3.3. Nonhomogeneous Equations

All solutions of a linear *homogeneous* equation can be obtained from only two solutions that are linearly independent, called fundamental solutions. Every other solution is a linear combination of these two. This is the general solution formula for homogeneous equations, and it is the main result in § **??**, Theorem **??**. This result is not longer true for *nonhomogeneous* equations. The superposition property, Theorem **??**, which played an important part to get the general solution formula for homogeneous equations, is not true for nonhomogeneous equations.

We start this section proving a general solution formula for nonhomogeneous equations. We show that all the solutions of the nonhomogeneous equation are a translation by a fixed function of the solutions of the homogeneous equation. The fixed function is one solution—it doesn't matter which one—of the nonhomogenous equation, and it is called a particular solution of the nonhomogeneous equation.

Later in this section we show two different ways to compute the particular solution of a nonhomogeneous equation—the undetermined coefficients method and the variation of parameters method. In the former method we guess a particular solution from the expression of the source in the equation. The guess contains a few unknown constants, the undetermined coefficients, that must be determined by the equation. The undetermined method works for constant coefficients linear operators and simple source functions. The source functions and the associated guessed solutions are collected in a small table. This table is constructed by trial and error. In the latter method we have a formula to compute a particular solution in terms of the equation source, and fundamental solutions of the homogeneous equation. The variation of parameters method works with variable coefficients linear operators and general source functions. But the calculations to find the solution are usually not so simple as in the undetermined coefficients method.

3.3.1. The General Solution Formula.

The general solution formula for homogeneous equations, Theorem **??**, is no longer true for nonhomogeneous equations. But there is a general solution formula for nonhomogeneous equations. Such formula involves three functions, two of them are fundamental solutions of the homogeneous equation, and the third function is any solution of the nonhomogeneous equation. Every other solution of the nonhomogeneous equation can be obtained from these three functions.

Theorem 3.3.1 (General Solution)**.** *Every solution y of the nonhomogeneous equation*

$$L(y) = f, \tag{3.3.1}$$

with $L(y) = y'' + a_1\, y' + a_0\, y$, where a_1, a_0, and f are continuous functions, is given by

$$y = c_1\, y_1 + c_2\, y_2 + y_p,$$

where the functions y_1 and y_2 are fundamental solutions of the homogeneous equation, $L(y_1) = 0$, $L(y_2) = 0$, and y_p is any solution of the nonhomogeneous equation $L(y_p) = f$.

Before we proof Theorem 3.3.1 we state the following definition, which comes naturally from this Theorem.

Definition 3.3.2. *The **general solution** of the nonhomogeneous equation $L(y) = f$ is a two-parameter family of functions*

$$y_{\rm gen}(t) = c_1\, y_1(t) + c_2\, y_2(t) + y_p(t), \tag{3.3.2}$$

where the functions y_1 and y_2 are fundamental solutions of the homogeneous equation, $L(y_1) = 0$, $L(y_2) = 0$, and y_p is any solution of the nonhomogeneous equation $L(y_p) = f$.

Remark: The difference of any two solutions of the nonhomogeneous equation is actually a solution of the homogeneous equation. This is the key idea to prove Theorem 3.3.1. In other words, the solutions of the nonhomogeneous equation are a *translation by a fixed function*, y_p, of the solutions of the homogeneous equation.

Proof of Theorem 3.3.1: Let y be any solution of the nonhomogeneous equation $L(y) = f$. Recall that we already have one solution, y_p, of the nonhomogeneous equation, $L(y_p) = f$. We can now subtract the second equation from the first,

$$L(y) - L(y_p) = f - f = 0 \quad \Rightarrow \quad L(y - y_p) = 0.$$

The equation on the right is obtained from the linearity of the operator L. This last equation says that the difference of any two solutions of the nonhomogeneous equation is solution of the homogeneous equation. The general solution formula for homogeneous equations says that all solutions of the homogeneous equation can be written as linear combinations of a pair of fundamental solutions, y_1, y_2. So the exist constants c_1, c_2 such that

$$y - y_p = c_1\, y_1 + c_2\, y_2.$$

Since for every y solution of $L(y) = f$ we can find constants c_1, c_2 such that the equation above holds true, we have found a formula for all solutions of the nonhomogeneous equation. This establishes the Theorem. □

3.3.2. **The Undetermined Coefficients Method.** The general solution formula in (3.3.2) is the most useful if there is a way to find a particular solution y_p of the nonhomogeneous equation $L(y_p) = f$. We now present a method to find such particular solution, the undetermined coefficients method. This method works for *linear operators L with constant coefficients* and for *simple source functions f*. Here is a summary of the undetermined coefficients method:

(1) Find fundamental solutions y_1, y_2 of the homogeneous equation $L(y) = 0$.
(2) Given the source functions f, guess the solutions y_p following the Table 1 below.
(3) If the function y_p given by the table satisfies $L(y_p) = 0$, then change the guess to $ty_{p.}$. If ty_p satisfies $L(ty_p) = 0$ as well, then change the guess to $t^2 y_p$.
(4) Find the undetermined constants k in the function y_p using the equation $L(y) = f$, where y is y_p, or ty_p or $t^2 y_p$.

$f(t)$ (Source) (K, m, a, b, given.)	$y_p(t)$ (Guess) (k not given.)
Ke^{at}	ke^{at}
$K_m t^m + \cdots + K_0$	$k_m t^m + \cdots + k_0$
$K_1 \cos(bt) + K_2 \sin(bt)$	$k_1 \cos(bt) + k_2 \sin(bt)$
$(K_m t^m + \cdots + K_0)\, e^{at}$	$(k_m t^m + \cdots + k_0)\, e^{at}$
$\big(K_1 \cos(bt) + K_2 \sin(bt)\big)\, e^{at}$	$\big(k_1 \cos(bt) + k_2 \sin(bt)\big)\, e^{at}$
$(K_m t^m + \cdots + K_0)\big(\tilde{K}_1 \cos(bt) + \tilde{K}_2 \sin(bt)\big)$	$(k_m t^m + \cdots + k_0)\big(\tilde{k}_1 \cos(bt) + \tilde{k}_2 \sin(bt)\big)$

TABLE 1. List of sources f and solutions y_p to the equation $L(y_p) = f$.

This is the undetermined coefficients method. It is a set of simple rules to find a particular solution y_p of an nonhomogeneous equation $L(y_p) = f$ in the case that the source function f is one of the entries in the Table 1. There are a few formulas in particular cases and a few generalizations of the whole method. We discuss them after a few examples.

Example 3.3.1 (First Guess Right): Find all solutions to the nonhomogeneous equation

$$y'' - 3y' - 4y = 3\,e^{2t}.$$

Solution: From the problem we get $L(y) = y'' - 3y' - 4y$ and $f(t) = 3\,e^{2t}$.

(1): Find fundamental solutions y_+, y_- to the homogeneous equation $L(y) = 0$. Since the homogeneous equation has constant coefficients we find the characteristic equation

$$r^2 - 3r - 4 = 0 \quad \Rightarrow \quad r_+ = 4, \quad r_- = -1, \quad \Rightarrow \quad y_+(t) = e^{4t}, \quad y_- = (t) = e^{-t}.$$

(2): The table says: For $f(t) = 3e^{2t}$ guess $y_p(t) = k\,e^{2t}$. The constant k is the undetermined coefficient we must find.

(3): Since $y_p(t) = k\,e^{2t}$ is not solution of the homogeneous equation, we do not need to modify our guess. (Recall: $L(y) = 0$ iff exist constants c_+, c_- such that $y(t) = c_+\,e^{4t} + c_-\,e^{-t}$.)

(4): Introduce y_p into $L(y_p) = f$ and find k. So we do that,

$$(2^2 - 6 - 4)ke^{2t} = 3\,e^{2t} \quad \Rightarrow \quad -6k = 3 \quad \Rightarrow \quad k = -\frac{1}{2}.$$

We guessed that y_p must be proportional to the exponential e^{2t} in order to cancel out the exponentials in the equation above. We have obtained that

$$y_p(t) = -\frac{1}{2}\,e^{2t}.$$

The undetermined coefficients method gives us a way to compute a particular solution y_p of the nonhomogeneous equation. We now use the general solution theorem, Theorem 3.3.1, to write the general solution of the nonhomogeneous equation,

$$y_{\text{gen}}(t) = c_+\,e^{4t} + c_-\,e^{-t} - \frac{1}{2}\,e^{2t}.$$

$\lhd$

Remark: The step (4) in Example 3.3.1 is a particular case of the following statement.

Theorem 3.3.3. *Consider the equation $L(y) = f$, where $L(y) = y'' + a_1\,y' + a_0\,y$ has constant coefficients and p is its characteristic polynomial. If the source function is $f(t) = K\,e^{at}$, with $p(a) \neq 0$, then a particular solution of the nonhomogeneous equation is*

$$y_p(t) = \frac{K}{p(a)}\,e^{at}.$$

Proof of Theorem 3.3.3: Since the linear operator L has constant coefficients, let us write L and its associated characteristic polynomial p as follows,

$$L(y) = y'' + a_1 y' + a_0 y, \qquad p(r) = r^2 + a_1 r + a_0.$$

Since the source function is $f(t) = K\,e^{at}$, the Table 1 says that a good guess for a particular soution of the nonhomogneous equation is $y_p(t) = k\,e^{at}$. Our hypothesis is that this guess is not solution of the homogenoeus equation, since

$$L(y_p) = (a^2 + a_1 a + a_0)\,k\,e^{at} = p(a)\,k\,e^{at}, \quad \text{and} \quad p(a) \neq 0.$$

We then compute the constant k using the equation $L(y_p) = f$,

$$(a^2 + a_1 a + a_0)\, k\, e^{at} = K\, e^{at} \quad \Rightarrow \quad p(a)\, k\, e^{at} = K\, e^{at} \quad \Rightarrow \quad k = \frac{K}{p(a)}.$$

We get the particular solution $y_p(t) = \dfrac{K}{p(a)}\, e^{at}$. This establishes the Theorem. $\square$

Remark: As we said, the step (4) in Example 3.3.1 is a particular case of Theorem 3.3.3,

$$y_p(t) = \frac{3}{p(2)}\, e^{2t} = \frac{3}{(2^2 - 6 - 4)}\, e^{2t} = \frac{3}{-6}\, e^{2t} \quad \Rightarrow \quad y_p(t) = -\frac{1}{2}\, e^{2t}.$$

In the following example our first guess for a particular solution y_p happens to be a solution of the homogenous equation.

Example 3.3.2 (First Guess Wrong): Find all solutions to the nonhomogeneous equation

$$y'' - 3y' - 4y = 3\, e^{4t}.$$

Solution: If we write the equation as $L(y) = f$, with $f(t) = 3\, e^{4t}$, then the operator L is the same as in Example 3.3.1. So the solutions of the homogeneous equation $L(y) = 0$, are the same as in that example,

$$y_+(t) = e^{4t}, \qquad y_-(t) = e^{-t}.$$

The source function is $f(t) = 3\, e^{4t}$, so the Table 1 says that we need to guess $y_p(t) = k\, e^{4t}$. However, this function y_p is solution of the homogeneous equation, because

$$y_p = k\, y_+ \quad \Rightarrow \quad L(y_p) = 0.$$

We have to change our guess, as indicated in the undetermined coefficients method, step (3)

$$y_p(t) = kt\, e^{4t}.$$

This new guess is not solution of the homogeneous equation. So we proceed to compute the constant k. We introduce the guess into $L(y_p) = f$,

$$y_p' = (1 + 4t)\, k\, e^{4t}, \qquad y_p'' = (8 + 16t)\, k\, e^{4t} \quad \Rightarrow \quad \big[8 - 3 + (16 - 12 - 4)t\big]\, k\, e^{4t} = 3\, e^{4t},$$

therefore, we get that

$$5k = 3 \quad \Rightarrow \quad k = \frac{3}{5} \quad \Rightarrow \quad y_p(t) = \frac{3}{5}\, t\, e^{4t}.$$

The general solution theorem for nonhomogeneous equations says that

$$y_{\text{gen}}(t) = c_+\, e^{4t} + c_-\, e^{-t} + \frac{3}{5}\, t\, e^{4t}.$$

$\lhd$

In the following example the equation source is a trigonometric function.

Example 3.3.3 (First Guess Right): Find all the solutions to the nonhomogeneous equation

$$y'' - 3y' - 4y = 2\sin(t).$$

Solution: If we write the equation as $L(y) = f$, with $f(t) = 2\sin(t)$, then the operator L is the same as in Example 3.3.1. So the solutions of the homogeneous equation $L(y) = 0$, are the same as in that example,

$$y_+(t) = e^{4t}, \qquad y_-(t) = e^{-t}.$$

Since the source function is $f(t) = 2\sin(t)$, the Table 1 says that we need to choose the function $y_p(t) = k_1 \cos(t) + k_2 \sin(t)$. This function y_p is not solution to the homogeneous equation. So we look for the constants k_1, k_2 using the differential equation,

$$y_p' = -k_1 \sin(t) + k_2 \cos(t), \qquad y_p'' = -k_1 \cos(t) - k_2 \sin(t),$$

and then we obtain

$$[-k_1 \cos(t) - k_2 \sin(t)] - 3[-k_1 \sin(t) + k_2 \cos(t)] - 4[k_1 \cos(t) + k_2 \sin(t)] = 2\sin(t).$$

Reordering terms in the expression above we get

$$(-5k_1 - 3k_2) \cos(t) + (3k_1 - 5k_2) \sin(t) = 2\sin(t).$$

The last equation must hold for all $t \in \mathbb{R}$. In particular, it must hold for $t = \pi/2$ and for $t = 0$. At these two points we obtain, respectively,

$$\left.\begin{aligned} 3k_1 - 5k_2 &= 2, \\ -5k_1 - 3k_2 &= 0, \end{aligned}\right\} \quad \Rightarrow \quad \begin{cases} k_1 = \dfrac{3}{17}, \\ k_2 = -\dfrac{5}{17}. \end{cases}$$

So the particular solution to the nonhomogeneous equation is given by

$$y_p(t) = \frac{1}{17} \big[3\cos(t) - 5\sin(t) \big].$$

The general solution theorem for nonhomogeneous equations implies

$$y_{\rm gen}(t) = c_+ \, e^{4t} + c_- \, e^{-t} + \frac{1}{17} \big[3\cos(t) - 5\sin(t) \big].$$

$\triangleleft$

The next example collects a few nonhomogeneous equations and the guessed function y_p.

Example 3.3.4: We provide few more examples of nonhomogeneous equations and the appropriate guesses for the particular solutions.

(a) For $y'' - 3y' - 4y = 3e^{2t} \sin(t)$, guess, $y_p(t) = \big[k_1 \cos(t) + k_2 \sin(t) \big] \, e^{2t}$.

(b) For $y'' - 3y' - 4y = 2t^2 \, e^{3t}$, guess, $y_p(t) = \big(k_2 t^2 + k_1 t + k_0 \big) \, e^{3t}$.

(c) For $y'' - 3y' - 4y = 2t^2 \, e^{4t}$, guess, $y_p(t) = \big(k_2 t^2 + k_1 t + k_0 \big) \, t \, e^{4t}$.

(d) For $y'' - 3y' - 4y = 3t \, \sin(t)$, guess, $y_p(t) = (k_1 t + k_0) \big[\tilde{k}_1 \cos(t) + \tilde{k}_2 \sin(t) \big]$.

$\triangleleft$

Remark: Suppose that the source function f does not appear in Table 1, but f can be written as $f = f_1 + f_2$, with f_1 and f_2 in the table. In such case look for a particular solution $y_p = y_{p_1} + y_{p_2}$, where $L(y_{p_1}) = f_1$ and $L(y_{p_2}) = f_2$. Since the operator L is linear,

$$L(y_p) = L(y_{p_1} + y_{p_2}) = L(y_{p_1}) + L(y_{p_2}) = f_1 + f_2 = f \quad \Rightarrow \quad L(y_p) = f.$$

Example 3.3.5: Find all solutions to the nonhomogeneous equation

$$y'' - 3y' - 4y = 3\, e^{2t} + 2\sin(t).$$

Solution: If we write the equation as $L(y) = f$, with $f(t) = 2\sin(t)$, then the operator L is the same as in Example 3.3.1 and 3.3.3. So the solutions of the homogeneous equation $L(y) = 0$, are the same as in these examples,

$$y_+(t) = e^{4t}, \qquad y_-(t) = e^{-t}.$$

The source function $f(t) = 3\,e^{2t} + 2\sin(t)$ does not appear in Table 1, but each term does, $f_1(t) = 3\,e^{2t}$ and $f_2(t) = 2\sin(t)$. So we look for a particular solution of the form

$$y_p = y_{p_1} + y_{p_2}, \quad \text{where} \quad L(y_{p_1}) = 3\,e^{2t}, \quad L(y_{p_2}) = 2\sin(t).$$

We have chosen this example because we have solved each one of these equations before, in Example 3.3.1 and 3.3.3. We found the solutions

$$y_{p_1}(t) = -\frac{1}{2}\,e^{2t}, \qquad y_{p_2}(t) = \frac{1}{17}\,\big(3\cos(t) - 5\sin(t)\big).$$

Therefore, the particular solution for the equation in this example is

$$y_p(t) = -\frac{1}{2}\,e^{2t} + \frac{1}{17}\,\big(3\cos(t) - 5\sin(t)\big).$$

Using the general solution theorem for nonhomogeneous equations we obtain

$$y_{\text{gen}}(t) = c_+\,e^{4t} + c_-\,e^{-t} - \frac{1}{2}\,e^{2t} + \frac{1}{17}\,\big(3\cos(t) - 5\sin(t)\big).$$

$\triangleleft$

3.3.3. **Exercises.**

3.3.1.- ***(HW due in recitation on 09/28)***

(a) Find the general solution to the homogeneous (unforced) equation corresponding to

$$\frac{d^2y}{dt^2} + 2y = \cos(\omega_0 t).$$

(b) Assume $\omega_0 \neq \sqrt{2}$. Find a particular solution, $y_p(t)$ to $\frac{d^2y}{dt^2} + 2y = \cos(\omega_0 t)$.

(c) Use the above to find the general solution to the forced equation $\frac{d^2y}{dt^2} + 2y = \cos(\omega_0 t)$.

(d) Discuss what will happen to the solution if we choose forcing with frequency ω_0 very close to the natural frequency, i.e. in this case, $\omega_0 \approx \sqrt{2}$. In particular, how does the amplitude of the oscillations depend on the forcing frequency?

3.3.2.- ***(HW due in recitation on 09/28)*** Consider the equation

$$m\frac{d^2y}{dt^2} + b\frac{dy}{dt} + ky = f(t). \tag{3.3.3}$$

Here m is the mass, b is the damping coefficient, and k is the spring constant and $m > 0$, $b > 0$, $k > 0$.

(a) Write down the characteristic polynomial corresponding to the **homogeneous** part of (3.3.3). Determine its roots. Does the real part of the roots have a particular sign (is it positive or negative or we cannot determine)?

(b) What can we conclude about the long-term behavior of the solution of an unforced harmonic oscillator (i.e., $f(t) = 0$) with damping, $y_h(t)$? Does this make sense from a physical point of view?

(c) Does the long-term behavior of $y_h(t)$ depend on the initial conditions? Explain.