \to \infty} L_{n_k}(u, x) = L_x(u), \quad \forall u \in C_c^\beta(M),$$

where L_x is a functional of Lévy-type based at x. Furthermore, the functional L_x inherits an analogue of (4.5), *namely, there is a universal C such that*

$$|L_x(u)| \leq C\|u\|_{C^\beta(B_{2r+3})} + C\omega(r)\|u\|_{L^\infty(M)}.$$

Remark 4.28. *The proof below will actually say more than what was stated in Lemma 4.27, and it shall highlight how Levy operators arise naturally as the limits of the Laplacian on sequences of weighted graphs that are becoming large as $n \to \infty$.*

Concretely, fix $x \in G_{n'}$. Let $C^n(x), A^{\epsilon,n}(x)$, and $B^{\epsilon,n}(x)$ be as in (4.8),(4.9), *and* (4.10). *Then, as shown below, there are subsequences $n_k \to \infty, \epsilon_j \to 0$ such that: 1) we have the limits*

$$A(x) := \lim_j \lim_k A^{\epsilon_j, n_k}(x), \quad B(x) := \lim_j \lim_k B^{\epsilon_j, n_k}(x), \quad C(x) := \lim_k C^{n_k}(x),$$

2) $\mu_x^{n_k}$ converges weakly in compact subsets of $M \setminus \{x\}$ to a positive measure μ_x and 3) for every $u \in C_c^\beta(M)$ we have

$$\lim_{k \to \infty} L_{n_k}(u,x) = \text{tr}(A(x)\nabla^2 u(x)) + (B(x), \nabla u(x))_{g_x} + C(x)u(x)$$

$$+ \int_{M \setminus \{x\}} u(y) - u(x) - \chi_{B_{r_0}}(y)(\nabla u(x), \exp_x^{-1}(y))_{g_x} \mu_x(y).$$

Proof of Lemma 4.27. For this proof, we only demonstrate the case of $C_b^\beta = C_b^2$ as it includes all of the details. The other four cases of β follow from a similar and simpler argument.

The case, $C_b^\beta = C_b^2$.

Let $x \in \cup_n G_n$ be fixed. Since G_n are increasing, we know $x \in G_n$ for all $n \geq n'$ for some n'. Also, by Lemma 4.21, we know that the measures m_x^n have bounded variation in $M \setminus \{x\}$, so we are free to use the Jordan decomposition to write

$$m_x^n = (m_x^n)^+ - (m_x^n)^-.$$

Furthermore, both of the measures $(m_x^n)^+$ and $(m_x^n)^-$ are uniformly bounded in x, and n, for $n \geq n'$ given by Lemma 4.21.

Step 1: extracting weak limits in n for ϵ fixed.

We can use the compactness of Radon measures, e.g. [20, p. 55] to extract weakly convergent subsequences of $(m_x^n)^+$ and $(m_x^n)^-$, and hence also m_x^n. We will label by n_k, and we will call the weak limiting signed measure as $\bar{m}_x$, i.e.

$$m_x^{n_k} \rightharpoonup \bar{m}_x,$$

but we note that a posteriori we will validate that $\bar{m}_x \geq 0$. For the moment, we keep ϵ fixed.

Let $u \in C_c^2(M)$. By Lemma 4.25, we have

$$L_n(u,x) = \text{tr}(A^{\epsilon,n}(x)\nabla^2 u(x)) + (B^{\epsilon,n}(x), \nabla u(x))_{g_x} + C^n(x)u(x) \tag{4.11}$$

$$+ \int_{M \setminus \{x\}} u(y) - T_x^{\epsilon,\beta}(u,y) \, \mu_x^n(dy) + (\text{Error})_{L_n, u, x}. \tag{4.12}$$

First we work on the nonlocal part, (4.12). We see that

$$\int_{M \setminus \{x\}} u(y) - T_x^{\epsilon,\beta}(u,y) \, \mu_x^n(dy)$$

$$= \int_{M \setminus \{x\}} \frac{u(y) - T_x^{\epsilon,\beta}(u,y)}{\rho((d(x,y)^2)} \rho((d(x,y)^2)\mu_x^n(dy)$$

$$= \int_{M \setminus \{x\}} \frac{u(y) - T_x^{\epsilon,\beta}(u,y)}{\rho((d(x,y)^2)} m_x^n(dy).$$

At this point, we note that by the C^2 nature of u, the function,

$$\frac{u(y) - T_x^{\epsilon,\beta}(u,y)}{\rho((d(x,y)^2)},$$

does in fact extend to a continuous function on M. Hence, by the weak limit of $m_x^{n_k}$, we see then that

$$\lim_{k\to\infty} \int_{M\setminus\{x\}} \frac{u(y) - T_x^{\epsilon,\beta}(u,y)}{\rho((d(x,y)^2)} m_x^{n_k}(dy) = \int_{M\setminus\{x\}} \frac{u(y) - T_x^{\epsilon,\beta}(u,y)}{\rho((d(x,y)^2)} \bar{m}_x(dy).$$

We define the limiting Lévy measure on $M \setminus \{x\}$ as

$$\bar{\mu}_x(dy) = (\rho(d(x,y)^2))^{-1}\bar{m}_x(dy),$$

and we note that by Lemma 4.26 we also know that $\bar{\mu}_x$ is indeed non-negative and satisfies, by definition the integrability condition independent of x

$$\int_{M\setminus\{x\}} \min(d(x,y)^2, 1)\bar{\mu}_x(dy) \leq C$$

because by definition $\bar{m}_x$ are finite measures with total mass independent of x.

Next, we move on to the local part of L_n, given in Lemma 4.25, which we recorded in (4.11). We will establish that the matrices $A^{\epsilon,n}(x)$ and vectors $B^{\epsilon,n}(x)$ are all uniformly bounded in ϵ, n, x. Thus, weak limits are immediate (as bounded sequences in Euclidean space). First, we note by a direct calculation that for x fixed, as functions of y, $\tilde{\eta}_x^\epsilon q(x, e_i \otimes e_j; y)$ are in $C_b^2(M)$, independent of x, n, and ϵ. Furthermore, the functions $\eta_x^\epsilon(y)l(x, e_i; y)$ have a bounded C^2 norm inside, e.g. $y \in \overline{B_{1/2}(x)}$. Thus the bounds for $A^{\epsilon,n}(x)$ follow from Lemma 4.10, and the bounds for $B^{\epsilon,n}(x)$ follow from Lemma 4.15, equation (4.7).

This means that we also have coefficients that depend on ϵ

$$\bar{A}^\epsilon(x), \quad \bar{B}^\epsilon(x), \quad \bar{C}(x),$$

such that along a subsequence, again labeled as n_k, we have (recall, x is fixed)

$$\lim_{k\to\infty} L_n(T_x^{\epsilon,\beta}u, x) = \text{tr}(\bar{A}^\epsilon(x)\nabla^2 u(x)) + (\bar{B}^\epsilon(x), \nabla u(x))_{g_x} + \bar{C}(x)u(x).$$

Furthermore, by Lemma 4.26, we see that

$$\bar{A}^\epsilon(x) \geq 0.$$

Step 2: removing the ϵ dependence.

We note that the definition of the ϵ-Taylor expansion (Definition 4.23) requires smooth approximations of $\mathbb{1}_{B_{r_0}(x)}$ and $\mathbb{1}_{\{x\}}$, with ϵ being a small parameter. First, we note that in the previous paragraph, it was established that $A^{\epsilon,n}(x)$ and $B^{\epsilon,n}(x)$ are bounded independently of ϵ, n, and x. Thus the limits $\bar{A}^\epsilon(x)$ and $\bar{B}^\epsilon(x)$ are still bounded independently of ϵ and x. Invoking once again the compactness of bounded closed sets in finite dimensional spaces, we obtain a subsequence in ϵ, along which

$$\lim_{\epsilon_k\to 0} \text{tr}(\bar{A}^{\epsilon_k}(x)\nabla^2 u(x)) + (\bar{B}^{\epsilon_k}(x), \nabla u(x))_{g_x} + \bar{C}(x)u(x)$$

$$= \text{tr}(\bar{A}(x)\nabla^2 u(x)) + (\bar{B}(x), \nabla u(x))_{g_x} + \bar{C}(x)u(x),$$

and again, we preserve

$$\bar{A}(x) \geq 0.$$

Next we conclude with the $\epsilon \to 0$ limits for

$$\int_{M\setminus\{x\}} u(y) - T_x^{\epsilon,\beta}(u,y)\bar{\mu}_x(dy).$$

Using the bound on the error term in the Taylor expansion (see Proposition A.12), and since $u \in C_c^2$, we have that as $y \to x$,

$$|u(y) - u(x) - \eta^\epsilon(y)l(x,\nabla u(x);y)| \leq \|u\|_{C^2}\rho(d(x,y)^2).$$

Hence, by dominated convergence, we see that (recall Definition 4.22 for η^ϵ)

$$\lim_{\epsilon\to 0}\int_{M\setminus\{x\}} u(y) - u(x) - \eta^\epsilon(y)l(x,\nabla u(x);y)\bar{\mu}_x(dy)$$

$$= \int_{M\setminus\{x\}} u(y) - u(x) - \mathbb{1}_{B_{r_0}(x)}l(x,\nabla u(x);y)\bar{\mu}_x(dy).$$

For the quadratic term, $q(x,\nabla^2 u(x);y)$, we note that

$$\left|\tilde{\eta}^\epsilon(y)q(x,\nabla^2 u(x);y)\right| \leq C\|u\|_{C^2(M)}\rho(d(x,y)^2)\mathbb{1}_{B_{2\epsilon}(x)}.$$

Hence, since $\bar{\mu}_x \geq 0$,

$$\int_{M\setminus\{x\}}\left|\tilde{\eta}^\epsilon(y)q(x,\nabla^2 u(x);y)\right|\bar{\mu}_x(dy) \leq C\|u\|_{C^2(M)}\int_{B_{2\epsilon}(x)\setminus\{x\}}\rho(d(x,y)^2)\bar{\mu}_x(dy).$$

Since $\bar{m}_x$ is a finite measure, we see by the continuity of $\bar{m}_x$ that necessarily

$$\bar{m}_x(B_{2\epsilon}(x)\setminus\{x\}) \to 0 \text{ as } \epsilon \to 0.$$

We conclude then that

$$\lim_{\epsilon\to 0}\int_{M\setminus\{x\}}\left|\tilde{\eta}^\epsilon(y)q(x,\nabla^2 u(x);y)\right|\bar{\mu}_x(dy) = 0.$$

This means that after the subsequential limits first in n followed by ϵ, we do indeed recover for $u \in C_c^2(M)$,

$$\lim_{\epsilon_j\to 0}\lim_{n_k\to\infty} L_{n_k}(u,x) = \bar{L}_x(u),$$

and $\bar{L}_x$ is a functional of the Lévy form (Definition 1.4, (1.2)). This concludes the lemma for the case $\beta = 2$.

Now we make a few remarks as to how the remaining cases follow from the proof for $\beta = 2$. This is the only part in the proof in which there is a true distinction between them, and it all rests on the ability to extend continuously the function

$$\frac{u(y) - T_x^{\epsilon,\beta}(u,y)}{\rho((d(x,y)^\beta)}.$$

The case, $\beta = 1$, $C_b^1(M)$. This case is completely analogous to $\beta = 2$, and on one hand simpler because $A^{\epsilon,n}(x) \equiv 0$, but on the other hand, complicated by Lemma 4.21. Now, we let $\varepsilon \in (0,1)$ be given, and we take $u \in C_b^{1,\varepsilon}$, and we invoke Lemma 4.21 with $1 + \varepsilon/2$. Taylor's theorem applies in exactly the same way for the continuity of the quantity

$$\frac{u(y) - T_x^{\epsilon,1}(u,y)}{\rho((d(x,y)^{1+\varepsilon/2})},$$

at $y = x$, where now the numerator has slightly stronger decay, by choice of $u \in C^{1,\varepsilon}$.

The cases of $C_b^\beta(M) = C^{0,1}(M)$ **and** $C_b^\beta(M) = C^{1,1}(M)$**.** These cases go in the same way as respectively the cases of C^1 and C^2 because we limit ourselves to only checking the formula for $u \in C^1$ and respectively C^2. Hence, the respective continuity of, e.g.

$$\frac{u(y) - T_x^{\epsilon,1}(u,y)}{\rho((d(x,y))} \text{ and } \frac{u(y) - T_x^{\epsilon,2}(u,y)}{\rho((d(x,y)^2)},$$

is unchanged.

The other cases of $\beta \in (0,2)$**.**

The only real difference here is that in these cases, we are applying the argument to $u \in C_b^{\beta+\varepsilon}(M)$ for some small $\varepsilon > 0$. In this case, the slightly larger Hölder exponent, $\beta + \varepsilon$, is what gives the continuity of

$$\frac{u(y) - T_x^{\epsilon,\beta}(u,y)}{\rho((d(x,y)^\beta)},$$

because the numerator is of the order $d(x,y)^{\beta+\varepsilon}$. $\qquad\square$

In the case I satisfies the equicontinuity assumption (1.5), one can do better than Lemma 4.27: one can show the compactness of the elements of $L_n \in \mathcal{D}I_n$ as linear operators. Moreover, the proof is rather straightforward, it boils down to the Arzelá-Ascoli theorem.

Lemma 4.29. *Suppose that I satisfies* (1.5)*. Then, given a sequence $\{L_n\}$ with $L_n \in \mathcal{D}I_n$ for every n, there exists a subsequence L_{n_k} and a bounded linear operator $L : C_b^\beta \to C_b$ such that*

$$\lim_{k\to\infty} L_{n_k}(u,x) = L(u,x), \ \ \forall\, u \in C_c^3(M), \ \ x \in M.$$

Proof. Fix $K \subset M$ be a compact set, and let $\mathcal{B}$ denote the set

$$\mathcal{B} := \{u \in C_b^3(M) \mid u \equiv 0 \text{ outside } K, \|u\|_{C^3(M)} \leq 1\}.$$

It is clear that $\mathcal{B}$ is a compact subset of C_b^β, for each $\beta < 3$. From the assumption (1.5), the continuity of π_n^β (Theorem 3.23), and the convergence of π_n^β to the identity in C_b^3 (Lemma 3.25), it follows that if $L_n \in \mathcal{D}I_n(v_n)$, where $v_n \in \mathcal{B}$, then

$$\{L_n(u,\cdot)\}_{u\in\mathcal{B}} \text{ is equicontinuous}$$

In other words, the real valued functions given by

$$(u,x) \in K \times \mathcal{B} \to L_n(u,x),$$

form an equicontinuous family of functions from $K \times \mathcal{B}$ to $\mathbb{R}$. In particular, this family of functions is precompact in $C(K \times \mathcal{B})$ with respect to uniform convergence. Therefore, there is some subsequence n_k and some $L \in C(K \times \mathcal{B})$ such that

$$L_{n_k} \to L \text{ uniformly in } K \times \mathcal{B}.$$

By homogeneity, L_{n_k} converges as a function defined for all functions $u \in C^3$ which are compactly supported on K. Moreover, using the linearity of the L_{n_k} it is clear that L is also a linear operator. Then, taking an increasing sequence of compacts K_n which cover M, one can apply a Cantor diagonalization argument to obtain the desired sequence. $\qquad\square$

4.4. **Limits of the finite dimensional min-max– the proof of Theorem 1.6 and Proposition 1.7.** Now that we have collected various facts about $\mathcal{D}I_n$, we have enough information to finish the proof of Theorems 1.6 and 1.8. The last remaining step is to pass to the limit "inside" of the min-max.

Proof of Theorem 1.6. The key point of this proof is to use the compactness established in Lemma 4.27 to go from the min-max formula for I_n to one for I. We introduce the family depending on I,

$$\mathcal{K}_{Levy}(I) := \text{hull}\big(\{L : C_b^\beta \to \mathbb{R} \ : \exists \ n_k \to \infty \text{ and } L_{n_k} \in \mathcal{D}I_{n_k},$$
$$x_k \in G_{n_k} \text{ s.t. } L(f) = \lim_{k \to \infty} L_{n_k}(f, x_k) \ \forall f \in C_c^\beta(M)\}\big). \tag{4.13}$$

Among the implications of Lemma 4.27, $\mathcal{K}_{Levy}(I) \neq \emptyset$, and every element of $\mathcal{K}_{Levy}(I)$ is an operator of Levy type based at some $x \in M$. Then, our aim is to prove the following: for every $x \in M$, and every pair $u, v \in C_b^\beta(M)$, there is some $L \in \mathcal{K}_{Levy}(I)$ based at x such that

$$I(u, x) \leq I(v, x) + L(u - v). \tag{4.14}$$

We proceed to prove (4.14) in increasing order of generality: 1) for all $u, v \in C_c^\beta$ and $x \in G_n$ for some n, 2) for all $u, v \in C_c^\beta$ and any $x \in M$, and finally 3) for all $u, v \in C_b^\beta$ (that is, u, v that may not be compactly supported) and any $x \in M$.

Fix $u, v \in C_c^\beta(M)$, and let $x \in G_{n'}$, for some $n' \in \mathbb{N}$. Since the G_n are increasing, we have that $x \in G_n$ for all $n \geq n'$. The min-max formula for I_n with $n \geq n'$ (Lemma 4.9) yields the existence of some $L_{n,u,x} \in \mathcal{D}I_n$ such that

$$I_n(u, x) \leq I_n(v, x) + L_{n,u,x}(u - v, x).$$

Given that $u, v \in C_c^\beta(M)$, Proposition 4.3 guarantees that

$$\lim_n I_n(u, x) = I(u, x), \ \ \lim_n I_n(v, x) = I(v, x),$$

and in particular,

$$I(u, x) \leq I(v, x) + \limsup_n L_{n,u,x}(u - v, x).$$

Applying Lemma 4.27, and the definition of $\mathcal{K}_{Levy}(I)$ (4.13), we conclude the following: for any $x \in \bigcup G_n$, and $u, v \in C_c^\beta(M)$, there is a functional $L \in \mathcal{K}_{Levy}(I)$, based at x, such that

$$I(u, x) \leq I(v, x) + L(u - v).$$

More generally, if $x \in M$, then we can choose a sequence of points x_m with $x_m \to x$ and $x_m \in \bigcup G_n$. Then, for each m there is some L_{x_m} based at x_m such that

$$I(u, x_m) \leq I(v, x_m) + L_{x_m}(u - v)$$

Once again, passing to the limit in m (and using again the compactness of $\mathcal{K}_{Levy}(I)$), and using the continuity of $I(u, \cdot)$ and $I(v, \cdot)$, we conclude that there exists some $L \in \mathcal{K}_{Levy}(I)$, based at x, and such that

$$I(u, x) \leq I(v, x) + L(u - v).$$

Finally, we need to extend (4.14) to all $u, v \in C_b^\beta(M)$, and not just those with compact support. Fix $u, v \in C_b^\beta(M)$, and $x \in M$. Consider sequences $u_k, v_k \in C_c^\beta(M), k \in \mathbb{N}$, which are such that

$$\|u_k - u\|_{C^\beta(B_{2k}(x_*))} \leq 1/k, \ \ \|v_k - v\|_{C^\beta(B_k(x_*))} \leq 1/k.$$

Then, for each k we have some $L_k \in \mathcal{K}_{Levy}(I)$ such that

$$I(u_k, x) \leq I(v_k, x) + L_k(u_k - v_k).$$

The assumption (1.3) and Lemma 4.12 imply that for all sufficiently large k,

$$I(u, x) \leq I(u_k, x) + C\|u - u_k\|_{C^\beta(B_{2k}(x_*))} + C\|u - u_k\|_{L^\infty(M)},$$
$$I(v_k, x) \leq I(v, x) + C\|v - v_k\|_{C^\beta(B_{2k}(x_*))} + C\|v - v_k\|_{L^\infty(M)},$$
$$L_k(u_k - v_k) \leq L_{x,k}(u - v) + C\|u - v - (u_k - v_k)\|_{C^\beta(B_{2k}(x_*))}$$
$$+ C\omega(k)\|u - v - (u_k - v_k)\|_{L^\infty(M)}$$

Therefore,

$$I(u, x) \leq I(v, x) + L_k(u - v) + C\frac{1}{k} + C\omega(k), \quad \forall\, k \in \mathbb{N}.$$

Then, after possibly taking a subsequence of the L_k, we obtain (4.14) in the limit in this final case. Since (4.14) trivially yields equality for $v = u$, we conclude that for any $x \in M$

$$I(u, x) = \min_{v \in C_b^\beta(M)} \max_{L \in \mathcal{K}_{Levy}(I)} \{I(v, x) + L(u - v)\},$$

and this finishes the proof.

$\square$

The nature of the set $\mathcal{K}_{Levy}(I)$ and its dependence on I is a direct and trivial outcome of the proof of Theorem 1.6, we record it as a Proposition.

Proposition 4.30. *The family $\mathcal{K}_{Levy}(I)$ appearing in Theorem 1.6 has the form*

$$\mathcal{K}_{Levy}(I) := \mathrm{hull}\big(\big\{L : C_b^\beta \to \mathbb{R} \;:\; \exists\, n_k \to \infty \text{ and } L_{n_k} \in \mathcal{D}I_{n_k},$$
$$x_k \in G_{n_k} s.t.\ L(f) = \lim_{k \to \infty} L_{n_k}(f, x_k)\ \forall f \in C_c^\beta(M)\big\}\big).$$

Finally, we comment on the minor modifications needed to obtain the stronger min-max result, under assumption (1.5).

Proof of Theorem 1.8. The proof is exactly as that of the previous Theorem, except we invoke Lemma 4.29 in place of Lemma 4.27, which is made possible once we have (1.5). In this case, we obtain convergence as operators of subsequences of L_n, where $L_n \in \mathcal{D}I_n$ for every n. We define

$$\mathcal{L} := \{L \mid \exists \{n_k\}_k, n_k \to \infty, \text{ and } L_{n_k} \in \mathcal{D}I_{n_k} \text{ such that } L(u, x) = \lim_k L_{n_k}(u, x) \forall u \in C_c^3(M)\}.$$

The min-max formula using the operators in $\mathcal{L}$ is proved as before, and the fact that for each x we have $L(\cdot, x) \in L_{Levy}(I)$ is immediate in light of Proposition 4.30.

$\square$

Remark 4.31. *In order to illustrate the difference between $I : C_b^\beta(M) \to C_b(M)$ and maps on a finite dimensional space, we point the reader to (4.14). If I were differentiable on a dense set of functions, one can basically go straight to this point– see e.g. Proposition 2.6 and the proof of Lemma 2.4. However, for generic Lipschitz I in infinite dimensional spaces, Fréchet differentiability on a dense set is not expected to hold. Thus, most of the difficulty was contained in obtaining (4.14).*

Remark 4.32. *One may ask how it is that including such a large set of linear functionals centered at x as $\mathcal{K}_{Levy}(I)$ in the max of the min-max formula (1.4) does not corrupt simpler operators that may not use all such linear functionals. Suppose that I is a simpler operator of the form*

$$I(u, x) = \max\{L_a(u, x), L_b(u, x)\},$$

where L_a and L_b are simply two fixed operators that have the GCP and properties (1.3)*. The reader can check in a straightforward fashion that indeed*

$$\min_{v \in C_b^\beta} \max_{L_x \in \mathcal{K}_{Levy}(I)} \left(I(v,x) + L_x(u-v)\right) = \max\{L_a(u,x), L_b(u,x)\}.$$

The main points are that choosing $v = u$ in the minimum immediately gives one inequality, and the reverse inequality comes from the fact that if L_a and L_b are linear maps from $C_b^\beta(M) \to C_b(M)$ with the GCP, then for x fixed, the linear functionals $L_a(\cdot, x)$ and $L_b(\cdot, x)$ are both of Lévy type, and hence in $\mathcal{K}_{Levy}(I)$.

4.5. **Convex operators.** If the Lipschitz operator I is assumed to be *convex*, then it may be represented simply as a maximum of linear operators of the same type as those appearing in the min-max formula from Theorem 1.6. First, let us recall what it means for an operator to be convex.

Definition 4.33. *An operator I is said to be convex if for any two functions u, v, and $x \in M$, and any $\lambda \in (0, 1)$ the following inequality holds*

$$I(\lambda u + (1-\lambda)v, x) \le \lambda I(u,x) + (1-\lambda)I(v,x).$$

The operator is said to be concave if the above inequality is reversed.

The convexity condition can clearly be restated as

$$t^{-1}\left(I(v + t(u-v), x) - I(v, x)\right) \le I(u) - I(v) \ \ \forall \, t \in [0, 1].$$

Taking $s \in [0, 1]$ and applying the above inequality to the functions v and $v + s(u - v)$, one sees that convexity of I is equivalent to the condition

$$t^{-1}\left(I(v + t(u-v), x) - I(v, x)\right) \le s^{-1}\left(I(v + s(u-v), x) - I(v, x)\right), \ \ \forall \, 0 \le t \le s \le 1.$$

Lemma 4.34. *Let M and I be as in Theorem 1.6. If in addition, I is known to be convex, then*

$$I(u, x) = \max_{v, L_x}\{I(v, x) + L_x(u - v)\}.$$

Here the maximum is over some family of pairs (v, L_x) where $v \in C_b^\beta$, and each L_x lies in the same family of functionals as in Theorem 1.6. Likewise, if I is concave, an analogous statement holds with a minimum instead of a maximum.

Proof. Let I_n be the finite dimensional approximation to I. By its construction, it is clear that I_n is convex if I is convex. We shall show that the min-max formula for I_n reduces to a max formula when I_n is convex. From this point on, the proof of the Lemma follows the argument used to obtain (4.14) in the proof of Theorem 1.6.

Fix $u, v \in C_b^\beta$ and $x \in \tilde{G}_n$ for some n. Assume further that v is such that I_n is differentiable at v, with derivative $L_v(\cdot)$. Then, due to the convexity of I_n, the function

$$t \to t^{-1}\left(I_n(v + t(u-v), x) - I(v, x)\right),$$

is nondecreasing for $t > 0$. Therefore,

$$t^{-1}\left(I_n(v + t(u-v), x) - I(v, x)\right) \ge \limsup_{t \to 0^+}\{t^{-1}\left(I_n(v + t(u-v), x) - I_n(v, x)\right)\}$$
$$= L_v(u - v, x).$$

In particular, for $t = 1$

$$I_n(u, x) \ge I(v, x) + L_v(u - v, x), \ \ \forall \, x \in \tilde{G}_n, \ u \in C_b^\beta(M).$$

If I_n is not differentiable at v, we take a sequence $v_k \to v$ with I_n differentiable at each v_k. Then,

$$I_n(u,x) \geq I_n(v_k,x) + L_{v_k}(u - v_k, x), \quad \forall\, k,$$

passing to the limit $k \to \infty$

$$I_n(u,x) \geq I_n(v,x) + \limsup_k L_{v_k}(u - v_k, x).$$

From here, it follows that for every $v \in C_b^\beta$ there is some L such that

$$I_n(u,x) \geq I_n(v,x) + L_x(u - v).$$

Since v is arbitrary and the above becomes an equality whenever $u = v$, it follows that we have

$$I_n(u,x) = \max_{v,L}\{I_n(v,x) + L(u - v, x)\},$$

the maximum being over some family of pairs (v,L). This proves the maximum for each of the finite dimensional approximations I_n. As mentioned at the beginning of the proof, to obtain the maximum formula for I, one proceeds by the same limiting argument used in the proof of Theorem 1.6, we leave the details to the reader. $\qquad\square$

4.6. **Extremal operators.** An elementary consequence of the min-max formula for I, is that one can bound the difference $I(u,x) - I(v,x)$ via "extremal operators". Namely, since

$$I(u,v) - I(v,x) = \min_{v'} \max_L \left\{ I(v',x) + L(u - v', x) \right\} - I(v,x),$$

$$(\text{take } v' = v) \leq \max_L \left\{ L(u - v, x) \right\}.$$

Likewise,

$$I(u,x) - I(v,x) \geq -\max_L \left\{ L(v - u, x) \right\},$$

$$\geq \min_L \left\{ L(u - v, x) \right\}.$$

Therefore, we call the following the extremal inequalities for I:

$$\min_L\{L(u - v, x)\} \leq I(u,x) - I(v,x) \leq \max_L\{L(u - v, x)\}; \tag{4.15}$$

and given a family of linear functionals, $\mathcal{L}$, we define the extremal operators

$$M_{\mathcal{L}}^+(u,x) = \sup_{L \in \mathcal{L}}\{L(u,x)\} \quad \text{and} \quad M_{\mathcal{L}}^-(u,x) = \inf_{L \in \mathcal{L}}\{L(u,x)\}. \tag{4.16}$$

Note, these extremal operators have made important appearances in PDE and control theory for decades (and most likely in other fields). For second order equations they can be traced back to Pucci [42], see also their importance in Caffarelli [8] or in the book of Caffarelli-Cabré [9, Chp 2-4]. They also play a fundamental role in much of the theory for integro-differential equations for both linear and nonlinear operators (a very abridged list is e.g. [7], [13], [12], [31], [45], etc...).

Now, one may ask if the reverse holds. Namely, if the (4.15) holds, does it follow that I can be written as a min-max of operators belonging to the class $\mathcal{L}$? The next lemma gives a partial answer to this question –which will be useful in a forthcoming work dealing with Dirichlet to Neumann maps.

Proposition 4.35. *Assume that I is as in Theorem 1.6, and suppose further that there exists a class of functionals $\mathcal{L}$, so that I obeys the extremal inequalities (4.15) with respect to $\mathcal{L}$. Then, with $\mathcal{K}_{Levy}(I)$ as in Proposition 4.30, it holds that for all $L_x \in \mathcal{K}_{Levy}(I)$*

$$\forall\, \phi \in C_c^3(M), \ \forall\, x \in M, \quad M_{\mathcal{L}}^-(\phi,x) \leq L_x(\phi).$$

We will prove this proposition via two more basic (and possibly also useful) facts separately, where both of them invoke the finite dimensional operators.

Lemma 4.36. *Let I and $\mathcal{L}$ be as in Proposition 4.30. Let I_n, $(M_{\mathcal{L}}^-)_n$, and $(M_{\mathcal{L}}^+)_n$ be the finite dimensional approximations defined in (4.2) for respectively I, $M_{\mathcal{L}}^-$, and $M_{\mathcal{L}}^+$ from (4.16), and let $u, v \in C_b^\beta(M)$. Then*

$$(M_{\mathcal{L}}^-)_n(u - v) \le \mathcal{I}_n(u) - \mathcal{I}_n(v) \le (M_{\mathcal{L}}^+)_n(u - v),$$

i.e. the approximation (4.2) preserves extremal inequalities.

Lemma 4.37. *Let I, $\mathcal{L}$, and $M_{\mathcal{L}}^-$ be as in Proposition 4.35. Let n be fixed, let I_n be defined in (4.2), and assume I_n is Fréchet differentiable at $u \in C_b^\beta(M)$ with derivative $DI_{n,u}$, let $x \in M$, and let $\phi \in C_b^3(M)$. Then the following estimate is true*

$$-h_n^\gamma \|\phi\|_{C^3} + M_{\mathcal{L}}^-(\phi, x) \le DI_{n,u}(\phi, x),$$

where $h_n^\gamma \to 0$ arises from Lemma 3.25 and is defined in (3.6).

For notational reasons, it will be easiest to simply present the proofs of Lemmas 4.36 and 4.36 together.

Proof of Lemmas 4.36 and 4.37. First, let $u, v \in C_b^\beta(M)$. We will use the fact that restriction/extension compositions

$$E^\beta \circ T_n \text{ and } E^0 \circ T_n$$

are both linear operators, and furthermore that $E^0 \circ T_n$ preserves ordering. Using the extremal inequality of (4.16), we see that since $E_n^\beta \circ T_n u$ and $E_n^\beta \circ T_n v$ are again in $C_b^\beta(M)$, it holds that

$$M_{\mathcal{L}}^-(E_n^\beta \circ T_n(u - v)) \le I(E_n^\beta \circ T_n u) - I(E_n^\beta \circ T_n v),$$

(and we have used linearity of $E_n^\beta \circ T_n$ on the left). Now we may apply $E^0 \circ T_n$ to both sides, and we use the monotonicity and linearity to conclude

$$E^0 \circ T_n \left(M_{\mathcal{L}}^-(E_n^\beta \circ T_n(u - v)) \right) \le E^0 \circ T_n \left(I(E_n^\beta \circ T_n u) \right) - E^0 \circ T_n \left(I(E_n^\beta \circ T_n v) \right).$$

Hence by the definition of $(M_{\mathcal{L}}^-)_n$ and I_n in (4.2), we have obtained half of Lemma 4.36. The other inequality follows the same proof.

Now to obtain the estimate on DI_n, let $t > 0$, and u and ϕ be as in Lemma 4.37. In the preceding equation, we may now replace u by $u + t\phi$ and v by u. Invoking the positive 1-homogeneity of $M_{\mathcal{L}}^-$ and $(M_{\mathcal{L}}^-)_n$, we obtain

$$t(M_{\mathcal{L}}^-)_n(\phi) \le I_n(u + t\phi) - I_n(u).$$

Now we can invoke the approximation estimate in Lemma 3.25 applied to $(M_{\mathcal{L}}^-)_n$, and rearrange to see that

$$-h_n \|\phi\|_{C^3} + M_{\mathcal{L}}^-(\phi, x) \le (M_{\mathcal{L}}^-)_n(\phi, x) \le \frac{1}{t} \left(I_n(u + t\phi, x) - I_n(u, x) \right).$$

Hence, taking the limit as $t \to 0^+$, we conclude Lemma 4.37. $\qquad\square$

Now we justify Proposition 4.35.

Proof of Proposition 4.35. Let ϕ and x be given. By the definition of $L_x \in \mathcal{K}_{Levy}(I)$ via Proposition 4.30, we see that L_x is a limit of convex combinations of operators, $\tilde{L}$ such that there exist u_n and x_k so that

$$\tilde{L}(\phi, x) = \lim_{k \to \infty} \lim_{n \to \infty} DI_{n,u_n}(\phi, x_k).$$

As Lemma 4.37 is independent of u_n, and from the fact that $M_{\mathcal{L}}^-(\phi, \cdot)$ is continuous in x, we see that

$$M_{\mathcal{L}}^-(\phi, x) \le \tilde{L}(\phi, x).$$

This inequality is preserved under further convex combinations over $\tilde{L}$, and thus we conclude it also holds that

$$M_{\mathcal{L}}^-(\phi, x) \le L_x(\phi).$$

$\square$

5. Some Questions

Here we take the time to mention some additional questions that arise from the min-max representation.

Question 5.1. *In the Introduction, among the examples for maps satisfying the GCP, we mentioned the Dirichlet-to-Neumann map for a fully nonlinear equation in a bounded smooth domain $\Omega \subset \mathbb{R}^d$. Our main theorem yields the representation*

$$\partial_\nu U = \min_a \max_b \left\{ f^{ab}(x) + L_{ab}(u, x)) \right\},$$

where $\{f^{ab}\}_{ab}$ is a bounded family of functions in $C(\partial\Omega)$, and each $L_{ab}(\cdot, x)$ has the form (1.2). Then, we ask: are the Levy measures μ^{ab} appearing in the min-max formula formula absolutely continuous with respect to the surface measure of $\partial\Omega$?. In other words, find out whether there are measurable functions $k^{ab} : \Omega \times \Omega \to \mathbb{R}$ such that

$$\mu_x^{ab}(dy) = k^{ab}(x, y)d\mathrm{vol}_g(y).$$

Furthermore, deriving further properties for the kernels k^{ab}, such as pointwise bounds with respect to the kernel $|x - y|^{-d}$, would be very useful. Such bounds would mean that the equation $\partial_\nu U = 0$ is closely related to existing regularity results for nonlocal elliptic equations, i.e. [4, 7, 13, 45].

Question 5.2. *Going in the opposite direction, are there Dirichlet to Neumann maps –even in the linear case– for which the resulting integro-differential operator on $\partial\Omega$ that has a singular Lévy measure? This seems a possibility for linear operators with low-regularity coefficients, as suggested by the existence of well known examples of elliptic operators for which the associated L-harmonic measure is singular.*

Question 5.3. *Let $M = \mathbb{R}^d$. If it is assumed that I is a translation invariant operator, can you show that it suffices to only use translation invariant linear operators in the min-max formula of Theorem 1.6?.*

Question 5.4. *Can the min-max formula be extended to degenerate or singular operators such as the infinity-Laplace or the p-Laplace? These operators are not bounded from $C^2 \to C$, but nonetheless they enjoy good existence / uniqueness and partial regularity theory for weak solutions of equations defined by them.*

Question 5.5. *The axiomatic image processing work of Alvarez-Guichard-Lions-Morel [1] showed that if a semi-group on the space of continuous functions satisfies certain axioms, most notably locality and comparison, then in fact the semi-group must be characterized as the (viscosity) solution operator for some fully nonlinear (degenerate) parabolic equation. This is notable because one recovers a representation using weak solutions. Is it possible to make an analog of the paper [1] to the context of Theorem 1.6 presented here? This would be an extension of Theorem 1.6 to both the parabolic setting and the setting of weak solutions.*

APPENDIX A. DISCRETIZATION OF THE GRADIENT AND THE HESSIAN ON M

First off, we shall construct proper discretizations for the covariant gradient and Hessian given M and $\tilde{G}_n$. Our point of view will be to think of a sufficiently smooth function $u : M \to \mathbb{R}$ as given. Then, the discrete gradient and Hessian of u will be defined at points in $\tilde{G}_n$ using only the values of u at points in $\tilde{G}_n$. We will see that the regularity of the original function u will control how far are these discrete operators from their continuum counterparts (Lemma A.14). Moreover, the regularity of u will control the regularity of discrete gradient and Hessian themselves, in a manner which is independent of the mesh size (Proposition A.15 and A.16).

Remark A.1. *Before proceeding further, it is worthwhile to note that the discrete gradient and Hessian defined below are standard, and that this appendix has been made with the chief purpose of making the paper as self contained as possible. In fact, as with the discussion of Whitney extension, we failed to find a direct reference where the discretization of the gradient and Hessian is done in the context of a Riemannian manifold. Furthermore, for the purposes of this paper, we only need rather minimal properties of our discretization –essentially, their "consistency". As such, the arguments and estimates here are far less optimal than what may be found in the numerical analysis literature where subtler issues are considered.*

As we can only use the values of u at points of $\tilde{G}_n$, our first order of business is to single out admissible directions at $x \in \tilde{G}_n$ along which a (discrete) derivative may be computed. This is done in the following proposition.

Proposition A.2. *Given $x \in \tilde{G}_n$ there are vectors*

$$V_{n,1}(x), \ldots, V_{n,d}(x) \in (TM)_x.$$

Satisfying the following properties,

(1) *For each k,*

$$\exp_x(V_{n,k}(x)) \in \tilde{G}_n.$$

(2) *Also for each k,*

$$98\tilde{h}_n \leq |V_{n,k}(x)|_{g_x} \leq 102\tilde{h}_n.$$

(3) *Finally, the family $\{V_{n,k}\}_{k=1}^d$ forms a basis which is "almost orthogonal". To be concrete, for sufficiently large n, we have*

$$|(\hat{V}_{n,l}(x), \hat{V}_{n,k}(x))_{g_x}| \leq \frac{1}{20}, \quad \text{if } k \neq l.$$

Here, $\hat{V}$ denotes the unit vector in the direction of V, that is $\hat{V} := V/|V|_{g_x}$.

Proof. Let us recall the constant $\delta \in (0,1)$ introduced in Remark 3.3, as well as $\tilde{h}_n$ (see (3.2)) which was given by

$$\tilde{h}_n := \sup_{x \in M} d(x, \tilde{G}_n), \quad \forall n,$$

and which is such that $\lim_n \tilde{h}_n = 0$. Next, recall that by (3.2), we have

$$500\tilde{h}_n < \delta.$$

Fix $x \in \tilde{G}_n$ and let $e_1, \ldots, e_d$ be an arbitrary orthonormal basis of $(TM)_x$. By definition of $\tilde{h}_n$,

$$d(\exp_x(100\tilde{h}_n e_k), \tilde{G}_n) \leq \tilde{h}_n, \quad k = 1, \ldots, d.$$

In particular, for each x and each k, it is possible to pick a point x_k such that

$$x_k \in \tilde{G}_n \text{ and } d(\exp_x(100\tilde{h}_n e_k), x_k) \le \tilde{h}_n.$$

Having made such a selection for each $x \in \tilde{G}_n$, we define

$$V_{n,k}(x) := (\exp_x)^{-1}(x_k), \quad k = 1, \ldots, d.$$

Thus, the first property holds by construction. Next, observe that since $100\tilde{h}_n < \delta$, both x_k, x and $\exp_x(100\tilde{h}_n e_k)$ all lie in a ball of radius $4\delta\sqrt{d}$. Therefore, using Remark 3.3 we can compare $|V_{n,k}(x)|_{g_x}$ and $|100\tilde{h}_n e_k|_{g_x}$. In particular, we have

$$|V_{n,k} - 100\tilde{h}_n e_k|_{g_x} \le \tfrac{101}{100} d(\exp_x(100\tilde{h}_n e_k), x_k) \le \tfrac{101}{100}\tilde{h}_n. \tag{A.1}$$

Then, the triangle inequality yields,

$$|V_{n,k}(x)|_{g_x} \le |100\tilde{h}_n e_k|_{g_x} + |V_{n,k}(x) - 100\tilde{h}_n e_k|_{g_x} \le 100\tilde{h}_n + \tfrac{101}{100}\tilde{h}_n, \le 102\tilde{h}_n,$$
$$|V_{n,k}(x)|_{g_x} \ge |100\tilde{h}_n e_k|_{g_x} - |V_{n,k}(x) - 100\tilde{h}_n e_k|_{g_x} \ge 100\tilde{h}_n - \tfrac{101}{100}\tilde{h}_n \ge 98\tilde{h}_n.$$

This proves the second property. It remains to prove the third one. For the sake of brevity, let us omit the x dependence in the computations below.

Let us express the inner product $(V_{n,l}, V_{n,k})_{g_x}$ in terms of the orthonormal basis e_k,

$$(V_{n,l}, V_{n,k})_{g_x} = (V_{n,l} - 100\tilde{h}_n e_l + 100\tilde{h}_n e_l, V_{n,k} - 100\tilde{h}_n e_k + 100\tilde{h}_n e_k)_{g_x}$$
$$= (V_{n,l} - 100\tilde{h}_n e_l, V_{n,k} - 100\tilde{h}_n e_k + 100\tilde{h}_n e_k)_{g_x}$$
$$\quad + (100\tilde{h}_n e_l, V_{n,k} - 100\tilde{h}_n e_k + 100\tilde{h}_n e_k)_{g_x}$$
$$= (V_{n,l} - 100\tilde{h}_n e_l, V_{n,k} - 100\tilde{h}_n e_k)_{g_x} + (V_{n,l} - 100\tilde{h}_n e_l, 100\tilde{h}_n e_k)_{g_x}$$
$$\quad + (100\tilde{h}_n e_l, V_{n,k} - 100\tilde{h}_n e_k)_{g_x} + (100\tilde{h}_n e_l, 100\tilde{h}_n e_k)_{g_x}.$$

Since the e_k are orthonormal, for $k \neq l$ it follows that

$$(V_{n,l}, V_{n,k})_{g_x} = (V_{n,l} - 100\tilde{h}_n e_l, V_{n,k} - 100\tilde{h}_n e_k)_{g_x} + (V_{n,l} - 100\tilde{h}_n e_l, 100\tilde{h}_n e_k)_{g_x}$$
$$\quad + (100\tilde{h}_n e_l, V_{n,k} - 100\tilde{h}_n e_k)_{g_x}, \quad k \neq l.$$

We apply the estimate (A.1) to this last identity, it follows that

$$|(V_{n,l}, V_{n,k})_{g_x}| \le |V_{n,l} - 100\tilde{h}_n e_l|_{g_x}|V_{n,k} - 100\tilde{h}_n e_k|_{g_x} + |V_{n,l} - 100\tilde{h}_n e_l|_{g_x}|100\tilde{h}_n e_k|_{g_x}$$
$$\quad + |100\tilde{h}_n e_l|_{g_x}|V_{n,k} - 100\tilde{h}_n e_k|_{g_x}$$
$$\le (\tfrac{101}{100})^2\tilde{h}_n^2 + 2(\tfrac{101}{100}\tilde{h}_n)(100\tilde{h}_n) \le 204\tilde{h}_n.$$

Since $|V_{n,l}|_{g_x}^{-1} \ge 98\tilde{h}_n$, it follows that

$$|(\hat{V}_{n,l}, \hat{V}_{n,k})_{g_x}| \le 204\tilde{h}_n^2|V_{n,l}|_{g_x}^{-1}|V_{n,k}|_{g_x}^{-1} \le 204(98)^{-2} \le \tfrac{1}{20},$$

and the third property is proved. $\qquad\square$

From here on, for each n and for every $x \in \tilde{G}_n$, we fix a selection of vectors $\{V_{n,1}(x), \ldots, V_{n,d}(x)\} \in (TM)_x$ as in the previous proposition. Moreover, we fix $u \in C_b^\beta(M)$ for the rest of this section.

Definition A.3. *(Discrete gradient) Given $x \in \tilde{G}_n$ and u, define $(\nabla_n)^1 u(x) \in (TM)_x$ by solving the system of linear equations*

$$(V_{n,k}(x), (\nabla_n)^1 u(x))_{g_x} = u(\exp_x(V_{n,k}(x))) - u(x), \quad k = 1, \ldots, d.$$

Note that, as the $V_{n,k}(x)$ are linearly independent, the above system always has a unique solution.

Remark A.4. *Let us illustrate the above definition in a simple case. Let us take,*

$$M = \mathbb{R}^d, \quad \tilde{G}_n = (2^{-n}\mathbb{Z}^d),$$

and write $\tilde{h}_n = 2^{-n}$ and $V_{n,l}(x) = h_n e_l$, where $\{e_1, \ldots, e_d\}$ denote the standard orthonormal basis of $\mathbb{R}^d$. Then,

$$u(x + \tilde{h}_n e_k) - u(x) = u(\exp_x(V_{n,k}(x))) - u(x)$$
$$= \sum_{l=1}^{d} (\nabla_n)_l^1 u(x)(V_{n,k}(x), \hat{V}_{n,l}(x))_{g_x}$$
$$= (\nabla_n)_k^1 u(x)\tilde{h}_n.$$

Thus, in this case we have

$$(\nabla_n)_k^1 u(x) = \frac{u(x + \tilde{h}_n e_k) - u(x)}{\tilde{h}_n} (\approx \partial_{x_k} u(x)),$$

and the vector $(\nabla_n)^1 u(x)$ is nothing but a discretization of the gradient.

Definition A.5. *Let $x, y \in M$ be such that $d(x, y) < r_0$. Then let $\Gamma_{x,y}$ denote the linear map*

$$\Gamma_{x,y} : (TM)_y \to (TM)_x,$$

given by parallel transport along the unique minimal geodesic connecting x to y. We should recall this map is an isometry with respect to the inner products g_x and g_y. If the point y is understood from context, we shall simply write Γ_x.

Definition A.6. *Let V be a section of the tangent bundle TM. We say V is of class C^α if*

$$[V]_{C^\alpha(M)} := \sup_{0 < d(x,y) < r_0} \frac{|V(x) - \Gamma_{x,y}V(y)|_{g_x}}{d(x,y)^\alpha} < \infty$$

Likewise, if $M : TM \to TM$, then

$$[M]_{C^\alpha(M)} := \sup_{0 < d(x,y) < r_0} \frac{|M(x) - M(y)\Gamma_{x,y}^{-1}|_{g_x}}{d(x,y)^\alpha} < \infty.$$

These seminorms, when applied to $V = \nabla u$ and $M = \nabla^2 u$ allows to define the C^β norm of u in the obvious manner.

Remark A.7. *Let $\beta \in [0, 3)$ be given. The following is a useful characterization of Hölder continuity that will be used later on. Let $x(t)$ denote a geodesic and $e(t)$ a parallel vector field along it with $|\dot{x}(t)|_{g_{x(t)}} = |e(t)|_{x(t)} = 1$. Then,*

$$|(\nabla u(x(t)), e(t))_{x(t)} - (\nabla u(x(s)), e(s))_{x(s)}| \leq \|u\|_{C^\beta} |t - s|^{\min\{\beta-1,1\}}, \quad \text{if } \beta \geq 1,$$

and

$$|(\nabla^2 u(x(t))e(t), e(t))_{x(t)} - (\nabla^2 u(x(s))e(s), e(s))_{x(s)}| \leq \|u\|_{C^\beta} |t - s|^{\min\{\beta-2,1\}}, \quad \text{if } \beta \geq 2.$$

Defining the discrete Hessian requires further preparation, we define first the following "second order difference",

$$\delta u_x(V_1, V_2) := u(\exp_{\exp_x(V_1)}(V_2)) - u(\exp_x(V_1)) - u(\exp_x(\Gamma_x V_2)) + u(x).$$

Here Γ_x denotes the operation of parallel transport, as introduced in Definition A.5.

Definition A.8. *(Discrete Hessian) Given $x \in \tilde{G}_n$ and u, we will define a linear transformation*

$$(\nabla_n)^2 u(x) : (TM)_x \to (TM)_x.$$

Given $k = 1, \ldots, d$, define $(\nabla_n)^2 u(x) V_{n,k}(x) \in (TM)_x$ as the solution V to the linear system

$$(V, \Gamma_x V_{n,l}(x_k))_{g_x} = \delta u_x(V_{n,k}(x), V_{n,l}(x_k)), \quad l = 1, \ldots, d.$$

Here, for the sake of brevity of notation, we have written

$$x_k = \exp_x(V_{n,k}(x)).$$

Having indicated how $(\nabla_n)^2 u(x)$ acts on the basis $\{V_{n,k}(x)\}_{k=1}^d$ of $(TM)_x$, the linear transformation is completely determined.

Let us elaborate on the linear algebra problem that was used to define $(\nabla_n)^2 u$. Given a linear transformation $D : (TM)_x \to (TM)_x$, and a family of pairs of vectors $\{(V_k, W_k)\}_{k=1}^N$ for some N, we seek to recover the full matrix D from the values

$$(DV_k, W_k).$$

We are given a basis V_k ($k = 1, \ldots, d$), and for each k another basis $\{W_{k,l}\}$ ($l = 1, \ldots, d$). Then, we seek to completely determine a linear transformation M given the values

$$(DV_k, W_{k,l}), \quad \text{for } k, l = 1, \ldots, d.$$

Remark A.9. *Let us again see what this definition says in a simple case. Let $M, \tilde{G}_n, \tilde{h}_n$ and $\{V_{n,k}(x)\}$ be as in Remark A.4. Then, given $x \in \tilde{G}_n$ and $k, l = 1, \ldots, d$ we have*

$$\delta u_x(V_{n,k}(x), V_{n,l}(x_k)) = u(x + 2^{-n} e_k + 2^{-n} e_l) - u(x + 2^{-n} e_k) - u(x + 2^{-n} e_l) + u(x)$$

$$= 2^{-n} 2^{-n} ((\nabla_n)^2 u(x) e_k, e_l).$$

It follows that the components of $(\nabla_n)^2 u(x)$ are given by

$$(\nabla_n)^2_{kl} u(x) = \frac{u(x + 2^{-n} e_k + 2^{-n} e_l) - u(x + 2^{-n} e_k) - u(x + 2^{-n} e_l) + u(x)}{2^{-n} 2^{-n}} (\approx \nabla^2_{kl} u(x))$$

and the matrix $(\nabla_n)^2_{kl} u(x)$ is nothing but a discretization of the standard Hessian.

Remark A.10. *Let $x \in \tilde{G}_n$. Using the upper bound in part (2) of Proposition A.2, one notes that all the values of u taken in evaluating $\nabla^1_n u(x)$ and $\nabla^2_n u(x)$ lie within a ball of radius $< 250 \tilde{h}_n$ centered at x. In particular, if $u \equiv 0$ in $B_{250 h_n}(x)$, then*

$$\nabla^1_n u(x) = 0, \quad \nabla^2_n u(x) = 0.$$

The previous remark guarantees that the extension operator is somewhat "local", the locality becoming more and more exact as n becomes larger, this is made rigorous in the following proposition.

Proposition A.11. *Let $u \in C^\beta$, and $x_0 \in M$. Then,*

$$u \equiv 0 \text{ in } B_{400 \tilde{h}_n}(x_0) \Rightarrow E_n^\beta(u, \cdot) \equiv 0 \text{ in } B_{100 \tilde{h}_n}(x_0).$$

Proof. First, we claim that

$$x \in B_{100 \tilde{h}_n}(x_0) \Rightarrow B_{250 \tilde{h}_n}(\hat{y}_{n,k}) \subset B_{400 \tilde{h}_n}(x_0), \quad \forall\, k \in K_x. \tag{A.2}$$

Let us see how (A.2) implies the proposition. Fix $x \in B_{100 \tilde{h}_n}(x_0)$, with $x \in M \setminus \tilde{G}_n$, then

$$E_n^\beta(u, x) = \sum_k p^\beta_{(u,k)}(x) \phi_{n,k}(x) = \sum_{k \in K_x} p^\beta_{(u,k)}(x) \phi_{n,k}(x).$$

Then, thanks to (A.2), we have that

$$u \equiv 0 \text{ in } B_{250\tilde{h}_n}(\hat{y}_{n,k}), \ \ \forall \, k \in K_x, \ \forall \, x \in B_{100\tilde{h}_n}(x_0).$$

In this case, Remark A.10 guarantees that

$$p^{\beta}_{(u,k)}(x) \equiv 0, \ \ \forall \, k \in K_x, \ \forall \, x \in B_{100\tilde{h}_n}(x_0).$$

In other words,

$$E^{\beta}_n(u,x) = 0, \ \ \forall \, x \in B_{100\tilde{h}_n}(x_0).$$

Which proves the proposition. It remains to prove (A.2). Fix $x \in B_{100\tilde{h}_n}(x_0)$ and $k \in K_x$. By the triangle inequality, and the definition of $\hat{y}_{n,k}$, we have

$$\begin{aligned} d(x, \hat{y}_{n,k}) \leq d(x, y_{n,k}) + d(\hat{y}_{n,k}, y_{n,k}) &= d(x, y_{n,k}) + d(y_{n,k}, \tilde{G}_n) \\ &\leq 2d(x, y_{n,k}) + d(x, \tilde{G}_n) \\ &\leq 2\text{diam}(P^*_{n,k}) + d(x, \tilde{G}_n). \end{aligned}$$

Then, thanks to Remark 3.10,

$$d(x, \hat{y}_{n,k}) \leq 15d(x, \tilde{G}_n) \leq 15\tilde{h}_n, \ \ \forall \, k \in K_x.$$

Furthermore,

$$\begin{aligned} d(\hat{y}_{n,k}, x_0) &\leq d(\hat{y}_{n,k}, x) + d(x, x_0) \\ &\leq d(\hat{y}_{n,k}, \hat{x}) + d(x, \hat{x}) + d(x, x_0). \end{aligned}$$

We now recall that $d(x, \hat{x}) = d(x, \tilde{G}_n) \leq \tilde{h}_n$, and $d(x, x_0) \leq 100\tilde{h}_n$. Furthermore, as shown in (3.16) in the proof of Proposition we have $d(\hat{y}_{n,k}, \hat{x}) \leq 16d(x, \tilde{G}_n)$ for $k \in K_x$. Gathering these inequalities it follows that

$$d(\hat{y}_{n,k}, x_0) \leq 117\tilde{h}_n, \ \ \forall \, k \in K_x.$$

From here, and the triangle inequality, we conclude that $B_{250\tilde{h}_n}(\hat{y}_{n,k})$ lies inside $B_{400\tilde{h}_n}(x_0)$, that is, (A.2). This proves the proposition. □

In what follows, we will be using the functions l and q, introduced in Definition 3.13. In $\mathbb{R}^d$ this is a completely straightforward calculation using the Taylor polynomial. On a Riemannian manifold, we shall use the coordinates given by the exponential map. For the next proposition, we recall that the functions "linear" and "quadratic" functions l and q introduced in Definition 3.13 are defined in a ball of of radius $4\delta\sqrt{d}$ around their base point, where δ is as in Remark 3.3

Proposition A.12. *Let $x_0, x \in M$ with $d(x, x_0) \leq 4\delta\sqrt{d}$, and $u \in C^{\beta}_b(M)$. Then,*
1) If $C^{\beta}_b = C^1_b$, then

$$u(x) - u(x_0) - l(\nabla u(x_0), x_0; x) = o(d(x, x_0)),$$

where the $o(d(x, x_0))$ term is controlled by the modulus of continuity of ∇u.
2) If $\beta \in [1, 2]$, then

$$|u(x) - u(x_0) - l(\nabla u(x_0), x_0; x)| \leq \|u\|_{C^{\beta}} d(x, x_0)^{\beta}.$$

3) If $C^{\beta}_b = C^2_b$, then

$$u(x) - u(x_0) - l(\nabla u(x_0), x_0; x) - q(\nabla^2 u(x_0), x_0; x) = o(d(x, x_0)^2),$$

where the $o(d(x, x_0))$ term is controlled by the modulus of continuity of $\nabla^2 u$.
4) If $\beta \in [2, 3]$, then

$$|u(x) - u(x_0) - l(\nabla u(x_0), x_0; x) - q(\nabla^2 u(x_0), x_0; x)| \leq \|u\|_{C^{\beta}} d(x, x_0)^{\beta}.$$

We omit the straightforward proof of Proposition A.12.

Remark A.13. *From Definition 3.13 it is immediate that Proposition A.12 has the following equivalent formulation which will also be useful: given a unit vector $e \in (TM)_{x_0}$ and $h \le 4\delta\sqrt{d}$, we have*

$$u(\exp_{x_0}(he)) - u(x_0) - h(\nabla u(x_0), e)_{g_{x_0}} = o(h), \text{ if } C_b^\beta = C_b^1,$$

$$|u(\exp_{x_0}(he)) - u(x_0) - h(\nabla u(x_0), e)_{g_{x_0}}| \le \|u\|_{C^\beta} h^\beta, \text{ if } \beta \in (1, 2],$$

$$u(\exp_{x_0}(he)) - u(x_0) - h(\nabla u(x_0), e)_{g_{x_0}} - \frac{h^2}{2}(\nabla^2 u(x_0)e, e)_{g_{x_0}} = o(h^2), \text{ if } C_b^\beta = C_b^2,$$

$$|u(\exp_{x_0}(he)) - u(x_0) - h(\nabla u(x_0), e)_{g_{x_0}} - \frac{h^2}{2}(\nabla^2 u(x_0)e, e)_{g_{x_0}}| \le \|u\|_{C^\beta} h^\beta, \text{ if } \beta \in [2, 3].$$

Proof of Remark A.13. First estimate. Fix a unit vector $e \in (TM)_{x_0}$. For $h \in [0, r_0]$ let $x(h) := \exp_{x_0}(he)$, and let

$$\varepsilon(h) := u(x(h)) - u(x_0) - (\nabla u(x_0), he)_{g_{x_0}}.$$

It is immediate that $\varepsilon(0) = 0$, $d(x_0, x(h)) = h$, and that

$$\varepsilon'(h) = (\nabla u(x(h)), \dot{x}(h))_{g_{x(h)}} - (\nabla u(x_0), e)_{g_{x_0}}.$$

Since $\dot{x}(0) = e$, we have $\varepsilon'(0) = 0$. Keeping in mind that $\dot{x}(h)$ is the parallel transport of e along $x(h)$, the Hölder regularity of $\nabla u(x)$ yields

$$|\varepsilon'(h)| = |(\nabla u(x(h)), \dot{x}(h))_{x(h)} - (\nabla u(x_0), e)_{g_{x_0}}| \le \|u\|_{C^\beta} d(x_0, x(h))^{\beta-1}$$
$$= \|u\|_{C^\beta} h^{\beta-1}.$$

Integrating this last inequality from 0 to h, we obtain the first estimate, since

$$|\varepsilon(h)| = |\varepsilon(h) - \varepsilon(0)| \le \|u\|_{C^\beta} h^\beta.$$

Second estimate. Let $x(h)$ be as before, with $h \in [0, r_0]$. This time we consider the function

$$\varepsilon(h) := u(x(h)) - u(x_0) - h(\nabla u(x_0), e)_{g_{x_0}} - \frac{h^2}{2}((\nabla^2 u(x_0))e, e)_{g_{x_0}}.$$

Then, as before it is clear that $\varepsilon(0) = \varepsilon'(0) = 0$ and

$$\varepsilon'(h) = (\nabla u(x(h)), \dot{x}(h))_{x(h)} - (\nabla u(x_0), e)_{g_{x_0}} - h((\nabla^2 u(x_0))e, e)_{g_{x_0}},$$
$$\varepsilon''(h) = ((\nabla^2 u(x(h)))\dot{x}(h), \dot{x}(h))_{g_{x(h)}} - ((\nabla^2 u(x_0))e, e)_{g_{x_0}}.$$

As before, we make use of the fact that $\dot{x}(h)$ is a parallel vector along $x(h)$, which leads to

$$|\varepsilon''(h)| = |((\nabla^2 u(x(h)))\dot{x}(h), \dot{x}(h))_{g_{x(h)}} - ((\nabla^2 u(x_0))e, e)_{g_{x_0}}| \le \|u\|_{C^\beta} h^{\beta-2}.$$

Integrating this inequality twice (and using that $\varepsilon(0) = \varepsilon'(0) = 0$) it follows that

$$|\varepsilon(h)| = |\varepsilon(h) - \varepsilon(0)| = \left|\int_0^h \varepsilon'(s)\, ds\right| \le \|u\|_{C^\beta} h^\beta,$$

which proves the second estimate.

□

The next Lemma consists of a very important fact, namely, that the discrete difference operators $(\nabla_n)^1 u$ and $(\nabla_n)^2 u$ are "consistent" –i.e. they converge to the differential operators ∇u and $\nabla^2 u$. Furthermore, we have that the error made when estimating the derivatives by the discrete operator is a quantity controlled by the C^β norm of $u \in C_b^\beta(M)$.

Lemma A.14. *Let $x \in \tilde{G}_n$ and $u \in C_b^\beta(M)$ then*

$$|(\nabla_n)^1 u(x) - \nabla u(x)|_{g_x} \leq C\|u\|_{C^\beta} \tilde{h}_n^{\beta-1}, \text{ if } \beta \in (1,2],$$

$$|(\nabla_n)^2 u(x) - \nabla^2 u(x)|_{g_x} \leq C\|u\|_{C^\beta} \tilde{h}_n^{\beta-2}, \text{ if } \beta \in (2,3].$$

Furthermore, if $C_b^\beta = C_b^1$ or $C_b^\beta = C_b^2$ then, we have, respectively

$$\lim_{n\to\infty} \sup_{x \in K \cap \tilde{G}_n} |(\nabla_n)^1 u(x) - \nabla u(x)|_{g_x} = 0,$$

$$\lim_{n\to\infty} \sup_{x \in K \cap \tilde{G}_n} |(\nabla_n)^2 u(x) - \nabla^2 u(x)|_{g_x} = 0,$$

where K is an arbitrary compact subset of M.

Proof. First estimate. We may write

$$\nabla u(x) = \sum_{l=1}^d \theta_l \hat{V}_{n,l}(x),$$

where the numbers $\theta_1, \ldots, \theta_d$ are determined from the system of equations

$$(\nabla u(x), \hat{V}_{n,k}(x))_{g_x} = \sum_{l=d}^d \theta_l (\hat{V}_{n,k}(x), \hat{V}_{n,l}(x))_{g_x}.$$

Now, Proposition A.12 says that

$$\left| \frac{u(\exp_x(V_{n,k})) - u(x)}{|V_{n,k}|_{g_x}} - \left(\nabla u(x), \hat{V}_{n,k}\right)_{g_x} \right| \leq C\|u\|_{C^\beta} |V_{n,k}|_{g_x}^{\beta-1},$$

and, if $C_b^\beta = C_b^1$, it says that for any compact K,

$$\lim_{n\to\infty} \sup_{x \in K \cap \tilde{G}_n} \max_{1 \leq k \leq d} \left| \frac{u(\exp_x(V_{n,k})) - u(x)}{|V_{n,k}|_{g_x}} - \left(\nabla u(x), \hat{V}_{n,k}\right)_{g_x} \right| = 0,$$

the convergence in the limit being determined by K, the continuity of ∇u, and M. Then,

$$|(\nabla_n)_l^1 u(x) - \theta_l| \leq C\|u\|_{C^\beta} \tilde{h}_n^{\beta-1} \quad \forall \, x \in M, \text{ if } \beta \in (0,1),$$

$$\lim_{n\to\infty} \sup_{x \in K \cap \tilde{G}_n} |(\nabla_n)_l^1 u(x) - \theta_l| = 0, \qquad \forall \, K \subset\subset M, \text{ if } C_b^\beta = C_b^1.$$

The above holds for each $l = 1, \ldots, d$. Combining these inequalities it is immediate that

$$|(\nabla_n)^1 u(x) - \nabla u(x)| \leq C\|u\|_{C^\beta} \tilde{h}_n^{\beta-1},$$

and, for $C_b^\beta = C_b^1$,

$$\lim_{n\to\infty} \sup_{x \in K \cap \tilde{G}_n} |(\nabla_n)^1 u(x) - \nabla u(x)| = 0.$$

Second estimate. First, we need an elementary observation about geodesics. Observe that

$$\exp_{\exp_x(V_{n,k}(x))}(V_{n,l}(x_k)) = \exp_x(V_{n,k}(x) + \Gamma_x V_{n,l}(x_k) + (\text{Error})_0). \tag{A.3}$$

Where the term $(\text{Error})_0$ is term appearing due to possibly non-zero curvature. It turns out that this error term is at least a cubic error in terms of $\tilde{h}_n$, which is proved as follows: let $J(t)$ be the Jacobi field along the geodesic $\gamma(t) = \exp_x(t\hat{V}_{n,k})$ determined by $J(0) = 0$ and $J(|V_{n,k}|_{g_x}) = \hat{V}_{n,l}(x_k)$. Then, define $\sigma(t,s) \in (TM)_x$ by

$$\exp_{\gamma(t)}(sJ(t)) = \exp_x(\sigma(t,s)).$$

Note that $\sigma\left(|V_{n,k}|_{g_x}, |V_{n,l}(x_k)|_{g_x}\right)$ must be equal to the argument in the exponential on the right hand side of (A.3). Then, note that

$$\sigma(0,s) = 0, \quad \forall\, s \Rightarrow \sigma(0,0) = \partial_s \sigma(0,0) = \partial_{ss} \sigma(0,0) = 0.$$

Furthermore, $\partial_t \sigma(0,0) = \hat{V}_{n,k}(x)$, so

$$\sigma(t,s) = t\hat{V}_{n,k} + st\partial_{ts}\sigma(0,0) + O((s^2 + t^2)^{3/2}).$$

Now, by contrasting the respective Jacobi and parallel transport equations, it can be shown that

$$|\partial_{ts}\sigma(0,0) - \Gamma_x \hat{V}_{n,k}(x_k)| \le C\tilde{h}_n.$$

Given that $|V_{n,k}|_{g_x}, |V_{n,l}(x_k)|_{g_x} \le h_n$, this leads to the bound

$$|(\text{Error})_0|_{g_x} \le C\tilde{h}_n^3. \tag{A.4}$$

The constant C depending only on the metric of M.

Let us analyze the first three terms appearing in the second order difference $\delta u_x(V_{n,k}(x), V_{n,l}(x_k))$. We consider the Taylor expansion and estimate the remainder via Proposition A.12. First of all, we have

$$u(\exp_{\exp_x(V_{n,k}(x))}(V_{n,l}(x_k))) = u(\exp_x(V_{n,k}(x) + \Gamma_x V_{n,l}(x_k) + (\text{Error})_0)).$$

The estimate (A.4) guarantees in particular that $|V_{n,k}(x) + \Gamma_x V_{n,l}(x_k) + (\text{Error})_0| \le C\tilde{h}_n$. With this in mind, we apply Proposition A.12 in order to obtain the expansion

$$
\begin{aligned}
&u(\exp_{\exp_x(V_{n,k}(x))}(V_{n,l}(x_k))) \\
&= u(x) + (\nabla u(x), V_{n,k}(x) + \Gamma_x V_{n,l}(x_k) + (\text{Error})_0)_{g_x} \\
&\quad + \tfrac{1}{2}\left(\nabla^2 u(x)(V_{n,k}(x) + \Gamma_x V_{n,l}(x_k) + (\text{Error})_0), V_{n,k}(x) + \Gamma_x V_{n,l}(x_k) + (\text{Error})_0\right)_{g_x} \\
&\quad + (\text{Error}),
\end{aligned}
$$

where (Error), which denotes the remainder in the Taylor expansion, satisfies the bound

$$|(\text{Error})| \le C\|u\|_{C^\beta}\tilde{h}_n^\beta.$$

Expanding, we see that

$$
\begin{aligned}
&u(\exp_{\exp_x(V_{n,k}(x))}(V_{n,l}(x_k))) \\
&= u(x) + (\nabla u(x), V_{n,k}(x))_{g_x} + (\nabla u(x), \Gamma_x V_{n,l}(x_k))_{g_x} \\
&\quad + \tfrac{1}{2}\left(\nabla^2 u(x)V_{n,k}(x), V_{n,k}(x)\right)_{g_x} + \tfrac{1}{2}\left(\nabla^2 u(x)\Gamma_x V_{n,l}(x_k), \Gamma_x V_{n,l}(x_k)\right)_{g_x} \\
&\quad + \left(\nabla^2 u(x)V_{n,k}(x), \Gamma_x V_{n,l}(x_k)\right)_{g_x} \\
&\quad + (\nabla u(x), (\text{Error})_0)_{g_x} + \frac{1}{2}\left(\nabla^2 u(x)(\text{Error})_0, (\text{Error})_0\right)_{g_x} \\
&\quad + \left(\nabla^2 u(x)(\text{Error})_0, V_{n,k}(x) + \Gamma_x V_{n,l}(x_k)\right)_{g_x} + (\text{Error}).
\end{aligned}
$$

The terms involving a factor of $(\text{Error})_0$ may be absorbed into (Error). To see why, we use the estimate (A.4) and bound term by term

$$|(\nabla u(x), (\text{Error})_0)_{g_x}| \le C\|u\|_{C^1} h_n^3,$$

$$\left|\left(\nabla^2 u(x)(\text{Error})_0, (\text{Error})_0\right)_{g_x}\right| \le C\|u\|_{C^2}\tilde{h}_n^6,$$

$$\left|\left(\nabla^2 u(x)(\text{Error})_0, V_{n,k}(x)\right)_{g_x}\right| \le C\|u\|_{C^2}\tilde{h}_n^4,$$

$$\left|\left(\nabla^2 u(x)(\text{Error})_0, \Gamma_x V_{n,l}(x_k)\right)_{g_x}\right| \le C\|u\|_{C^2}\tilde{h}_n^4.$$

Since $\beta \geq 2$, each of the above terms is bounded by $C\|u\|_{C^\beta}\tilde{h}_n^\beta$. Then, absorbing the terms involving $(\text{Error})_0$ into (Error) we obtain

$$u(\exp_{\exp_x(V_{n,k}(x))}(V_{n,l}(x_k)))$$
$$= u(x) + (\nabla u(x), V_{n,k}(x))_{g_x} + (\nabla u(x), \Gamma_x V_{n,l}(x_k))_{g_x}$$
$$+ \tfrac{1}{2}\left(\nabla^2 u(x)V_{n,k}(x), V_{n,k}(x)\right)_{g_x} + \tfrac{1}{2}\left(\nabla^2 u(x)\Gamma_x V_{n,l}(x_k), \Gamma_x V_{n,l}(x_k)\right)_{g_x}$$
$$+ \left(\nabla^2 u(x)V_{n,k}(x), \Gamma_x V_{n,l}(x_k)\right)_{g_x} + (\text{Error}).$$

As for the other two terms, we have

$$u(\exp_x(V_{n,k}(x))) = u(x) + (\nabla u(x), V_{n,k}(x))_{g_x}$$
$$+ \tfrac{1}{2}\left(\nabla^2 u(x)V_{n,k}(x), V_{n,k}(x)\right)_{g_x} + (\text{Error}),$$

and

$$u(\exp_x(\Gamma_x V_{n,l}(x_k))) = u(x) + (\nabla u(x_0), \Gamma_x V_{n,l}(x_k))_{g_x}$$
$$+ \tfrac{1}{2}\left(\nabla^2 u(x)\Gamma_x V_{n,l}(x_k), \Gamma_x V_{n,l}(x_k)\right)_{g_x} + (\text{Error}).$$

In each case, $|(\text{Error})|$ is no larger than $C\|u\|_{C^\beta}\tilde{h}_n^\beta$, thanks to Proposition A.12.

Combining the last three formulas, it follows that

$$\delta u_x(V_{n,k}(x), V_{n,l}(x_k))$$

is equal to

$$u(x) + (\nabla u(x), V_{n,k}(x))_{g_x} + (\nabla u(x), \Gamma_x(V_{n,l}(x_k))_{g_x} + \tfrac{1}{2}((\nabla^2 u(x))V_{n,k}(x), V_{n,k}(x))_{g_x}$$
$$+ ((\nabla^2 u(x))V_{n,k}(x), \Gamma_x V_{n,l}(x_k))_{g_x} + \tfrac{1}{2}((\nabla^2 u(x))\Gamma_x V_{n,l}(x_k), \Gamma_x V_{n,l}(x_k))_{g_x}$$
$$- u(x) - (\nabla u(x), V_{n,k}(x))_{g_x} - \tfrac{1}{2}((\nabla^2 u(x))V_{n,k}(x), V_{n,k}(x))_{g_x} - u(x) - (\nabla u(x), \Gamma_x(V_{n,l}(x_k))_{g_x}$$
$$- \tfrac{1}{2}((\nabla^2 u(x))\Gamma_x(V_{n,l}(x_k), \Gamma_x(V_{n,l}(x_k))_{g_x} + u(x) + (\text{Error}).$$

From the above, it is clear all but one of the terms in the first two lines above is cancelled out with a term in the last two lines. We then arrive at the formula

$$\delta u_x(V_{n,k}(x), V_{n,l}(x_k)) = ((\nabla^2 u(x))V_{n,k}(x), \Gamma_x V_{n,l}(x_k))_{g_x} + (\text{Error}),$$

where –thanks to Proposition A.12, as pointed out earlier– we have

$$|(\text{Error})| \leq C\|u\|_{C^\beta}\tilde{h}_n^\beta.$$

Then, solving the linear problem corresponding to $(\nabla_n)^2 u(x)$ and $\nabla^2 u(x)$ it follows that

$$|(\nabla_n)^2 u(x) - \nabla^2 u(x)| \leq C\|u\|_{C^\beta}\tilde{h}_n^{\beta-2}.$$

Finally, if $C_b^\beta = C_b^2$, the convergence of $(\nabla_n)^2 u(x)$ to $\nabla^2 u(x)$ follows analogously to the convergence of $\nabla_n u(x)$ to $\nabla u(x)$ for $C_b^\beta = C_b^1$, we omit the details. $\square$

Given the proof of Lemma A.14 it should be clear that the $L^\infty(\tilde{G}_n)$ norm of $(\nabla_n)^i u$ $(i=1,2)$ is controlled by the appropriate C^β norm of u in a manner which is independent of n. This fact is the content of the next proposition.

Proposition A.15. *Let $x \in \tilde{G}_n$, then we have the estimates*

$$|(\nabla_n)^1 u(x)|_{g_x} \leq C\|u\|_{C^1},$$
$$|(\nabla_n)^2 u(x)|_{g_x} \leq C\|u\|_{C^2}.$$

Proof of Proposition A.15. This is an immediate consequence of the previous proposition. Indeed, fix $u \in C_b^\beta(M)$ and $x \in \tilde{G}_n$. Then, we have

$$|(\nabla_n)^1 u(x)|_{g_x} \leq |(\nabla_n)^1 u(x) - \nabla u(x)|_{g_x} + |\nabla u(x)|_{g_x}, \quad \beta \geq 1.$$

$$|(\nabla_n)^2 u(x)|_{g_x} \leq |(\nabla_n)^2 u(x) - \nabla^2 u(x)|_{g_x} + |\nabla^2 u(x)|_{g_x}, \quad \beta \geq 2.$$

Then, using the two estimates in Proposition A.12, we have

$$|(\nabla_n)^1 u(x)|_{g_x} \leq C \tilde{h}_n^{\min\{\beta-1,1\}} \|u\|_{C^\beta} + \|u\|_{C^1} \leq C \|u\|_{C^\beta}, \quad \beta \geq 1.$$

$$|(\nabla_n)^2 u(x)|_{g_x} \leq C \tilde{h}_n^{\min\{\beta-2,1\}} \|u\|_{C^\beta} + \|u\|_{C^2} \leq C \|u\|_{C^\beta}, \quad \beta \geq 2.$$

$\square$

The next proposition yields a quantitative control on the "continuity" of $(\nabla_n)^i u$ in terms of the regularity of the original function u. As one may expect, if $\nabla u(x)$ and $\nabla^2 u(x)$ are Hölder continuous in M, then $(\nabla_n)^1 u$ and $(\nabla_n)^2 u$ enjoy a respective modulus of "continuity" on $\tilde{G}_n$, this being uniform in n.

Proposition A.16. *Consider points* $x, y \in M \setminus \tilde{G}_n$ *and* $\hat{y}, \hat{x}$ *the corresponding points in* $\tilde{G}_n$ *with* $d(x, \tilde{G}_n) = d(x, \hat{x})$, $d(y, \tilde{G}_n) = d(y, \hat{y})$, *we have the following estimates with a universal* C.

(1) *For* $1 \leq \beta \leq 2$,

$$|(\nabla_n)_a^1 u(\hat{x}) - (\nabla_n)_a^1 u(\hat{y})| \leq C \|u\|_{C^\beta} d(\hat{x}, \hat{y})^{\beta-1}.$$

(2) *For* $2 \leq \beta \leq 3$,

$$|(\nabla_n)_{ab}^2 u(\hat{x}) - (\nabla_n)_{ab}^2 u(\hat{y})| \leq C \|u\|_{C^\beta} d(\hat{x}, \hat{y})^{\beta-2}.$$

Proof. If $\hat{x} = \hat{y}$ both inequalities are trivial and there is nothing to prove, so let us assume $\hat{x}, \hat{y}$ are two different points in $\tilde{G}_n$. In this case, and thanks to (3.3), we have

$$d(\hat{x}, \hat{y}) \geq \lambda \tilde{h}_n. \tag{A.5}$$

First estimate. The triangle inequality yields,

$$|(\nabla_n)_a^1 u(\hat{x}) - (\nabla_n)_a^1 u(\hat{y})|$$
$$\leq |(\nabla_n)_a^1 u(\hat{x}) - \nabla_a u(\hat{x})| + |\nabla_a u(\hat{x}) - \nabla_a u(\hat{y})| + |\nabla_a u(\hat{y}) - (\nabla_n)_a^1 u(\hat{y})|.$$

Let us estimate each of the three terms on the right. The middle term is straightforward,

$$|\nabla_a u(\hat{x}) - \nabla_a u(\hat{y})| \leq C \|u\|_{C^\beta} d(\hat{x}, \hat{y})^{\beta-1}.$$

For the first and third term, we use the first part of Lemma A.14, which says that

$$|(\nabla_n)_a^1 u(\hat{x}) - \nabla_a u(\hat{x})| \leq C \|u\|_{C^\beta} \tilde{h}_n^{\beta-1},$$
$$|(\nabla_n)_a^1 u(\hat{y}) - \nabla_a u(\hat{y})| \leq C \|u\|_{C^\beta} \tilde{h}_n^{\beta-1}.$$

Using (A.5) it follows that

$$|(\nabla_n)_a^1 u(\hat{x}) - \nabla_a u(\hat{x})| \leq C \|u\|_{C^\beta} d(\hat{x}, \hat{y})^{\beta-1},$$
$$|(\nabla_n)_a^1 u(\hat{y}) - \nabla_a u(\hat{y})| \leq C \|u\|_{C^\beta} d(\hat{x}, \hat{y})^{\beta-1}.$$

Combining the bounds for the three terms the first estimate follows.

Second estimate. As before, we start by breaking the difference in three parts, so

$$|(\nabla_n)_{ab}^2 u(\hat{x}) - (\nabla_n)_{ab}^2 u(\hat{y})|$$
$$\leq |(\nabla_n)_{ab}^2 u(\hat{x}) - \nabla_{ab}^2 u(\hat{x})| + |\nabla_{ab}^2 u(\hat{x}) - \nabla_{ab}^2 u(\hat{y})| + |\nabla_{ab}^2 u(\hat{y}) - (\nabla_n)_{ab}^2 u(\hat{y})|.$$

The middle term is bounded by

$$|\nabla_{ab}^2 u(\hat{x}) - \nabla_{ab}^2 u(\hat{y})| \le C\|u\|_{C^\beta} d(\hat{x}, \hat{y})^{\beta-2}.$$

Next, thanks to the second part of Lemma A.14,

$$|(\nabla_n)_{ab}^2 u(\hat{x}) - \nabla_{ab}^2 u(\hat{x})| \le C\|u\|_{C^\beta} \tilde{h}_n^{\beta-2},$$
$$|(\nabla_n)_{ab}^2 u(\hat{y}) - \nabla_{ab}^2 u(\hat{y})| \le C\|u\|_{C^\beta} \tilde{h}_n^{\beta-2}.$$

Using (A.5) again, we conclude that

$$|(\nabla_n)_{ab}^2 u(\hat{x}) - \nabla_{ab}^2 u(\hat{x})| \le C\|u\|_{C^\beta} d(\hat{x}, \hat{y})^{\beta-2},$$
$$|(\nabla_n)_{ab}^2 u(\hat{y}) - \nabla_{ab}^2 u(\hat{y})| \le C\|u\|_{C^\beta} d(\hat{x}, \hat{y})^{\beta-2}.$$

As in the previous case, the combined bounds for the three terms yields the estimate. $\qquad\square$

APPENDIX B. THE PROOF OF PROPOSITION 3.21

This section is dedicated to proving Proposition 3.21, which we re-record right here for the reader's convenience.

Proposition. *Let $x \in M \setminus \tilde{G}_n$ and $u \in C^\beta$. There is a universal constant C such that the following bounds hold. First, if $0 \le \beta < 1$,*

$$|\nabla(E_n^\beta \circ T_n)u(x)| \le C\|u\|_{C^\beta} d(x, \tilde{G}_n)^{\beta-1}.$$

If $1 \le \beta < 2$, we have

$$|\nabla^2(E_n^\beta \circ T_n)u(x)| \le C\|u\|_{C^\beta} d(x, \tilde{G}_n)^{\beta-2}.$$

Finally, if $2 \le \beta < 3$, we have

$$|\nabla^3(E_n^\beta \circ T_n)u(x)| \le C\|u\|_{C^\beta} d(x, \tilde{G}_n)^{\beta-3}.$$

Proof. As done throughout Section 3, for the sake of brevity we shall write $f = \pi_n^\beta u$.

The case $\beta \in [0, 1)$. Since the sum defining f is locally finite, we may differentiate term by term, which yields

$$\nabla f(x) = \sum_k u(\hat{y}_{n,k}) \nabla \phi_{n,k}(x).$$

Using (3.13) with $i = 1$ we may rewrite the above as

$$\nabla f(x) = \sum_k (u(\hat{y}_{n,k}) - u(\hat{x})) \nabla \phi_{n,k}(x), \quad \forall\, x \in M \setminus \tilde{G}_n.$$

Then, since the only non-zero terms are those with $k \in K_x$ (K_x was introduced in Lemma 3.9),

$$|\nabla f(x)|_{g_x} \le \sum_k |u(\hat{y}_{n,k}) - u(\hat{x})||\nabla\phi_{n,k}(x)|_{g_x}$$
$$\le N \sup_{k \in K_x} |u(\hat{y}_{n,k}) - u(\hat{x})||\nabla\phi_{n,k}(x)|_{g_x}.$$

For $k \in K_x$, using Remark 3.10, and the Hölder regularity of u one can check that

$$|u(\hat{y}_{n,k}) - u(\hat{x})||\nabla\phi_{n,k}(x)|_{g_x} \le C\|u\|_{C^\beta} d(x, \tilde{G}_n)^\beta d(x, \tilde{G}_n)^{-1}.$$

From here, it follows that

$$|\nabla f(x)| \le C\|u\|_{C^\beta} d(x, \tilde{G}_n)^{\beta-1}.$$

The case $\beta \in [1, 2)$. This time, we shall compute the Hessian $\nabla^2 f$ using a local system of coordinates $\{x^1, \ldots, x^d\}$. Then, for any pair of indices a, b we have

$$\nabla^2_{ab}\phi = \partial^2_{x_a x_b}\phi - \sum_{k=1}^d \Gamma^k_{ab}\partial_{x_k}\phi.$$

Then

$$\nabla^2_{ab}f(x) = \sum_k \nabla^2_{ab}\left((u(\hat{y}_{n,k}) + l(\nabla^1_n u(\hat{y}_{n,k}), \hat{y}_{n,k}; x))\phi_{n,k}(x)\right).$$

We expand each term using the Leibniz rule, and conclude $\nabla^2_{ab}f(x)$ is equal to

$$\mathrm{I}(x) + \mathrm{II}(x) + \mathrm{III}(x),$$

where, for the sake of brevity, we have written

$$\mathrm{I}(x) = \sum_k (u(\hat{y}_{n,k}) + l(\nabla^1_n u(\hat{y}_{n,k}), \hat{y}_{n,k}; x))\nabla^2_{ab}\phi_{n,k}(x),$$

$$\mathrm{II}(x) = \sum_k \nabla_a l(\nabla^1_n u(\hat{y}_{n,k}), \hat{y}_{n,k}; x)\nabla_b\phi_{n,k}(x) + \sum_k \nabla_b\phi_{n,k}(x)\nabla_a l(\nabla^1_n u(\hat{y}_{n,k}), \hat{y}_{n,k}; x),$$

$$\mathrm{III}(x) = \sum_k \nabla^2_{ab}l(\nabla^1_n u(\hat{y}_{n,k}), \hat{y}_{n,k}; x)\phi_{n,k}(x).$$

Since $x \in M \setminus \tilde{G}_n$, we can use (3.13) with $i = 1, 2$ to obtain

$$\sum_k (u(\hat{y}_{n,k}) + l(\nabla^1_n u(\hat{y}_{n,k}), \hat{y}_{n,k}; x))\nabla^2_{ab}\phi_{n,k}(x)$$

$$= \sum_k (u(\hat{y}_{n,k}) + l(\nabla^1_n u(\hat{y}_{n,k}), \hat{y}_{n,k}; x) - u(\hat{x}))\nabla^2_{ab}\phi_{n,k}(x),$$

and

$$\sum_k \nabla_a l(\nabla^1_n u(\hat{y}_{n,k}), \hat{y}_{n,k}; x)\nabla_b\phi_{n,k}(x)$$

$$= \sum_k \left(\nabla_a l(\nabla^1_n u(\hat{y}_{n,k}), \hat{y}_{n,k}; x) - \nabla_a l(\nabla^1_n u(\hat{x}), \hat{x}; x)\right)\nabla_b\phi_{n,k}(x).$$

Let us bound each of these. The triangle inequality says

$$|u(\hat{y}_{n,k}) + l((\nabla_n)^1 u(\hat{y}_{n,k}), \hat{y}_{n,k}; x) - u(x)|$$

$$\leq |u(\hat{y}_{n,k}) + l(\nabla u(\hat{y}_{n,k}), \hat{y}_{n,k}; x) - u(x)| + |l((\nabla_n)^1 u(\hat{y}_{n,k}), \hat{y}_{n,k}; x) - l(\nabla u(\hat{y}_{n,k}), \hat{y}_{n,k}; x)|.$$

By Proposition A.12 the first term on the right is no larger than $C\|u\|_{C^\beta}d(x, \hat{y}_{n,k})^\beta$. On the other hand, from the definition of $l(\cdot, \cdot; \cdot)$, it is immediate that the second term is no larger than

$$|(\nabla_n)^1 u(\hat{y}_{n,k}) - \nabla u(\hat{y}_{n,k})|_{g_{\hat{y}_{n,k}}}d(x, \hat{y}_{n,k}).$$

Now, Lemma A.14 says that $|(\nabla_n)^1 u(\hat{y}_{n,k}) - \nabla u(\hat{y}_{n,k})|_{g_{\hat{y}_{n,k}}} \leq C\|u\|_{C^\beta}\tilde{h}_n^{\beta-1}$. Noting that $d(x, \hat{y}_{n,k})$ is no larger than $C\tilde{h}_n$ for $x \in P^*_{n,k}$, we obtain the estimate

$$|(\nabla_n)^1 u(\hat{y}_{n,k}) - \nabla u(\hat{y}_{n,k})|_{g_{\hat{y}_{n,k}}} \leq C\|u\|_{C^\beta}d(x, \hat{y}_{n,k})^{\beta-1}.$$

Combining the last three estimates, we conclude that

$$|u(\hat{y}_{n,k}) + l((\nabla_n)^1 u(\hat{y}_{n,k}), \hat{y}_{n,k}; x) - u(x)| \leq C\|u\|_{C^\beta}d(x, \tilde{G}_n)^\beta, \quad \forall\, x \in P^*_{n,k}.$$

Using the estimates for the size of $\nabla^2 \phi_{n,k}$, the above implies that

$$|u(\hat{y}_{n,k}) + l((\nabla_n)^1 u(\hat{y}_{n,k}), \hat{y}_{n,k}; x) - u(x)||\nabla^2 \phi_{n,k}(x)| \leq C\|u\|_{C^\beta} d(x, \tilde{G}_n)^{\beta-2}, \ \ \forall \, x \in P_{n,k}^*.$$

Finally, let us recall that the only nonzero terms appearing in the sum I(x) are those with $k \in K_x$ (i.e. $x \in P_{n,k}^*$), and that there at most N of these terms. Then, we conclude that

$$\text{I}(x) \leq C\|u\|_{C^\beta} d(x, \tilde{G}_n)^{\beta-2}.$$

Let us now bound II(x), observe that

$$\left|\nabla_a l((\nabla_n) u(\hat{y}_{n,k}), \hat{y}_{n,k}; x) - \nabla_a l(\nabla_n^1 u(\hat{x}), \hat{x}; x)\right| \leq C\|u\|_{C^\beta} d(\hat{x}, \hat{y}_{n,k})^{\beta-1}, \ \ \forall \, x \in P_{n,k}^*.$$

Therefore

$$\sup_{x \in P_{n,k}^*} \left|\nabla_a l(\nabla_n^1 u(\hat{y}_{n,k}), \hat{y}_{n,k}; x) - \nabla_a l(\nabla_n^1 u(\hat{x}), \hat{x}; x)\right| \leq C\|u\|_{C^\beta} d(x, \tilde{G}_n)^{\beta-1}.$$

As before, the only nonzero terms adding up to II(x) are those with $x \in P_{n,k}^*$, therefore, the above bound implies that

$$\left|\sum_k \nabla_a l(\nabla_n^1 u(\hat{y}_{n,k}), \hat{y}_{n,k}; x) \nabla_b \phi_{n,k}(x)\right| \leq \sum_{k \in K_x} \|u\|_{C^\beta} d(x, \tilde{G}_n)^{\beta-1} C d(x, \tilde{G}_n)^{-1}$$
$$\leq CN\|u\|_{C^\beta} d(x, \tilde{G}_n)^{\beta-2}.$$

Therefore,

$$\text{II}(x) \leq C\|u\|_{C^\beta} d(x, \tilde{G}_n)^{\beta-2}.$$

It remains to bound III(x). According to Proposition 3.16 and Proposition A.15,

$$|\nabla_{ab}^2 l(\nabla_n^1 u(\hat{y}_{n,k}), \hat{y}_{n,k}; x)| \leq C\|u\|_{C^\beta(M)}.$$

Therefore, using (3.9) (from Lemma 3.9) it follows that

$$\left|\sum_k \nabla_{ab}^2 l(\nabla_n^1 u(\hat{y}_{n,k}), \hat{y}_{n,k}; x) \phi_{n,k}(x)\right| \leq C \sum_{k \in K_x} \|u\|_{C^\beta} \phi_{n,k}(x)$$
$$\leq CN\|u\|_{C^\beta}.$$

Gathering the last three estimates, we conclude that

$$|\nabla_{ab}^2 f(x)| \leq C\|u\|_{C^\beta} (d(x, \tilde{G}_n)^{\beta-2} + 1).$$

Moreover, since the indices a, b were arbitrary, and since $d(x, \tilde{G}_n)$ is bounded from above for $x \in M \setminus \tilde{G}_n$ by a constant C, we conclude that

$$|\nabla^2 f(x)| \leq C\|u\|_{C^\beta} d(x, \tilde{G}_n)^{\beta-2}.$$

The case $\beta \in [2,3)$. The proof is entirely analogous to the previous case, and we only highlight the overall steps of the proof: as before, we pick a local system of coordinates $\{x_1, \ldots, x_d\}$ and use the identity

$$\nabla_{abc}^3 f(x) = \sum_k \nabla_{abc}^3 \left[\left(u(\hat{y}_{n,k}) + l(\nabla_n^1 u(\hat{y}_{n,k}), \hat{y}_{n,k}; x) + q(\nabla_n^2 u(\hat{y}_{n,k}), \hat{y}_{n,k}; x)\right) \phi_{n,k}(x) \right],$$

which holds for any three indices a, b, and c. The expression on the right may be expanded via Leibniz rule, resulting in terms mixing various derivatives of $\phi_{n,k}$, $l(\nabla_n^1 u(\hat{y}_{n,k}), \hat{y}_{n,k}; \cdot)$, and $q(\nabla_n^2 u(\hat{y}_{n,k}), \hat{y}_{n,k}; \cdot)$.

It can then be checked that $\nabla_{abc}^3 f(x)$ is given by a sum in k of terms involving $\phi_{n,k}$ and values of u on $\tilde{G}_n$ –in a manner analogue to the case $\beta \in [1,2)$. Now, to bound each of the resulting

terms we will use (3.13) with $i = 1, 2$ as before, but this time also with $i = 3$. The bounds will follow by applying at difference instances Propositions A.12 and A.15, as well as Lemma A.14. All throughout, we will make us of the fact that the only non-zero terms appearing in the sums are those with $k \in P_{n,k}^*$. At the end, we arrive at the bound,

$$|\nabla_{abc}^3 f(x)| \leq C\|u\|_{C^\beta}(d(x, \tilde{G}_n)^{\beta-3} + 1),$$

which holds for any choice of the indices a, b and c. This means that

$$|\nabla^3 f(x)| \leq C\|u\|_{C^\beta} d(x, \tilde{G}_n)^{\beta-3},$$

where we have used again that $d(x, \tilde{G}_n)$ is bounded from above for $x \in M \setminus \tilde{G}_n$. $\qquad \square$

References

[1] Luis Alvarez, Frédéric Guichard, Pierre-Louis Lions, and Jean-Michel Morel. Axioms and fundamental equations of image processing. *Arch. Rational Mech. Anal.*, 123(3):199–257, 1993.

[2] Guy Barles and Cyril Imbert. Second-order elliptic integro-differential equations: viscosity solutions' theory revisited. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 25(3):567–585, 2008.

[3] Guy Barles and Panagiotis E. Souganidis. A new approach to front propagation problems: theory and applications. *Arch. Rational Mech. Anal.*, 141(3):237–296, 1998.

[4] Richard F. Bass and David A. Levin. Transition probabilities for symmetric jump processes. *Trans. Amer. Math. Soc.*, 354(7):2933–2953 (electronic), 2002.

[5] Richard Bellman. *Dynamic programming*. Princeton University Press, Princeton, N. J., 1957.

[6] Jean-Michel Bony, Philippe Courrège, and Pierre Priouret. Sur la forme intégro-différentielle du générateur infinitésimal d'un semi-groupe de Feller sur une variété différentiable. *C. R. Acad. Sci. Paris Sér. A-B*, 263:A207–A210, 1966.

[7] Luis Caffarelli and Luis Silvestre. Regularity theory for fully nonlinear integro-differential equations. *Comm. Pure Appl. Math.*, 62(5):597–638, 2009.

[8] Luis A. Caffarelli. Interior a priori estimates for solutions of fully nonlinear equations. *Ann. of Math. (2)*, 130(1):189–213, 1989.

[9] Luis A. Caffarelli and Xavier Cabré. *Fully nonlinear elliptic equations*, volume 43 of *American Mathematical Society Colloquium Publications*. American Mathematical Society, Providence, RI, 1995.

[10] Jeffrey S Case and Sun-Yung Alice Chang. On fractional gjms operators. *Communications on Pure and Applied Mathematics*, 69(6):1017–1061, 2016.

[11] Sun-Yung Alice Chang and Maria del Mar Gonzalez. Fractional laplacian in conformal geometry. *Advances in Mathematics*, 226(2):1410 – 1432, 2011.

[12] Héctor Chang Lara and Gonzalo Dávila. Regularity for solutions of non local parabolic equations. *Calc. Var. PDE*, 2012. published online.

[13] Héctor Chang Lara and Gonzalo Dávila. Regularity for solutions of nonlocal, nonsymmetric equations. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 29(6):833–859, 2012.

[14] Frank H Clarke. *Optimization and nonsmooth analysis*, volume 5. Siam, 1990.

[15] Philippe Courrege. Sur la forme intégro-différentielle des opérateurs de c_k^∞ dans c satisfaisant au principe du maximum. *Séminaire Brelot-Choquet-Deny. Théorie du Potentiel*, 10(1):1–38, 1965.

[16] Robert J. Elliott and Nigel J. Kalton. *The existence of value in differential games*. American Mathematical Society, Providence, R.I., 1972. Memoirs of the American Mathematical Society, No. 126.

[17] L. C. Evans and H. Ishii. Differential games and nonlinear first order PDE on bounded domains. *Manuscripta Math.*, 49(2):109–139, 1984.

[18] Lawrence C. Evans. On solving certain nonlinear partial differential equations by accretive operator methods. *Israel J. Math.*, 36(3-4):225–247, 1980.

[19] Lawrence C. Evans. Some min-max methods for the Hamilton-Jacobi equation. *Indiana Univ. Math. J.*, 33(1):31–50, 1984.

[20] Lawrence C. Evans and Ronald F. Gariepy. *Measure theory and fine properties of functions*. Studies in Advanced Mathematics. CRC Press, Boca Raton, FL, 1992.

[21] L.C. Evans and Panagiotis E. Souganidis. Differential games and representation formulas for solutions of Hamilton-Jacobi-Isaacs equations. *Indiana Univ. Math. J.*, 33(5):773–797, 1984.

[22] Charles Fefferman, Arie Israel, and Garving K Luli. Interpolation of data by smooth non-negative functions. *arXiv preprint arXiv:1603.02330*, 2016.

[23] Wendell H. Fleming. The Cauchy problem for a nonlinear first order partial differential equation. *J. Differential Equations*, 5:515–530, 1969.

[24] Wendell H. Fleming and Raymond W. Rishel. *Deterministic and stochastic optimal control*. Springer-Verlag, Berlin-New York, 1975. Applications of Mathematics, No. 1.

[25] Avner Friedman. The Cauchy problem for first order partial differential equations. *Indiana Univ. Math. J.*, 23:27–40, 1974.

[26] Guy Gilboa and Stanley Osher. Nonlocal operators with applications to image processing. *Multiscale Model. Simul.*, 7(3):1005–1028, 2008.

[27] Emmanuel Hebey. *Nonlinear analysis on manifolds: Sobolev spaces and inequalities*, volume 5. American Mathematical Soc., 2000.

[28] Pei Hsu. On excursions of reflecting Brownian motion. *Trans. Amer. Math. Soc.*, 296(1):239–264, 1986.

[29] Rufus Isaacs. *Differential games. A mathematical theory with applications to warfare and pursuit, control and optimization.* John Wiley & Sons, Inc., New York-London-Sydney, 1965.

[30] Espen R. Jakobsen and Kenneth H. Karlsen. A "maximum principle for semicontinuous functions" applicable to integro-partial differential equations. *NoDEA Nonlinear Differential Equations Appl.*, 13(2):137–165, 2006.

[31] Moritz Kassmann, Marcus Rang, and Russell W. Schwab. Hölder regularity for integro-differential equations with nonlinear directional dependence. *Indiana Univ. Math. J.*, To Appear, 2014.

[32] Markos A. Katsoulakis. A representation formula and regularizing properties for viscosity solutions of second-order fully nonlinear degenerate parabolic equations. *Nonlinear Anal.*, 24(2):147–158, 1995.

[33] Robert V. Kohn and Sylvia Serfaty. A deterministic-control-based approach to motion by curvature. *Comm. Pure Appl. Math.*, 59(3):344–407, 2006.

[34] Robert V. Kohn and Sylvia Serfaty. A deterministic-control-based approach to fully nonlinear parabolic and elliptic equations. *Comm. Pure Appl. Math.*, 63(10):1298–1350, 2010.

[35] Shigeaki Koike and Andrzej Święch. Representation formulas for solutions of Isaacs integro-PDE. *Indiana Univ. Math. J.*, 62(5):1473–1502, 2013.

[36] N. V. Krylov. On the rate of convergence of finite-difference approximations for elliptic Isaacs equations in smooth domains. *Comm. Partial Differential Equations*, 40(8):1393–1407, 2015.

[37] Hung-Ju Kuo and Neil S. Trudinger. New maximum principles for linear elliptic equations. *Indiana Univ. Math. J.*, 56(5):2439–2452, 2007.

[38] John M. Lee. *Riemannian manifolds*, volume 176 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1997. An introduction to curvature.

[39] YanYan Li and Aobing Li. A fully nonlinear version of the yamabe problem on manifolds with boundary. *Journal of the European Mathematical Society*, 8(2):295–316, 2006.

[40] P L Lions, G Papanicolaou, and S R S Varadhan. Homogenization of Hamilton-Jacobi equations. *unpublished*, circa 1988.

[41] P.-L. Lions and P. E. Souganidis. Differential games, optimal control and directional derivatives of viscosity solutions of Bellman's and Isaacs' equations. *SIAM J. Control Optim.*, 23(4):566–583, 1985.

[42] Carlo Pucci. Su le equazioni ellittiche estremanti. *Rend. Sem. Mat. Fis. Milano*, 35:12–20, 1965.

[43] Russell W. Schwab. Periodic homogenization for nonlinear integro-differential equations. *SIAM J. Math. Anal.*, 42(6):2652–2680, 2010.

[44] Russell W. Schwab. Stochastic homogenization for some nonlinear integro-differential equations. *Communications in Partial Differential Equations*, 38(2):171–198, 2012.

[45] Russell W. Schwab and Luis Silvestre. Regularity for parabolic integro-differential equations with very irregular kernels. *Anal. PDE*, 9(3):727–772, 2016.

[46] Luis Silvestre. On the differentiability of the solution to the Hamilton-Jacobi equation with critical fractional diffusion. *Adv. Math.*, 226(2):2020–2039, 2011.

[47] Panagiotis E. Souganidis. Max-min representations and product formulas for the viscosity solutions of Hamilton-Jacobi equations with applications to differential games. *Nonlinear Anal.*, 9(3):217–257, 1985.

[48] E. M. Stein. *Singular Integrals and Differentiability Properties of Functions*. Princeton University Press, Princeton, 1971.

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