MTH 133 MAKEUP FINAL EXAM DECEMBER 14, 2004

NAME:______________________ STUDENT NUMBER: __________

SECTION NUMBER: _____ RECITATION INSTRUCTOR: __________

SCORE: _____________

##

1. Compute the derivative of each of the following functions. You do not need to simplify your answer.

product / chain rules

(a) (9 pts) $y = x^2 \tan^{-1}(2x^3)$ $\quad (2x)(\tan^{-1}(2x^3)) + (x^2)\left(\frac{1}{1+(2x^3)^2}\right)\cdot(6x^2)$

chain rule

(b) (9 pts) $y = \{\ln(e^x+1)\}^3$ $\quad 3\cdot(\ln(e^x+1))^2 \cdot \frac{1}{e^x+1}\cdot e^x$

(c) (9 pts) $y = (\cos x)^{\sin x}$ logrithmic differentiation

$$(\ln(y))' = (\ln((\cos x)^{\sin x}))' = (\sin x \cdot \ln(\cos x))'$$

$$= \cos x \cdot \ln(\cos x) + \sin x \cdot \frac{1}{\cos x}\cdot(-\sin x)$$

$$y' = y\cdot(\ln(y))' = (\cos x)^{\sin x}\left[\cos x \cdot \ln(\cos x) + \sin x \cdot \frac{1}{\cos x}\cdot(-\sin x)\right]$$

2. Evaluate the following integrals. Please show your work and circle your answer.

(a) (9 pts) $\int \frac{1}{x^2+2x}\,dx$

partial fractions $\frac{1}{x^2+2x} = \frac{1}{x(x+2)} = \frac{A}{x} + \frac{B}{x+2}$

$\Rightarrow A = \frac{1}{2},\ B = -\frac{1}{2}$

$$\int \frac{1}{x^2+2x}\,dx = \int \frac{1/2}{x} + \frac{-1/2}{x+2}\,dx = \frac{1}{2}\ln|x| - \frac{1}{2}\ln|x+2| + C$$

(b) (9 pts) $\int \frac{dx}{(r^2-x^2)^{3/2}}$; $r>0$, $-r<x<r$

trig sub $x = r\sin\theta$ $dx = r\cos\theta\, d\theta$

$$\int \frac{dx}{(r^2-x^2)^{3/2}} = \int \frac{r\cos\theta\, d\theta}{(r^2-(r\sin\theta)^2)^{3/2}}$$

$$= \int \frac{r\cos\theta\, d\theta}{(r^2(1-\sin^2\theta))^{3/2}}$$

$$= \int \frac{r\cos\theta\, d\theta}{r^3\cos^3\theta}$$

$$= \frac{1}{r^2}\int \frac{1}{\cos^2\theta}\, d\theta = \frac{1}{r^2}\int \sec^2\theta\, d\theta = \frac{1}{r^2}\tan\theta + C$$

ans in terms of x

$\frac{1}{r^2}\tan\theta + C$

$x = r\sin\theta$

$\Rightarrow \theta = \sin^{-1}\left(\frac{x}{r}\right)$

$\frac{1}{r^2}\tan\left(\sin^{-1}\left(\frac{x}{r}\right)\right) + C$

(c) (9 pts) $\int xe^{-2x}\, dx$

integration by parts

$u = x$ $dv = e^{-2x}dx$

$du = dx$ $v = -\frac{1}{2}e^{-2x}$

$$\int xe^{-2x}dx = (x)\left(-\frac{1}{2}e^{-2x}\right) - \int\left(-\frac{1}{2}e^{-2x}\right)dx = -\frac{xe^{-2x}}{2} - \frac{e^{-2x}}{4} + C$$

(d) (9 pts) $\int_1^\infty x^{-\frac{4}{3}} dx$

improper integral

$$= \lim_{b\to\infty}\int_1^b x^{-4/3}dx = \lim_{b\to\infty}\left[\frac{x^{-1/3}}{-1/3}\right]_1^b$$

$$= \lim_{b\to\infty}\left(\frac{b^{-1/3}}{-1/3} - \frac{1}{-1/3}\right)$$

$$= 3$$

3. Evaluate the following limits. Please show your work.

L'Hop $\frac{\infty}{\infty}$

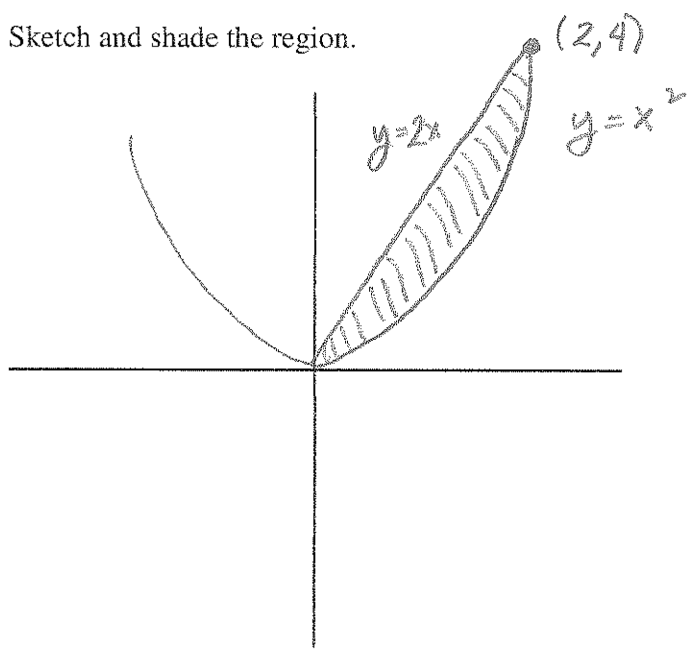
(a) (8 pts) $\lim_{x \to 0^+} x^2 \ln x$ $= \lim_{x \to 0^+} \frac{\ln(x)}{1/x^2} = \lim_{x \to 0^+} \frac{1/x}{-2/x^3} = \lim_{x \to 0^+} -\frac{x^2}{2} = 0$

(b) (10 pts) $\lim_{x \to +\infty} \left(\frac{x}{x+1}\right)^x$ $= \lim_{x \to \infty} \left(\frac{x+1}{x}\right)^{-x} = \lim_{x \to \infty} \left(1+\frac{1}{x}\right)^{-x}$

$= \lim_{x \to \infty} \frac{1}{\left(1+\frac{1}{x}\right)^x} = \frac{1}{e}$

4. Consider the finite region in the first quadrant bounded by the curves $y = x^2$ and $y = 2x$.

(a) (5 pts) Sketch and shade the region.



Set up completely, but <u>do not evaluate</u>, an integral for the following quantities:

(b) (5 pts) The area of the region in (a).

$$\int_0^2 (2x - x^2)\,dx$$

(c) (5 pts) The volume of the solid obtained by rotating the region in (a) around the x-axis.

washer method $R = 2x$ $r = x^2$

$$\int_0^2 \pi(R^2 - r^2)\,dx = \int_0^2 \pi\left((2x)^2 - (x^2)^2\right)dx$$

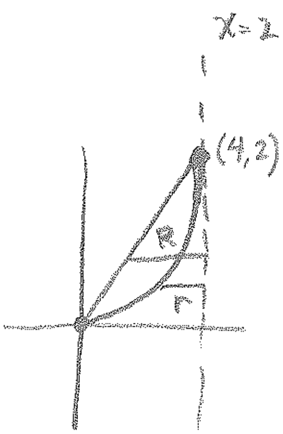
(d) (5 pts) The perimeter of the region in (a).

length of a curve $\int \sqrt{1+(f')^2}\,dx$

length of line length of parabola

$$\int_0^2 \sqrt{1+((2x)')^2}\,dx + \int_0^2 \sqrt{1+((x^2)')^2}\,dx$$

$$= \int_0^2 \sqrt{1+2^2}\,dx + \int_0^2 \sqrt{1+(2x)^2}\,dx$$



(e) (5 pts) The volume of the solid obtained by rotating the region in (a) around the line x=2.

washer method $R = 2 - \frac{y}{2}$ $r = 2 - \sqrt{y}$

$$\int_0^2 \pi\left(\left(2 - \frac{y}{2}\right)^2 - \left(2 - \sqrt{y}\right)^2\right)dy$$

5. Determine whether each of the series is convergent or divergent. Show your work and name the test(s) you are using.

(a) (8 pts) $\sum_{n=1}^{\infty} \frac{n^2+2n}{100n^2+5n-3}$

test for divergence

$$\lim_{n\to\infty} \frac{n^2+2n}{100n^2+5n-3} = \frac{1}{100} \neq 0$$

So $\sum_{n=1}^{\infty} \frac{n^2+2n}{100n^2+5n-3}$ diverges

(b) (8 pts) $\sum_{n=0}^{\infty} \frac{\sqrt{n}+2}{3n^2-n+2}$

limit comparison to $\sum \frac{1}{n^{3/2}}$ convergent p-series

$$\lim_{n\to\infty} \frac{1/n^{3/2}}{\sqrt{n}+2/3n^2-n+2} = \lim_{n\to\infty} \frac{3n^{1/2}-n^{-1/2}+2n^{-3/2}}{n^{1/2}+2} = 3$$

non-zero
not infinity

$\Rightarrow$ Converges

(c) (8 pts) $\sum_{k=1}^{\infty} \frac{3k^2}{2^k}$

ratio test

$$\rho = \lim_{n\to\infty} \frac{a_{k+1}}{a_k} = \lim_{n\to\infty} \frac{3(k+1)^2/2^{k+1}}{3k^2/2^k} = \lim_{n\to\infty} \frac{(k+1)^2}{2k^2} = \frac{1}{2}$$

$\rho = \frac{1}{2} < 1$ Converges

(d) (8 pts) $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^3}$

integral test

$u = \ln(n)$
$du = \frac{1}{n}\,dn$

$$\int_2^{\infty} \frac{1}{n(\ln(n))^3}\,dn = \int_{n=2}^{\infty} \frac{1}{u^3}\,du = \left[\frac{-1}{2u^2}\right]_{n=2}^{\infty}$$

$$= \left[\frac{-1}{2(\ln(n))^2}\right]_{n=2}^{\infty} = -0 - \frac{-1}{2(\ln(2))^2} < \infty$$

$\Rightarrow$ converges

6. (12 pts) Determine the open interval of convergence of the series $\sum_{k=0}^{\infty} \frac{(x+2)^k}{(k+1)3^k}$.

ratio test for absolute convergence

$$\rho = \lim_{n \to \infty} \left|\frac{a_{k+1}}{a_k}\right| = \lim_{n \to \infty} \left|\frac{(x+2)^{k+1}/((k+1)+1)\cdot 3^{k+1}}{(x+2)^k/(k+1)\cdot 3^k}\right|$$

$$= \lim_{n \to \infty} \left|\frac{(x+2)(k+1)}{3\cdot(k+2)}\right| = \frac{|x+2|}{3}$$

converges $\rho < 1 \Rightarrow \frac{|x+2|}{3} < 1 \Rightarrow -3 < x+2 < 3 \Rightarrow -5 < x < 1$

7.(10 pts) Find the Maclaurin series expansion for $\frac{x}{1-x^2}$. Determine the open interval of convergence of the series.

Know

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$$

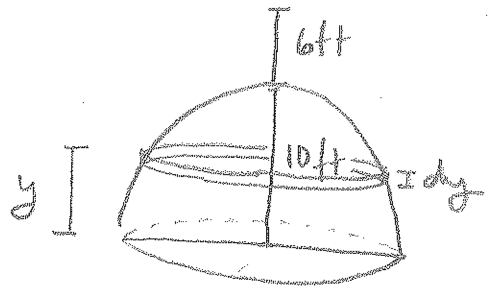
Sub in x^2 for x

$$\frac{1}{1-x^2} = \sum_{n=0}^{\infty} (x^2)^n$$

multiply by x

$$\frac{x}{1-x^2} = x \cdot \sum_{n=0}^{\infty} (x^2)^n = \sum_{n=0}^{\infty} x(x^{2n}) = \sum_{n=0}^{\infty} x^{2n+1}$$

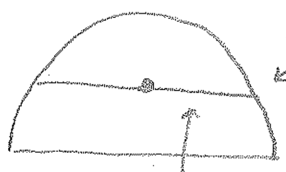
8. (15 pts) A hemispherical tank with radius 10 feet (opening down, like a dome) is filled with water weighing 62.5 lbs/ft^3. Find the amount of work required to pump all of the water out to a height of 6 feet above the top of the tank.



$$\text{Work} = \int_0^{10} dW = \int_0^{10} \underbrace{62.5}_{\text{density}} \underbrace{(16-y)}_{\text{dist}} \underbrace{A(y)\,dy}_{\text{volume}}$$

$A(y) = \pi \cdot (\text{radius})^2$

Cross section



radius $= \sqrt{10^2-y^2}$

$A(y) = \pi(10^2-y^2)$

$$\text{Work} = \int_0^{10} 62.5(16-y)(\pi(10^2-y^2))\,dy$$

$$= 62.5\pi \int_0^{10} 1600 - 16y^2 - 100y + y^3\,dy$$

$$= 62.5\pi \left[1600y - \frac{16y^3}{3} - \frac{100y^2}{2} + \frac{y^4}{4} \right]_0^{10}$$

$$= 62.5\pi \left[1600(10) - \frac{16(10)^3}{3} - \frac{100(10)^2}{2} + \frac{(10)^4}{4} \right]$$

9. (10 pts) Solve the following differential equations.

$$\frac{dy}{dx} = y + xy \quad , \quad y(0) = 1$$

$$\frac{dy}{dx} = y(1+x)$$

$$\frac{dy}{y} = (1+x)\,dx$$

$$\int \frac{dy}{y} = \int (1+x)\,dx$$

$$\ln|y| = x + \frac{x^2}{2} + C$$

initial $y(0)=1$

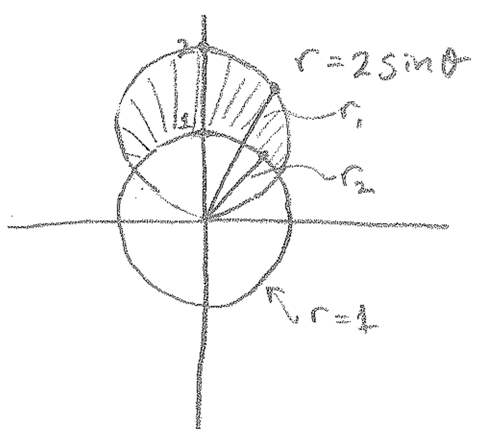
$$\ln|1| = (0) + \frac{(0)^2}{2} + C$$

$$C = 0$$

Solution

$$\ln|y| = x + \frac{x^2}{2}$$

10. (15 pts) On the same polar coordinate system, sketch the graphs of the equations $r = 2\sin\theta$ and $r = 1$, carefully locating all intersection points. Find the area inside the first curve and outside the second curve.



intersection points

$r = 2\sin\theta$, $r = 1$

$$1 = 2\sin\theta$$

$$\frac{1}{2} = \sin\theta$$

$$\Rightarrow \theta = \frac{\pi}{6} \quad \text{or} \quad \theta = \frac{5\pi}{6}$$

polar area

$$\int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} \frac{r_1^2}{2}\,d\theta - \int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} \frac{r_2^2}{2}\,d\theta$$

$$\int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} 2\sin^2\theta\,d\theta - \int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} \frac{1}{2}\,d\theta = \int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} 2\left(\frac{1-\cos 2\theta}{2}\right)d\theta - \int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} \frac{1}{2}\,d\theta$$

$$= \left[\theta - \frac{\sin(2\theta)}{2}\right]_{\pi}^{5\pi/6} - \left[\frac{\theta}{2}\right]_{\pi}^{5\pi/6} = \frac{\pi}{\;} + \sqrt{3}$$