A multitude of orders in the field $\mathbb{F}_\infty$*

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Abstract

The field $\mathbb{F}_\infty$ appeared in [2] in the construction of the first non-classical orthomodular space. It has a non-archimedean order. The purpose of this short paper is to show that it admits uncountable different orders, archimedean as well as non-archimedean.

1 Preliminaries

In all this paper $\mathbb{Q}$ is the field of rational numbers, and $\mathbb{R}$ the field of real numbers.

Definition. Following [2] we define the field $\mathbb{F}_\infty$ in a recursive way. Let $X_1, X_2, \ldots$ be variables. Put
$F_0 = \mathbb{Q}$
$F_n = F_{n-1}(X_n)$ para $n \geq 1$.
And $\mathbb{F}_\infty = \cup_{n=0}^{\infty} F_n$

H. Keller ordered this field by $\leq_0$ as follows.

i) $F_0 = \mathbb{Q}$ has its unique order, which is induced by the order of $\mathbb{R}$.

ii) Suppose the order $\leq_0$ has been extended to F_{n-1}. We define now $\leq_0$ in F_n.
Let $f,\ g\ \in F_{n-1}[X_n]$, $f = a_m X_n^m + a_{m-1} X_n^{m-1} + ... + a_0$ with $a_i \in F_{n-1}$, then

$$0 \leq_0 f \iff 0 \leq_0 a_m \text{ in } F_{n-1}$$

$$0 \leq_0 \frac{f}{g} \iff 0 \leq_0 fg$$

- Clearly this order $\leq_0$ in F_n is an extension of the order $\leq_0$ in F_{n-1}

iii) Hence $(F_\infty,\ \leq_0)$ is an ordered field.

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That this is a non-archimedean order can easily be seen, since for $X_1, X_1^2 \in \mathbb{F}$ and every $n \in \mathbb{N}$ $nX_1 \leq_0 X_1^2$. Even more, we have that for all $n \in \mathbb{N}$ and $i,\ j,\ k,\ l \in \mathbb{N}$

a) $i > k \Rightarrow X_i^j > nX_k^l$

b) $i = k,\ j > l \Rightarrow X_i^j > nX_k^l$

We are interested in the following questions. Can this field be ordered in a non-finite number of ways? Is the same true if the orders are required to be non-archimedean?

We refer the reader to [1] for the general theory of Ordered Fields, and to [3] Ch. 19 for facts about Trascendental Extensions of Fields.

2 Non-archimedean extensions of archimedean orders in $\mathbb{F}_\infty$

The main idea in the construction of the order $\leq_0$ was to give F_0 its usual order as a subfield of $\mathbb{R}$, and then define a non-archimedean extension of this order to F_n for every $n \in \mathbb{N}$. But this "cut" could well be done at a higher level, that is we can order the field F_n (via isomorphism) as a subfield of $\mathbb{R}$ and then define our unconventional order from F_{n+1} on. According to this scheme we will define the order $\leq_j$ for any fixed but arbitrary $j \in \mathbb{N}$ in the following way.

i) We have that $F_j = \mathbb{Q}(X_1, ..., X_j)$. Consider a fixed set $\{y_n\}_1^j$ of positive real numbers, algebraically independent over $\mathbb{Q}$. Then (see [3] 19.5) $F_j \cong \mathbb{Q}(y_1, ..., y_j) \subset \mathbb{R}$. Therefore the order of $\mathbb{R}$ induces an archimedean order $\leq_j$ in the field F_j (and clearly in any F_i with $0 \leq i < j$).

ii) Assume that the order has been constructed up to F_{n-1} where $n - 1 \geq j$. We define $\leq_j$ in F_n in the following way.
Let $f \in F_{n-1}[X_n]$ and $f = a_m X_n^m + a_{m-1} X_n^{m-1} + ... + a_0$ with $a_i \in F_{n-1}$ and $g \in F_{n-1}[X_n]$

$$0 \leq_j f \Longleftrightarrow 0 \leq_j a_m \text{ in } F_{n-1}$$

$$0 \leq_j \tfrac{f}{g} \Longleftrightarrow 0 \leq_j fg$$

- Therefore $\leq_j$ in F_n is an extension of the order $\leq_j$ in F_{n-1}

iii) Then $(F_\infty, \leq_j)$ is a non-archimedeanly ordered field.

We shall now prove that $\leq_i$ and $\leq_j$ are different orders whenever $i \neq j$. Without loss of generality, assume that $j > i$ then there exists an $n \in \mathbb{Z}$ such that

$$X_j^2 + n \leq_j 0.$$

But $0 \leq_i X_j^2 + m$ for all $m \in \mathbb{Z}$ in particular $0 \leq_i X_j^2 + n$, therefore the positive cones of this orders are different, and $\leq_i$ is not equal to $\leq_j$.
In this way we have constructed a countable family of different non-archimedean orders in $\mathbb{F}_\infty$.

2.1 Archimedean orders in $\mathbb{Q}(X_1)$

We will explicitly show that $\mathbb{Q}(X_1)$ can be given an archimedean order in uncountable different ways.
Let a be number in $\mathbb{R}$ transcendental over $\mathbb{Q}$, as it is well known

$$\mathbb{Q}(X_1) \cong \mathbb{Q}(a) \subset \mathbb{R}$$

The restriction of the order of $\mathbb{R}$ is an archimedean order in $\mathbb{Q}(a)$. It induces an order $<_a$, in $\mathbb{Q}(X_1)$ by,

$$p(X_1) \geq 0 \text{ in } \mathbb{Q}[X_1] \Leftrightarrow p(a) \geq 0 \text{ in } \mathbb{R}$$
$$\frac{p(X_1)}{q(X_1)} \geq 0 \text{ in } \mathbb{Q}[X_1] \Leftrightarrow p(X_1)q(X_1) \geq 0 \text{ in } \mathbb{Q}[X_1]$$

We notice that if $a \neq b$ are transcendental numbers in $\mathbb{R}$, then there exists a rational number r such that

$$a < r < b \;\vee\; b < r < a$$

Assume $a < r < b$, and consider $p(X_1) = X_1 - r$

In $\mathbb{Q}(a)$

$$a < r \Rightarrow a - r < 0 \Rightarrow p(X_1) <_a 0$$

In $\mathbb{Q}(b)$

$$r < b \Rightarrow 0 < b - r \Rightarrow 0 <_b p(X_1)$$

Therefore the positive cones of the orders $\leq_a$ and $\leq_b$ are different. This implies that

$$\leq_a \neq \leq_b .$$

Hence we have constructed an uncountable family of archimedean orders in $\mathbb{Q}(X_1)$.

3 Back to $\mathbb{F}_\infty$

We are now in the position to prove the existence of a non-countable family of non-archimedean orders, as well as non-countable family of archimedean orders, in $\mathbb{F}_\infty$.

3.1 Non-Archimedean orders

Let us put together the results of both subsections of the previous section. Each of the different orders of $F_1 = Q(X_1)$ can be extended to an order $\leq_1$ in $\mathbb{F}_\infty$. Therefore we obtain an uncountable family of non-archimedean orders in $\mathbb{F}_\infty$.

3.2 Archimedean orders

We show now that the collection of archimedean orders in $\mathbb{F}_\infty$ is also uncountable.
Let $S \subset \mathbb{R}$ a transcendence basis for $\mathbb{R}$ over $\mathbb{Q}$, we can assume that $S \subset \mathbb{R}^+$. Choose a sequence

$$T = t_1 < t_2 < \ldots < t_m < \ldots$$

of elements of S. For every $n \in \mathbb{N}$ the set $t_1, t_2, \ldots t_n$ is algebraically independent, thus there exists a field isomorphism $\varphi_n : F_n = \mathbb{Q}(X_1, X_2, \ldots X_n) \to \mathbb{Q}(t_1, t_2, \ldots t_n)$ with $\varphi_n(X_i) = t_i$ for $i = 1 \ldots n$.
Let P_n be the positive cone of $\mathbb{Q}(t_1, t_2, \ldots t_n)$ as a subfield of $(\mathbb{R}, \leq)$, then $\varphi_n^{-1}(P_n)$ is a positive cone of F_n. Hence $\mathbb{F}_\infty$ is ordered with positive cone

$$\bigcup_{n \in \mathbb{N}} \varphi_n^{-1}(P_n).$$

This order, denoted by $\leq_T$ is archimedean since it is induced by the archimedean order of a subfield of $(\mathbb{R}, \leq)$.

Since S is a non-denumerable set, we can choose a non-denumerable family of sequences of elements of S, each one ordered as in the previous paragraph, whose first elements are all different. Each one induces an archimedean order in $\mathbb{F}_\infty$ and we contend that those orders are all different.

The proof goes along the same lines as in the case of $\mathbb{Q}(x)$.

Let $T = t_1, t_2, ...$ and $Z = z_1, z_2, ...$ be two sequences as above. We can assume, without loss of generality, that $t_1 < z_1$. Let r be a rational number such that $t_1 < r < z_1$
Consider $p(x_1) = x_1 - r$

In $\mathbb{Q}(T)$

$$t_1 < r \Rightarrow t_1 - r < 0. \text{ Therefore } p(x_1) <_T 0$$

In $\mathbb{Q}(Z)$

$$r < z_1 \Rightarrow 0 < z_1 - r. \text{ Therefore } 0 <_Z p(x_1)$$

Hence the positive cone of the order $\leq_T$ is different from the positive cone of the order induced by $\leq_Z$. Therefore we have constructed an uncountable family of archimedean orders in $\mathbb{F}_\infty$.

Remark. Clearly this procedure implies that for any $j \in \mathbb{N}$, the field F_j also admits uncountable different archimedean orders. And any of these can be extended to a non-archimedean order $\leq_j$ as in subsection 2.1.

Hence there are really a multitude of different orders of the field $\mathbb{F}_\infty$.

References

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[3] Morandi, P. *Field and Galois Theory.* Springer, New York, 1996.