6.4.1. **The Inverse Matrix.** We now introduce the concept of the inverse of a square matrix. Not every square matrix is invertible. The inverse of a matrix is useful to compute solutions to linear systems of algebraic equations.

Definition 6.4.1. *The matrix $I_n \in \mathbb{F}^{n,n}$ is the* ***identity matrix*** *iff $I_n \boldsymbol{x} = \boldsymbol{x}$ for all $\boldsymbol{x} \in \mathbb{F}^n$.*

It is simple to see that the components of the identity matrix are given by

$$I_n = [I_{ij}] \quad \text{with} \quad \begin{cases} I_{ii} = 1 \\ I_{ij} = 0 \quad i \neq j. \end{cases}$$

The cases $n = 2, 3$ are given by

$$I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Definition 6.4.2. *A matrix $A \in \mathbb{F}^{n,n}$ is called* ***invertible*** *iff there exists a matrix, denoted as A^{-1}, such that $\big(A^{-1}\big)A = I_n$, and $A\big(A^{-1}\big) = I_n$.*

Example 6.4.1: Verify that the matrix and its inverse are given by

$$A = \begin{bmatrix} 2 & 2 \\ 1 & 3 \end{bmatrix}, \qquad A^{-1} = \frac{1}{4}\begin{bmatrix} 3 & -2 \\ -1 & 2 \end{bmatrix}.$$

Solution: We have to compute the products,

$$A\big(A^{-1}\big) = \begin{bmatrix} 2 & 2 \\ 1 & 3 \end{bmatrix} \frac{1}{4}\begin{bmatrix} 3 & -2 \\ -1 & 2 \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix} \quad \Rightarrow \quad A\big(A^{-1}\big) = I_2.$$

It is simple to check that the equation $\big(A^{-1}\big)A = I_2$ also holds. ◁

Theorem 6.4.3. *Given a 2×2 matrix A introduce the number Δ as follows,*

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \qquad \Delta = ad - bc.$$

The matrix A is invertible iff $\Delta \neq 0$. Furthermore, if A is invertible, its inverse is given by

$$\mathsf{A}^{-1} = \frac{1}{\Delta}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}. \tag{6.4.1}$$

The number Δ is called the determinant of A, since it is the number that determines whether A is invertible or not.

Example 6.4.2: Compute the inverse of matrix $A = \begin{bmatrix} 2 & 2 \\ 1 & 3 \end{bmatrix}$, given in Example 6.4.1.

Solution: Following Theorem 6.4.3 we first compute $\Delta = 6 - 4 = 4$. Since $\Delta \neq 0$, then A^{-1} exists and it is given by

$$A^{-1} = \frac{1}{4}\begin{bmatrix} 3 & -2 \\ -1 & 2 \end{bmatrix}.$$

◁

Example 6.4.3: Compute the inverse of matrix $A = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}$.

Solution: Following Theorem 6.4.3 we first compute $\Delta = 6 - 6 = 0$. Since $\Delta = 0$, then matrix A is not invertible. ◁

The matrix operations we have introduced are useful to solve matrix equations, where the unknown is a matrix. We now show an example of a matrix equation.

EXAMPLE 6.4.4: Find a matrix X such that $AXB = I$, where

$$A = \begin{bmatrix} 1 & 3 \\ 2 & 1 \end{bmatrix}, \qquad B = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}, \qquad I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

SOLUTION: There are many ways to solve a matrix equation. We choose to multiply the equation by the inverses of matrix A and B, if they exist. So first we check whether A is invertible. But

$$\det(A) = \begin{vmatrix} 1 & 3 \\ 2 & 1 \end{vmatrix} = 1 - 6 = -5 \neq 0,$$

so A is indeed invertible. Regarding matrix B we get

$$\det(B) = \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} = 4 - 1 = 3 \neq 0,$$

so B is also invertible. We then compute their inverses,

$$A^{-1} = \frac{1}{-5} \begin{bmatrix} 1 & -3 \\ -2 & 1 \end{bmatrix}, \qquad B = \frac{1}{3} \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}.$$

We can now compute X,

$$AXB = I \quad \Rightarrow \quad A^{-1}(AXB)B^{-1} = A^{-1}IB^{-1} \quad \Rightarrow \quad X = A^{-1}B^{-1}.$$

Therefore,

$$X = \frac{1}{-5} \begin{bmatrix} 1 & -3 \\ -2 & 1 \end{bmatrix} \frac{1}{3} \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} = -\frac{1}{15} \begin{bmatrix} 5 & -7 \\ -5 & 4 \end{bmatrix}$$

so we obtain

$$X = \begin{bmatrix} -\frac{1}{3} & \frac{7}{15} \\ \frac{1}{3} & -\frac{4}{15} \end{bmatrix}.$$

◁

6.4.2. **Overview of Determinants.** A determinant is a scalar computed form a square matrix that gives important information about the matrix, for example if the matrix is invertible or not. We now review the definition and properties of the determinant of 2×2 and 3×3 matrices.

Definition 6.4.4. *The **determinant of a** 2×2 matrix $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ is given by*

$$\det(A) = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}.$$

*The **determinant of a** 3×3 matrix $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ is given by*

$$\det(A) = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}.$$

Example 6.4.5: The following three examples show that the determinant can be a negative, zero or positive number.

$$\begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} = 4 - 6 = -2, \qquad \begin{vmatrix} 2 & 1 \\ 3 & 4 \end{vmatrix} = 8 - 3 = 5, \qquad \begin{vmatrix} 1 & 2 \\ 2 & 4 \end{vmatrix} = 4 - 4 = 0.$$

The following is an example shows how to compute the determinant of a 3×3 matrix,

$$\begin{aligned} \begin{vmatrix} 1 & 3 & -1 \\ 2 & 1 & 1 \\ 3 & 2 & 1 \end{vmatrix} &= (1) \begin{vmatrix} 1 & 1 \\ 2 & 1 \end{vmatrix} - 3 \begin{vmatrix} 2 & 1 \\ 3 & 1 \end{vmatrix} + (-1) \begin{vmatrix} 2 & 1 \\ 3 & 2 \end{vmatrix} \\ &= (1-2) - 3\,(2-3) - (4-3) \\ &= -1 + 3 - 1 \\ &= 1. \end{aligned}$$

◁

Remark: The determinant of upper or lower triangular matrices is the product of the diagonal coefficients.

Example 6.4.6: Compute the determinant of a 3×3 upper triangular matrix.

Solution:

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{vmatrix} = a_{11} \begin{vmatrix} a_{22} & a_{23} \\ 0 & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} 0 & a_{23} \\ 0 & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} 0 & a_{22} \\ 0 & 0 \end{vmatrix} = a_{11} a_{22} a_{33}.$$

◁

The absolute value of the determinant of a 2×2 matrix $A = [\boldsymbol{a}_1, \boldsymbol{a}_2]$ has a geometrical meaning: It is the area of the parallelogram whose sides are given by $\boldsymbol{a}_1$ and $\boldsymbol{a}_2$, that is, by the columns of the matrix A; see Fig. 1. Analogously, the absolute value of the determinant of a 3×3 matrix $A = [\boldsymbol{a}_1, \boldsymbol{a}_2, \boldsymbol{a}_3]$ also has a geometrical meaning: It is the volume of the parallelepiped whose sides are given by $\boldsymbol{a}_1$, $\boldsymbol{a}_2$ and $\boldsymbol{a}_3$, that is, by the columns of the matrix A; see Fig. 1.

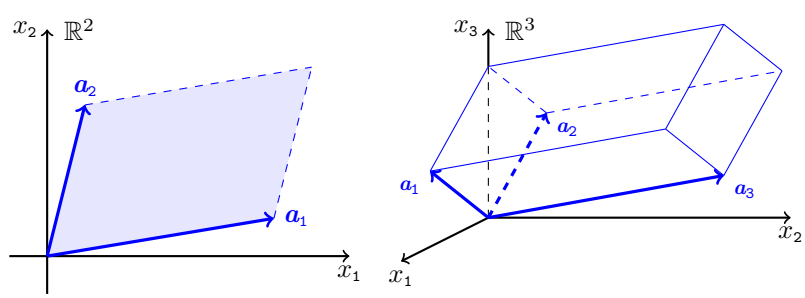

Figure 1. Geometrical meaning of the determinant.

The determinant of an $n \times n$ matrix A can be defined generalizing the properties that areas of parallelogram have in two dimensions and volumes of parallelepipeds have in three dimensions. One of these properties is the following: if one of the column vectors of the matrix A is a linear combination of the others, then the figure determined by these column vectors is not n-dimensional but $(n-1)$-dimensional, so its volume must vanish. We highlight this property of the determinant of $n \times n$ matrices in the following result.

Theorem 6.4.5. *The set of n-vectors $\{\boldsymbol{v}_1, \cdots, \boldsymbol{v}_n\}$, with $n \geqslant 1$, is linearly dependent iff*

$$\det[\boldsymbol{v}_1, \cdots, \boldsymbol{v}_n] = 0.$$

EXAMPLE 6.4.7: Show whether the set of vectors below linearly independent,

$$\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} -3 \\ 2 \\ 7 \end{bmatrix} \right\}.$$

The determinant of the matrix whose column vectors are the vectors above is given by

$$\begin{vmatrix} 1 & 3 & -3 \\ 2 & 2 & 2 \\ 3 & 1 & 7 \end{vmatrix} = (1)(14-2) - 3(14-6) + (-3)(2-6) = 12 - 24 + 12 = 0.$$

Therefore, the set of vectors above is linearly dependent. ◁

The determinant of a square matrix also determines whether the matrix is invertible or not.

Theorem 6.4.6. *An $n \times n$ matrix A is invertible iff holds $\det(A) \neq 0$.*

EXAMPLE 6.4.8: Is matrix $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 7 \\ 3 & 7 & 9 \end{bmatrix}$ invertible?

SOLUTION: We only need to compute the determinant of A.

$$\det(A) = \begin{vmatrix} 1 & 2 & 3 \\ 2 & 5 & 7 \\ 3 & 7 & 9 \end{vmatrix} = (1)\begin{vmatrix} 5 & 7 \\ 7 & 9 \end{vmatrix} - (2)\begin{vmatrix} 2 & 7 \\ 3 & 9 \end{vmatrix} + (3)\begin{vmatrix} 2 & 5 \\ 3 & 7 \end{vmatrix}.$$

Using the definition of determinant for 2×2 matrices we obtain

$$\det(A) = (45 - 49) - 2(18 - 21) + 3(14 - 15) = -4 + 6 - 3.$$

Since $\det(A) = -1$, that is, non-zero, matrix A is invertible. ◁

We continue with the review on Linear Algebra we started in § **??** and § **??**. We saw that a square matrix is a function on the space of vectors, since it acts on a vector and the result is another vector. In this section we see that, given an $n \times n$ matrix, there may exist lines through the origin in $\mathbb{R}^n$ that are left invariant under the action of the matrix. This means that such a matrix acting on any vector in such a line is a vector on the same line. The vector is called an eigenvector of the matrix, and the proportionality factor is called an eigenvalue. If there is a linearly independent set of n eigenvectors of an $n \times n$ matrix, then this matrix is diagonalizable. In § **??** we will see that the exponential of a matrix is particularly simple to compute in the case that the matrix is diagonalizable. In Chapter **??** we use the exponential of a matrix to write the solutions of systems of linear differential equations with constant coefficients.

6.4.3. **Eigenvalues and Eigenvectors.** When a square matrix acts on a vector the result is another vector that, more often than not, points in a different direction from the original vector. However, there may exist vectors whose direction is not changed by the matrix. These will be important for us, so we give them a name.

Definition 6.4.7. *A number λ and a nonzero n-vector $\boldsymbol{v}$ are an **eigenvalue** with corresponding **eigenvector** (eigenpair) of an $n \times n$ matrix A iff they satisfy the equation*

$$A\boldsymbol{v} = \lambda \boldsymbol{v}.$$

Remark: We see that an eigenvector $\boldsymbol{v}$ determines a particular direction in the space $\mathbb{R}^n$, given by $(a\boldsymbol{v})$ for $a \in \mathbb{R}$, that remains invariant under the action of the matrix A. That is, the result of matrix A acting on any vector $(a\boldsymbol{v})$ on the line determined by $\boldsymbol{v}$ is again a vector on the same line, since

$$A(a\boldsymbol{v}) = a\,A\boldsymbol{v} = a\lambda\boldsymbol{v} = \lambda(a\boldsymbol{v}).$$

Example 6.4.9: Verify that the pair λ_1, $\boldsymbol{v}_1$ and the pair λ_2, $\boldsymbol{v}_2$ are eigenvalue and eigenvector pairs of matrix A given below,

$$A = \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix}, \qquad \begin{cases} \lambda_1 = 4 & \boldsymbol{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \\ \lambda_2 = -2 & \boldsymbol{v}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}. \end{cases}$$

Solution: We just must verify the definition of eigenvalue and eigenvector given above. We start with the first pair,

$$A\boldsymbol{v}_1 = \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 4 \end{bmatrix} = 4 \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \lambda_1 \boldsymbol{v}_1 \quad \Rightarrow \quad A\boldsymbol{v}_1 = \lambda_1 \boldsymbol{v}_1.$$

A similar calculation for the second pair implies,

$$A\boldsymbol{v}_2 = \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ -2 \end{bmatrix} = -2 \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \lambda_2 \boldsymbol{v}_2 \quad \Rightarrow \quad A\boldsymbol{v}_2 = \lambda_2 \boldsymbol{v}_2.$$

◁

Example 6.4.10: Find the eigenvalues and eigenvectors of the matrix $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.

Solution: This is the matrix given in Example **??**. The action of this matrix on the plane is a reflection along the line $x_1 = x_2$, as it was shown in Fig. **??**. Therefore, this line $x_1 = x_2$ is left invariant under the action of this matrix. This property suggests that an eigenvector is any vector on that line, for example

$$\boldsymbol{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad \Rightarrow \quad \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad \Rightarrow \quad \lambda_1 = 1.$$

So, we have found one eigenvalue-eigenvector pair: $\lambda_1 = 1$, with $\boldsymbol{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. We remark that any nonzero vector proportional to $\boldsymbol{v}_1$ is also an eigenvector. Another choice fo eigenvalue-eigenvector pair is $\lambda_1 = 1$, with $\boldsymbol{v}_1 = \begin{bmatrix} 3 \\ 3 \end{bmatrix}$. It is not so easy to find a second eigenvector which does not belong to the line determined by $\boldsymbol{v}_1$. One way to find such eigenvector is noticing that the line perpendicular to the line $x_1 = x_2$ is also left invariant by matrix A. Therefore, any nonzero vector on that line must be an eigenvector. For example the vector $\boldsymbol{v}_2$ below, since

$$\boldsymbol{v}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \quad \Rightarrow \quad \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} = (-1) \begin{bmatrix} -1 \\ 1 \end{bmatrix} \quad \Rightarrow \quad \lambda_2 = -1.$$

So, we have found a second eigenvalue-eigenvector pair: $\lambda_2 = -1$, with $\boldsymbol{v}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$. These two eigenvectors are displayed on Fig. 2. ◁

There exist matrices that do not have eigenvalues and eigenvectors, as it is show in the example below.

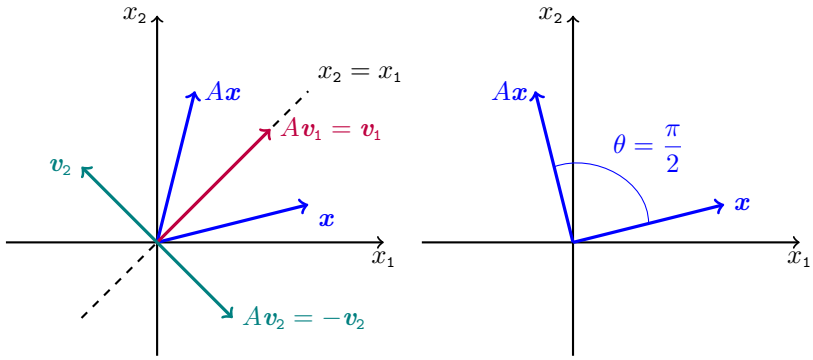

FIGURE 2. The first picture shows the eigenvalues and eigenvectors of the matrix in Example 6.4.10. The second picture shows that the matrix in Example 6.4.11 makes a counterclockwise rotation by an angle θ, which proves that this matrix does not have eigenvalues or eigenvectors.

EXAMPLE 6.4.11: Fix any number $\theta \in (0, 2\pi)$ and define the matrix $A = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$. Show that A has no real eigenvalues.

SOLUTION: One can compute the action of matrix A on several vectors and verify that the action of this matrix on the plane is a rotation counterclockwise by and angle θ, as shown in Fig. 2. A particular case of this matrix was shown in Example **??**, where $\theta = \pi/2$. Since eigenvectors of a matrix determine directions which are left invariant by the action of the matrix, and a rotation does not have such directions, we conclude that the matrix A above does not have eigenvectors and so it does not have eigenvalues either. $\triangleleft$

Remark: We will show that matrix A in Example 6.4.11 has complex-valued eigenvalues.

We now describe a method to find eigenvalue-eigenvector pairs of a matrix, if they exit. In other words, we are going to solve the eigenvalue-eigenvector problem: Given an $n \times n$ matrix A find, if possible, all its eigenvalues and eigenvectors, that is, all pairs λ and $\boldsymbol{v} \neq \boldsymbol{0}$ solutions of the equation

$$A\boldsymbol{v} = \lambda \boldsymbol{v}.$$

This problem is more complicated than finding the solution $\boldsymbol{x}$ to a linear system $A\boldsymbol{x} = \boldsymbol{b}$, where A and $\boldsymbol{b}$ are known. In the eigenvalue-eigenvector problem above neither λ nor $\boldsymbol{v}$ are known. To solve the eigenvalue-eigenvector problem for a matrix A we proceed as follows:

(a) First, find the eigenvalues λ;
(b) Second, for each eigenvalue λ, find the corresponding eigenvectors $\boldsymbol{v}$.

The following result summarizes a way to solve the steps above.

Theorem 6.4.8 (Eigenvalues-Eigenvectors)**.**

(a) All the eigenvalues λ of an $n \times n$ matrix A are the solutions of

$$\det(A - \lambda I) = 0. \tag{6.4.2}$$

(b) Given an eigenvalue λ of an $n \times n$ matrix A, the corresponding eigenvectors $\boldsymbol{v}$ are the nonzero solutions to the homogeneous linear system

$$(A - \lambda I)\boldsymbol{v} = \boldsymbol{0}. \tag{6.4.3}$$

Proof of Theorem 6.4.8: The number λ and the nonzero vector $\boldsymbol{v}$ are an eigenvalue-eigenvector pair of matrix A iff holds

$$A\boldsymbol{v} = \lambda\boldsymbol{v} \quad \Leftrightarrow \quad (A - \lambda I)\boldsymbol{v} = \boldsymbol{0},$$

where I is the $n \times n$ identity matrix. Since $\boldsymbol{v} \neq \boldsymbol{0}$, the last equation above says that the columns of the matrix $(A - \lambda I)$ are linearly dependent. This last property is equivalent, by Theorem 6.4.5, to the equation

$$\det(A - \lambda I) = 0,$$

which is the equation that determines the eigenvalues λ. Once this equation is solved, substitute each solution λ back into the original eigenvalue-eigenvector equation

$$(A - \lambda I)\boldsymbol{v} = \boldsymbol{0}.$$

Since λ is known, this is a linear homogeneous system for the eigenvector components. It always has nonzero solutions, since λ is precisely the number that makes the coefficient matrix $(A - \lambda I)$ not invertible. This establishes the Theorem. $\square$

Example 6.4.12: Find the eigenvalues λ and eigenvectors $\boldsymbol{v}$ of the matrix $A = \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix}$.

Solution: We first find the eigenvalues as the solutions of the Eq. (6.4.2). Compute

$$A - \lambda I = \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} = \begin{bmatrix} (1-\lambda) & 3 \\ 3 & (1-\lambda) \end{bmatrix}.$$

Then we compute its determinant,

$$0 = \det(A - \lambda I) = \begin{vmatrix} (1-\lambda) & 3 \\ 3 & (1-\lambda) \end{vmatrix} = (\lambda - 1)^2 - 9 \quad \Rightarrow \quad \begin{cases} \lambda_+ = 4, \\ \lambda_- = -2. \end{cases}$$

We have obtained two eigenvalues, so now we introduce $\lambda_+ = 4$ into Eq. (6.4.3), that is,

$$A - 4I = \begin{bmatrix} 1-4 & 3 \\ 3 & 1-4 \end{bmatrix} = \begin{bmatrix} -3 & 3 \\ 3 & -3 \end{bmatrix}.$$

Then we solve for $\boldsymbol{v}^+$ the equation

$$(A - 4I)\boldsymbol{v}^+ = \boldsymbol{0} \quad \Leftrightarrow \quad \begin{bmatrix} -3 & 3 \\ 3 & -3 \end{bmatrix} \begin{bmatrix} v_1^+ \\ v_2^+ \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

The solution can be found using Gauss elimination operations, as follows,

$$\begin{bmatrix} -3 & 3 \\ 3 & -3 \end{bmatrix} \to \begin{bmatrix} 1 & -1 \\ 3 & -3 \end{bmatrix} \to \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} \quad \Rightarrow \quad \begin{cases} v_1^+ = v_2^+, \\ v_2^+ \quad \text{free}. \end{cases}$$

Al solutions to the equation above are then given by

$$\boldsymbol{v}^+ = \begin{bmatrix} v_2^+ \\ v_2^+ \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} v_2^+ \quad \Rightarrow \quad \boldsymbol{v}^+ = \begin{bmatrix} 1 \\ 1 \end{bmatrix},$$

where we have chosen $v_2^+ = 1$. A similar calculation provides the eigenvector $\boldsymbol{v}^-$ associated with the eigenvalue $\lambda_- = -2$, that is, first compute the matrix

$$A + 2I = \begin{bmatrix} 3 & 3 \\ 3 & 3 \end{bmatrix}$$

then we solve for $\boldsymbol{v}^-$ the equation

$$(A + 2I)\boldsymbol{v}^- = \boldsymbol{0} \quad \Leftrightarrow \quad \begin{bmatrix} 3 & 3 \\ 3 & 3 \end{bmatrix} \begin{bmatrix} v_1^- \\ v_2^- \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

The solution can be found using Gauss elimination operations, as follows,

$$\begin{bmatrix} 3 & 3 \\ 3 & 3 \end{bmatrix} \to \begin{bmatrix} 1 & 1 \\ 3 & 3 \end{bmatrix} \to \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \quad \Rightarrow \quad \begin{cases} v_1^- = -v_2^-, \\ v_2^- \quad \text{free.} \end{cases}$$

All solutions to the equation above are then given by

$$\boldsymbol{v}^- = \begin{bmatrix} -v_2^- \\ v_2^- \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix} v_2^- \quad \Rightarrow \quad \boldsymbol{v}^- = \begin{bmatrix} -1 \\ 1 \end{bmatrix},$$

where we have chosen $v_2^- = 1$. We therefore conclude that the eigenvalues and eigenvectors of the matrix A above are given by

$$\lambda_+ = 4, \quad \boldsymbol{v}^+ = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \qquad \lambda_- = -2, \quad \boldsymbol{v}^- = \begin{bmatrix} -1 \\ 1 \end{bmatrix}.$$

◁

It is useful to introduce few more concepts, that are common in the literature.

Definition 6.4.9. *The **characteristic polynomial** of an $n \times n$ matrix A is the function*

$$p(\lambda) = \det(A - \lambda I).$$

Example 6.4.13: Find the characteristic polynomial of matrix $A = \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix}$.

Solution: We need to compute the determinant

$$p(\lambda) = \det(A - \lambda I) = \begin{vmatrix} (1-\lambda) & 3 \\ 3 & (1-\lambda) \end{vmatrix} = (1-\lambda)^2 - 9 = \lambda^2 - 2\lambda + 1 - 9.$$

We conclude that the characteristic polynomial is $p(\lambda) = \lambda^2 - 2\lambda - 8$. ◁

Since the matrix A in this example is 2×2, its characteristic polynomial has degree two. One can show that the characteristic polynomial of an $n \times n$ matrix has degree n. The eigenvalues of the matrix are the roots of the characteristic polynomial. Different matrices may have different types of roots, so we try to classify these roots in the following definition.

Definition 6.4.10. *Given an $n \times n$ matrix A with real eigenvalues λ_i, where $i = 1, \cdots, k \leqslant n$, it is always possible to express the characteristic polynomial of A as*

$$p(\lambda) = (\lambda - \lambda_1)^{r_1} \cdots (\lambda - \lambda_k)^{r_k}.$$

*The number r_i is called the **algebraic multiplicity** of the eigenvalue λ_i. Furthermore, the **geometric multiplicity** of an eigenvalue λ_i, denoted as s_i, is the maximum number of eigenvectors of λ_i that form a linearly independent set.*

Example 6.4.14: Find the eigenvalues algebraic and geometric multiplicities of the matrix

$$A = \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix}$$

Solution: In order to find the algebraic multiplicity of the eigenvalues we need first to find the eigenvalues. We now that the characteristic polynomial of this matrix is given by

$$p(\lambda) = \begin{vmatrix} (1-\lambda) & 3 \\ 3 & (1-\lambda) \end{vmatrix} = (\lambda - 1)^2 - 9.$$

The roots of this polynomial are $\lambda_1 = 4$ and $\lambda_2 = -2$, so we know that $p(\lambda)$ can be rewritten in the following way,

$$p(\lambda) = (\lambda - 4)(\lambda + 2).$$

We conclude that the algebraic multiplicity of the eigenvalues are both one, that is,

$$\lambda_1 = 4, \quad r_1 = 1, \quad \text{and} \qquad \lambda_2 = -2, \quad r_2 = 1.$$

In order to find the geometric multiplicities of matrix eigenvalues we need first to find the matrix eigenvectors. This part of the work was already done in the Example 6.4.12 above and the result is

$$\lambda_1 = 4, \quad \boldsymbol{v}^{(1)} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \qquad \lambda_2 = -2, \quad \boldsymbol{v}^{(2)} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}.$$

From this expression we conclude that the geometric multiplicities for each eigenvalue are just one, that is,

$$\lambda_1 = 4, \quad s_1 = 1, \quad \text{and} \quad \lambda_2 = -2, \quad s_2 = 1.$$

◁

The following example shows that two matrices can have the same eigenvalues, and so the same algebraic multiplicities, but different eigenvectors with different geometric multiplicities.

Example 6.4.15: Find the eigenvalues and eigenvectors of the matrix $A = \begin{bmatrix} 3 & 0 & 1 \\ 0 & 3 & 2 \\ 0 & 0 & 1 \end{bmatrix}$.

Solution: We start finding the eigenvalues, the roots of the characteristic polynomial

$$p(\lambda) = \begin{vmatrix} (3-\lambda) & 0 & 1 \\ 0 & (3-\lambda) & 2 \\ 0 & 0 & (1-\lambda) \end{vmatrix} = -(\lambda-1)(\lambda-3)^2 \quad \Rightarrow \quad \begin{cases} \lambda_1 = 1, & r_1 = 1, \\ \lambda_2 = 3, & r_2 = 2. \end{cases}$$

We now compute the eigenvector associated with the eigenvalue $\lambda_1 = 1$, which is the solution of the linear system

$$(A - I)\boldsymbol{v}^{(1)} = \boldsymbol{0} \quad \Leftrightarrow \quad \begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1^{(1)} \\ v_2^{(1)} \\ v_3^{(1)} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

After the few Gauss elimination operation we obtain the following,

$$\begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 2 \\ 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & \frac{1}{2} \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \quad \Rightarrow \quad \begin{cases} v_1^{(1)} = -\dfrac{v_3^{(1)}}{2}, \\ v_2^{(1)} = -v_3^{(1)}, \\ v_3^{(1)} \quad \text{free}. \end{cases}$$

Therefore, choosing $v_3^{(1)} = 2$ we obtain that

$$\boldsymbol{v}^{(1)} = \begin{bmatrix} -1 \\ -2 \\ 2 \end{bmatrix}, \qquad \lambda_1 = 1, \qquad r_1 = 1, \qquad s_1 = 1.$$

In a similar way we now compute the eigenvectors for the eigenvalue $\lambda_2 = 3$, which are all solutions of the linear system

$$(A - 3I)\boldsymbol{v}^{(2)} = \boldsymbol{0} \quad \Leftrightarrow \quad \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 2 \\ 0 & 0 & -2 \end{bmatrix} \begin{bmatrix} v_1^{(2)} \\ v_2^{(2)} \\ v_3^{(2)} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

After the few Gauss elimination operation we obtain the following,

$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 2 \\ 0 & 0 & -2 \end{bmatrix} \to \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \Rightarrow \quad \begin{cases} v_1^{(2)} & \text{free}, \\ v_2^{(2)} & \text{free}, \\ v_3^{(2)} = 0. \end{cases}$$

Therefore, we obtain two linearly independent solutions, the first one $\boldsymbol{v}^{(2)}$ with the choice $v_1^{(2)} = 1$, $v_2^{(2)} = 0$, and the second one $\boldsymbol{w}^{(2)}$ with the choice $v_1^{(2)} = 0$, $v_2^{(2)} = 1$, that is

$$\boldsymbol{v}^{(2)} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \qquad \boldsymbol{w}^{(2)} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \qquad \lambda_2 = 3, \qquad r_2 = 2, \qquad s_2 = 2.$$

Summarizing, the matrix in this example has three linearly independent eigenvectors. $\triangleleft$

Example 6.4.16: Find the eigenvalues and eigenvectors of the matrix $A = \begin{bmatrix} 3 & 1 & 1 \\ 0 & 3 & 2 \\ 0 & 0 & 1 \end{bmatrix}$.

Solution: Notice that this matrix has only the coefficient a_{12} different from the previous example. Again, we start finding the eigenvalues, which are the roots of the characteristic polynomial

$$p(\lambda) = \begin{vmatrix} (3-\lambda) & 1 & 1 \\ 0 & (3-\lambda) & 2 \\ 0 & 0 & (1-\lambda) \end{vmatrix} = -(\lambda-1)(\lambda-3)^2 \quad \Rightarrow \quad \begin{cases} \lambda_1 = 1, & r_1 = 1, \\ \lambda_2 = 3, & r_2 = 2. \end{cases}$$

So this matrix has the same eigenvalues and algebraic multiplicities as the matrix in the previous example. We now compute the eigenvector associated with the eigenvalue $\lambda_1 = 1$, which is the solution of the linear system

$$(A - I)\boldsymbol{v}^{(1)} = \boldsymbol{0} \quad \Leftrightarrow \quad \begin{bmatrix} 2 & 1 & 1 \\ 0 & 2 & 2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1^{(1)} \\ v_2^{(1)} \\ v_3^{(1)} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

After the few Gauss elimination operation we obtain the following,

$$\begin{bmatrix} 2 & 1 & 1 \\ 0 & 2 & 2 \\ 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \quad \Rightarrow \quad \begin{cases} v_1^{(1)} = 0, \\ v_2^{(1)} = -v_3^{(1)}, \\ v_3^{(1)} \quad \text{free}. \end{cases}$$

Therefore, choosing $v_3^{(1)} = 1$ we obtain that

$$\boldsymbol{v}^{(1)} = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}, \qquad \lambda_1 = 1, \qquad r_1 = 1, \qquad s_1 = 1.$$

In a similar way we now compute the eigenvectors for the eigenvalue $\lambda_2 = 3$. However, in this case we obtain only one solution, as this calculation shows,

$$(A - 3I)\boldsymbol{v}^{(2)} = \boldsymbol{0} \quad \Leftrightarrow \quad \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 2 \\ 0 & 0 & -2 \end{bmatrix} \begin{bmatrix} v_1^{(2)} \\ v_2^{(2)} \\ v_3^{(2)} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

After the few Gauss elimination operation we obtain the following,

$$\begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 2 \\ 0 & 0 & -2 \end{bmatrix} \to \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \quad \Rightarrow \quad \begin{cases} v_1^{(2)} \quad \text{free}, \\ v_2^{(2)} = 0, \\ v_3^{(2)} = 0. \end{cases}$$

Therefore, we obtain only one linearly independent solution, which corresponds to the choice $v_1^{(2)} = 1$, that is,

$$\boldsymbol{v}^{(2)} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \qquad \lambda_2 = 3, \qquad r_2 = 2, \qquad s_2 = 1.$$

Summarizing, the matrix in this example has only two linearly independent eigenvectors, and in the case of the eigenvalue $\lambda_2 = 3$ we have the strict inequality

$$1 = s_2 < r_2 = 2.$$

$\triangleleft$