# Chapter 5. Systems of Differential Equations

## 5.1. Modeling with First Order Systems of Differential Equations

**Section Objective(s):**

- Predator-Prey System Revisited.
- Equilibrium Solutions.
- Graphical Representations of Solutions. Phase Portraits.
- Spring-Mass System as a First Order System

### 5.1.1. **Predator-Prey System Revisited.**

Example 1: Consider a system of rabbits and foxes. Let us make the following assumptions:

- Rabbits grow exponentially in the absence of foxes.
- Foxes die out at a rate proportional to their numbers in the absence of rabbits.
- The population of rabbits decreases at a rate proportional to the interaction between rabbits and foxes.
- The population of foxes increases at a rate proportional to the interaction between rabbits and foxes.

Under the above assumptions, the following differential equations might govern the changes in the population of these two species.

**Definition 1.** (Equilibrium Solutions) Given an autonomous system

$$\frac{dx}{dt} = f(x, y)$$
$$\frac{dy}{dt} = g(x, y),$$

_____ are solutions of the form

$x(t) \equiv x_0$, $y(t) \equiv y_0$, which satisfy

_____

In this case $(x_0, y_0)$ is also referred to as

_____

Example 1a: Now consider a particular example, with fixed values of the parameters.

$$\frac{dR}{dt} = 5R - 2RF$$
$$\frac{dF}{dt} = -F + 0.5RF$$

Find the equilibrium solutions of the system. Discuss what they represent in the physical situation of the model.

Solution:

5.1.2. **Graphical Representations of Solutions. Phase Portraits.**

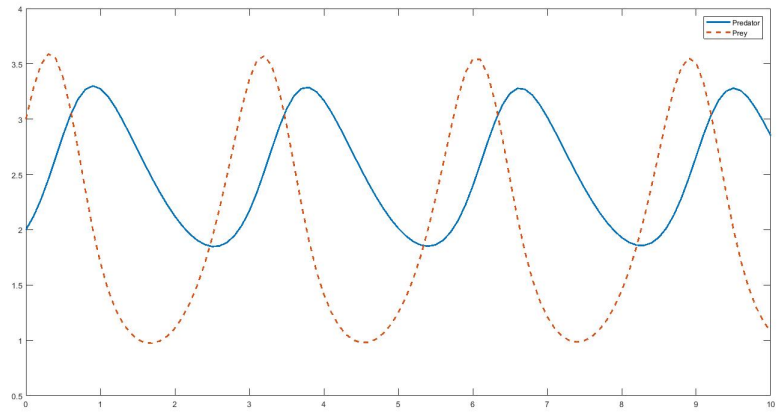


FIGURE 1. $R(t)$ and $F(t)$ as functions of time.

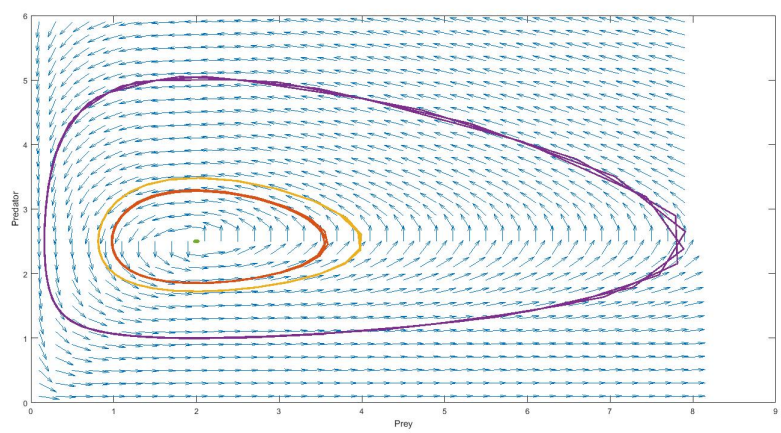


FIGURE 2. Phase portrait of the predator-prey system.

Identify the solution curves in the phase portrait corresponding to the equilibrium solutions and the ones corresponding to initial condition $R(0) = 3$, $F(0) = 2$.

Given a system of differential equations,

$$\frac{dx}{dt} = f(x, y)$$
$$\frac{dy}{dt} = g(x, y),$$

we can plot $x$ and $y$ as functions of $t$ (as in Figure 5.1.2), or we can plot the solution in the $xy-$plane (as in Figure 5.1.2).

**Definition 1.** (Phase Portrait) Given an autonomous system

$$\frac{dx}{dt} = f(x, y)$$
$$\frac{dy}{dt} = g(x, y),$$

the $xy$-plane is called _____

If we simultaneously plot many solution curves, corresponding to different _____, in the phase plane, this is referred to as a _____

EXAMPLE 2: Find the equilibrium solutions of the following competing species system.

$$\frac{dR}{dt} = 3R\,(1 - H - R)$$
$$\frac{dH}{dt} = 2H\,(2 - H - 3R)$$

5.1.3. **Spring-Mass System as a First Order System.** Recall that the motion of a body of mass $m$, attached to a spring with stiffness $k$, under the assumption of no damping/friction is modeled by the following differential equation

where $y(t)$ denotes the displacement of the body away from rest position at time $t$.

Now, let $v(t)$ denote the _____ at time $t$. Then, the above second order differential equation can be rewritten as the following **system of two first order differential equations.**

For simplicity, assume $k = m$. We then actually know the general solution to $y'' + y = 0$. It is given by

(a) Use this to draw the solution curves to the system, corresponding to initial conditions $y(0) = 1$ and $v(0) = 0$. In addition, draw a phase portrait of the system.

(b) Now consider the solution corresponding to initial conditions $y(0) = 0$ and $v(0) = 1$. Plot the solution curve in the $yv-$ plane. In what ways is this solution curve the same as the one in part (a)?

Example 3: Convert the following ODE into a system of first order differential equations.

$$y'' + 5y' + 7y = u(t-3)e^{-2t}.$$

Example 4: Consider the two systems of differential equations.

$$\frac{dx}{dt} = 0.3x - 0.1xy \qquad \frac{dx}{dt} = 0.3x - 3xy$$
$$\frac{dy}{dt} = -0.1y + 2xy \qquad \frac{dy}{dt} = -2y + 0.1xy.$$

One of these systems refers to a predator-prey system with very lethargic predators - those who seldom catch prey but can live for a long time on a single prey (e.g. boa constrictors). The other system refers to a very active predator that requires many prey to stay healthy (e.g. a bob cat). The prey in each case is the same. Identify which system is which and justify your answer.