Chapter 3. Second Order Linear Equations

3.1. General Second Order Linear Equations. Superposition Property

Section Objective(s):

- Second Order ODEs in Science and Engineering
- Definitions and Examples.
- Solutions to the Initial Value Problem.
- Properties of Homogeneous Equations.
- The General Solution Theorem.

3.1.1. Definitions and Examples.

Definition 2.1.1. A ***second order*** differential equation on y is

_____,

where f is a given function. An initial value problem (IVP) for a second order ODE needs _____.

Usually, they are _____ - **initial position and initial velocity**.

Definition 2.1.2. A ***second order linear*** differential equation on y is

_____,

where a_1, a_0, b are given functions. The differential equation above:

(a) is ***homogeneous*** iff the source _____ for all $t \in \mathbb{R}$;

(b) has ***constant coefficients*** iff _____ are constants;

(c) has ***variable coefficients*** iff either _____ is not constant.

Remark: The homogeneous equations here _____ the Euler homogeneous equations.

In physical problems y' can represent _____ and y'' is then the _____.

Arguably the most important equation in dynamics is ***Newton's Second Law*** _____.

For a point particle of mass m moving in one space dimension under a force f it becomes,

_____.

EXAMPLE (SPRING-MASS SYSTEM, NO DAMPING): Consider mass hanging at the bottom of a spring. **Hooke's Law** states the force is proportional to the stretching distance y

_____.

Thus, the ODE modeling this motion is

_____. (A)

This is a _____.

EXAMPLE (CONSERVATION OF ENERGY): Multiply equation (A) by y' and integrate.

The **total energy** in a mechanical system is equal to the sum of _____
The kinetic energy for a body with mass m moving at a velocity v is _____. The potential energy stored in the spring with spring constant k and displacement y from its resting position is _____.

Energy in a spring-mass system with no damping is conserved!

EXAMPLE (SPRING-MASS SYSTEM WITH DAMPING): Now, if we **include damping**, the damping force is proportional to _____ and acts in the direction _____ so it is given by _____, where $b > 0$. Thus, the ODE modeling this motion is

_____. (B)

The total mechanical energy in a spring-mass system with damping _____.
Indeed, multiply equation (B) by y' and integrate.

EXAMPLE (LRC CIRCUIT): Consider an electrical network consisting of resistor, inductor and capacitor, connected in series with the voltage supply. ***Kirchhoff's Voltage Law*** states that the sum of voltage drops around any closed loop is zero.

Let $I(t)$ denote the _____, L - _____, R - _____, and C - _____

Kirchhoff's Law then implies

$$L\frac{dI}{dt} + RI + \frac{1}{C}\int I\,dt = \text{applied voltage.}$$

This is equivalent to the following second order ODE:

Remark: The equations for spring-mass system (with applied force) and LRC circuit are _____. We have the following correspondence:

(1)

(2)

(3)

EXAMPLE (SCHRÖDINGER EQUATION): Schrödinger equation in Quantum Mechanics, in one space dimension, stationary, is

_____,

where ___ is the probability density of finding a particle of mass ____ at the position ___ having energy ____ under a potential ____, where $\hbar$ is the Planck constant divided by 2π. ◁

3.1.2. **Solutions to the Initial Value Problem.**

Theorem 2.1.2. (IVP) If the a_1, a_0, b are continuous on (t_1, t_2) and $t_0 \in (t_1, t_2)$, then there is _____ solution of the initial value problem

_____.

Proof: Based on the Picard iteration. Two integrations, two initial conditions.

EXAMPLE 2.1.5: Find the domain of the solution to the initial value problem

$$(t-1)\,y'' - 3t\,y' + \frac{4(t-1)}{(t-3)}\,y = t(t-1), \qquad y(2) = 1, \qquad y'(2) = 0.$$

SOLUTION:

◁

3.1.3. **Properties of Homogeneous Equations.**

Remark: We introduce a new notation. We write

$$y'' + a_1(t)\, y' + a_0(t)\, y = b(t) \qquad \text{as} \qquad \underline{\qquad\qquad\qquad\qquad},$$

where

$$\underline{\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad}.$$

Here ___ is an operator, that is,

$$\underline{\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad}.$$

The operator above is a linear operator.

Definition 2.1.3. A ***linear operator*** is an operator L such that for every pair of functions y_1, y_2 and constants c_1, c_2 holds

$$\underline{\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad}.$$

EXAMPLE : Show that the operator $L(y) = y'' + a_1\, y' + a_0\, y$ is a linear operator.

SOLUTION:

◁

Is $L(y) = y'' + a_1 y + a_0 y - b$ a linear operator?

Remark:
Linearity of an operator and the superposition property are two sides of the same coin.

Theorem 2.1.5. (Superposition Property) If L is a linear operator and y_1, y_2 are solutions of the homogeneous equations

_____,

then for every constants c_1, c_2 holds

_____.

Remark: This result _____ for nonhomogeneous equations.

Proof of Theorem 2.1.5:

□

3.1.4. **The General Solution Theorem.**

Definition . The functions y_1, y_2 are ***linearly dependent*** iff

_____.

_____ the functions are ***linearly independent***.

Remark: Two functions y_1, y_2 are proportional iff there is a constant c such that

_____.

Theorem (General Solution). If y_1 and y_2 are linearly independent solutions of the variable coefficients equation

_____, $\qquad L(y) = y'' + a_1\, y' + a_0\, y,$

then _____ solution y of $L(y) = 0$ can be written as

_____.

Definition .

(a) The functions y_1 and y_2 are ***fundamental solutions*** of $L(y) = 0$ if and only if $L(y_1) = 0$, $L(y_2) = 0$ and y_1, y_2 are _____.

(b) The ***general solution*** of the homogeneous equation $L(y) = 0$ is a two-parameter family of functions

_____,

where y_1, y_2 are _____ of $L(y) = 0$.

EXAMPLE (FUNDAMENTAL SOLUTIONS): Show that $y_1 = e^t$ and $y_2 = e^{-2t}$ are fundamental solutions to the equation

$$y'' + y' - 2y = 0.$$

SOLUTION:

◁

Remark: The fundamental solutions to the equation above are not unique. For example, show that another set of fundamental solutions to the equation above is given by,

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EXAMPLE : Find the differential equation satisfied by the family of functions

$$y(t) = \frac{c_1}{t} + c_2\, t, \qquad c_1, c_2 \in \mathbb{R}.$$

SOLUTION:

◁