Chapter 1. Modeling with Ordinary Differential Equations

1.1. Population Models: Exponential Growth, Logistic Equation, Predator-Prey

Section Objective(s):
- Mathematical Models.
- Exponential Growth.
- Logistic Equation.
- Predator-Prey Models.

The following three steps are central to creating a mathematical model in almost any setting.

Step 1. Clearly state all the assumptions.

Step 2. Describe all the ____,
____, and __________________________ to be used in the model.

Step 3. Use your assumptions to derive equations relating the variables and parameters.

Step 4. Analyze the predictions of the model.

1.1.1. **Exponential Growth.**

We model population growth under the assumption of ____.
Examples where this model applies include:

Remark: Assume that the rate of growth of the population is

____.

Clearly denote all our variables and parameters.

- Let t denote time (independent variable),
- let $P(t)$ denote ______________________________
- let ____

Recall that the rate of growth of $P(t)$ is given by ____________.

Given our assumption that the rate of growth of the population P is proportional to $P(t)$ with ______________________________, we arrive at the following model for **exponential growth**:

If the constant of proportionality is $k = 1$, what do the solutions of this differential equation with initial conditions $P(0) = 0$, $P(0) = 1$, and $P(0) = 2$ look like?

What if $P(0) = 1$ and we vary k? How will solutions to the following equations differ?

$$\frac{dP}{dt} = P \qquad \frac{dP}{dt} = 2P$$

Could we choose $k < 0$? What physical situation would this represent?

1.1.2. **Logistic Population Model.**

Under what conditions does the exponential growth model make physical sense? Is it realistic for the number of rabbits to grow to infinity as t grows?

For this reason we consider the so called **logistic model** to account for limited resources (in terms of food, space, etc.).

We assume that if the population is small, the rate of growth of the population is ____. However, if the population is too large to be supported by the resources in the environment, we assume ____.

As in the previous model, we denote all our variables and parameters.

- Let t denote time,
- let $P(t)$ denote the population at time t,
- let k denote the growth-rate coefficient for small values of P,
- ____

Given the above assumptions, $P(t)$ will decrease if ________________________ and if P is relatively small, then ________________________________. We are looking for the simplest equation that would satisfy the above conditions. Thus,

$$\frac{dP}{dt} =$$

1.1.3. **Predator-Prey Systems.**

In many cases we might be interested in modeling the interaction between two or more species.

Here we will model the population of one species of predator and one species of prey living in the same environment. Consider, for example, foxes and rabbits living in a forest. We make the following assumptions.

(1) In the absence of foxes, the rabbits ________________________________.

(2) The death rate of rabbits is proportional to ________________________ ____.

(3) In the absence of rabbits, the foxes _________________________________ ____.

(4) The birth rate of foxes is proportional to _________________________ ____.

We will use the following notation:

- As usual, t denotes time.
- $F(t)$ - ____
- $R(t)$ - ____
- α - growth rate coefficient of the rabbits,
- β - the death rate coefficient of rabbits due to the fox-rabbit interaction,
- γ - the death rate coefficient of foxes,
- δ - the birth rate coefficient of foxes (the constant of proportionality which measures the benefit to the fox due to the rabit-fox interaction).

Note that all the parameters in the model are positive. Taking the above assumptions into account, we arrive at the following model

$$\begin{aligned} \frac{dR}{dt} &= \underline{\hspace{6cm}} \\ \frac{dF}{dt} &= \underline{\hspace{6.5cm}}. \end{aligned} \tag{1.1.1}$$