1.2. Direction Fields: Graphical Representation of the ODE and its Solution

Let us consider a first order differential equation of the form $\frac{dy}{dx} = f(x, y)$. In this section we aim to understand the solution $y(x)$ graphically. Questions about existence and uniqueness, i.e.,

(1) Given an initial condition $y(0) = y_0$, does a solution exist?
(2) If a solution does exist, is it unique?

will be discussed in Chapter **??**. For now it suffices to say that the answer to both questions is "yes", provided $f(x, y)$ is sufficiently nice. Moreover, the property of existence and uniqueness of the solution to a differential equation of the form $\frac{dy}{dx} = f(x, y)$ still holds if we start "counting time" at any point t_0, i.e., we could choose any initial condition of the form $y(t_0) = y_0$. This can be rigorously stated in the following theorem.

Theorem 1.2.1. *(Existence and Uniqueness of Solutions) Suppose that both the function $f(x, y)$ and its partial derivative $\frac{\partial}{\partial y} f(x, y)$ are continuous on some rectangle R in the $xy-$plane containing the point (x_0, y_0) in its interior. Then, for some open interval I containing the point x_0, the initial value problem*

$$\frac{dy}{dx} = f(x, y), \qquad y(x_0) = y_0$$

has one and only one solution that is defined on the interval I.

As a first example, consider the equation $\frac{dy}{dx} = 2x$. The general solution to this ODE is $y(x) = x^2 + C$. For each initial condition $y(0) = y_0$, there exists a unique solution. In fact, given any point (x, y), there is a unique curve through this point, which has slope $y'(x) = 2x$.

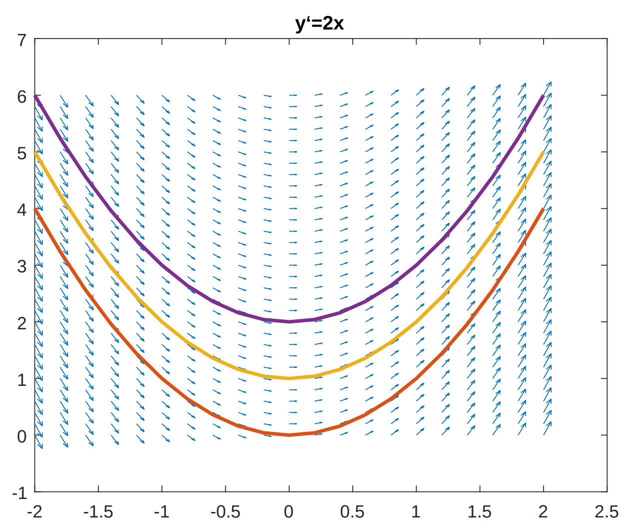

Figure 1. Direction field for the equation $\frac{dy}{dx} = 2x$ and particular solutions with initial conditions $y(0) = 0$, $y(0) = 1$, and $y(0) = 2$.

Similarly, for the ODE $\frac{dy}{dx} = y$, one can draw solution curves for different initial conditions, using the slope information provided by the ODE.

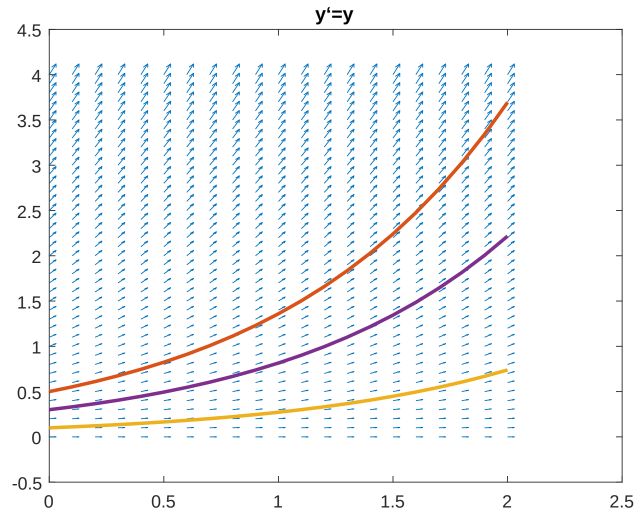

FIGURE 2. Direction field for the equation $\frac{dy}{dx} = y$ and particular solutions with initial conditions $y(0) = 0.1$, $y(0) = 0.3$, and $y(0) = 0.5$.

Remark: Note that the solution curves corresponding to different initial conditions *do not cross*. Is this just a coincidence for the above two examples, or is this a property of solution curves in general?

Recall the discussion of uniqueness of the solution of the differential equation, given any initial condition $y(t_0) = y_0$. Now, if we assume that two solution curves are allowed to cross, consider the point where they do cross and denote it by (a, b). Now, consider the same differential equation with initial condition $y(a) = b$. The fact that we assumed there are two solution curves passing through (a, b) would violate the theorem of uniqueness of the solution to the initial value problem with initial condition $y(a) = b$.

In general, the fact that the function $f(x, y)$ provides the slope of the tangent line to the solution curves at each point (x, y) gives us a simple way to determine the overall shape of the solution curves. Pick several points (x, y) in the plane and compute $f(x, y)$ for each of them. Draw a short line segment with slope $m = f(x, y)$ at each of your chosen points (x, y). The resulting sketch is called the **slope field** of the ODE. This process is quite tedious to do by hand, but computer algebra systems like Matlab and Mathematica do a wonderful job.

This approach was not necessary for the simple differential equations we considered above, as we could easily solve them analytically. However, in many cases finding an analytic solution is not possible, so one should use qualitative, graphical and numerical approaches. For example, an equation as simple as $\frac{dy}{dx} = x + \sin(y)$ does not have an analytic solution. Note that it does have a solution, but this solution cannot be expressed in terms of standard trigonometric, exponential, polynomial functions. Consider the slope field of this equation, given below and draw the solution curve satisfying $y(0) = 1$.

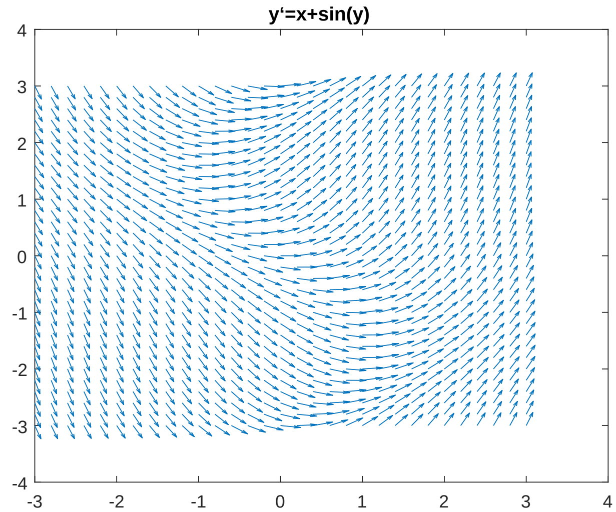

FIGURE 3. Direction field for the equation $\frac{dy}{dx} = x + \sin(y)$.

1.3. AUTONOMOUS EQUATIONS: QUALITATIVE APPROACH

In the case of an **autonomous differential equation**

$$\frac{dy}{dx} = f(y), \tag{1.3.1}$$

where the right hand side of the equation does not depend on the independent variable, x, the slope field has a special property. Notice, for example in figure 2, that the vectors in the direction field are parallel to each other along each horizontal line. This will be the case for every autonomous ODE as the right hand side of the differential equation only depends on y.

As discussed above, there are many equations which cannot be solved explicitly. Often, however, we can extract qualitative information about general properties of the solution. For example, we may be able to say something about the long-term behavior of a solution $y(t)$, $(\lim_{t\to\infty} y(t))$ - does the solution become unbounded, does it settle down at a finite limit, etc. Here we will focus our attention on gaining information about the qualitative behavior of autonomous differential equations.

The solutions of the equation $f(y) = 0$ are called **critical points** of the autonomous ODE and play an important role in analyzing its qualitative behavior. If $y = c$ is a critical point of equation 1.3.1, then $y(t) \equiv c$ is a solution, which we call an **equilibrium solution**.

For example, consider again the logistic equation **??**, with $k = 2$ and $N = 10$:

$$\frac{dP}{dt} = 2P\left(1 - \frac{P}{10}\right). \tag{1.3.2}$$

It has two equilibrium solutions, $P(t) = 0$ and $P(t) = 10$. We can interpret these in the context of population dynamics in the following way. If we start with 0 individuals, the population will be 0 at any point in time. Similarly, if the population is exactly at its carrying capacity, in this case $P(0) = N = 10$, then the solution is again a constant-valued function, $P(t) = 10$ for all t.

Can we say anything about the solution if $P(0) = 1$ or $P(0) = 11$, or in the case of the general logistic equation, if $0 < P(0) < N$ or if $P(0) > N$?

Let $f(P) = kP\left(1 - \dfrac{P}{N}\right)$ denote the right hand side of **??**.

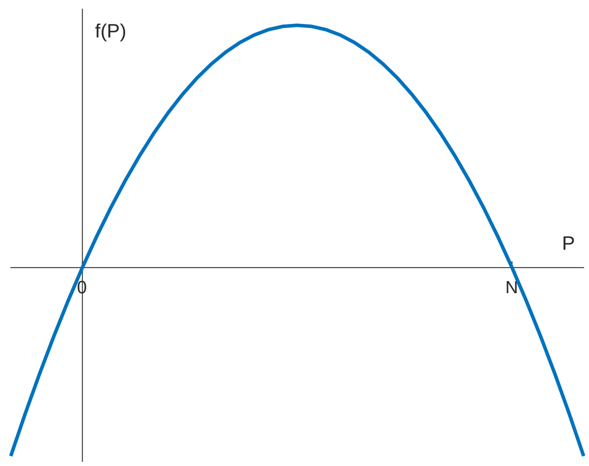

FIGURE 4. Graph of the right hand side of the logistic equation **??**.

Since the logistic equation is an autonomous ODE, its slope field only depends on the dependent variable (in this notation, P) and does not depend on the independent variable (t), we can summarize its behavior in a so-called **phase diagram**. It indicates the direction, or "phase" of change of P as a function of itself.

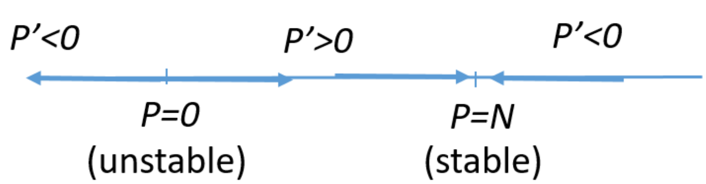

FIGURE 5. Phase diagram for the logistic equation.

Note that $P' = f(P) > 0$ for $0 < P < N$, so $P(t)$ is increasing whenever the initial population, $P(0)$ is less than the carrying capacity, N. As the population approaches the carrying capacity, $P' = f(P)$ approaches 0, so the population levels off at the carrying capacity number. Similarly, if $P(0) > N$, $f(P) < 0$, so the population is decreasing, approaching N as t goes to infinity.

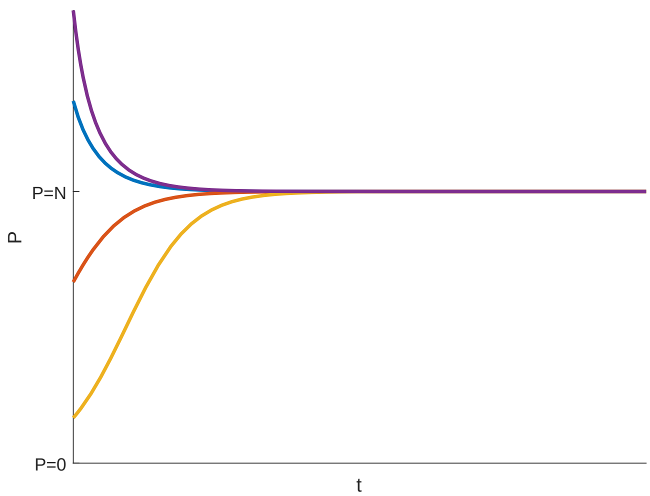

FIGURE 6. Solutions of the logistic equation **??** for different values of the initial condition.

A critical point $y = c$ of an autonomous first order differential equation $y' = f(y)$ is called **stable** provided that for any initial condition which is sufficiently close to c, the solution $y(t)$ remains close to c for all $t > 0$. Mathematically we express this in the following way. For each $\varepsilon > 0$, there exists $\delta > 0$ such that $|y(0) - c| < \delta$ implies $|y(t) - c| < \varepsilon$ for all $t > 0$. The equilibrium solution $y = c$ is **unstable** if it is not stable.