MTH 320 Section 004 Midterm 2 Sample

Instructions: You have 120 minutes to complete the exam. There are eight problems, worth a total of 80 points. You may not use any books or notes. Partial credit will be given for progress toward correct proofs.

Write your solutions in the space below the questions. If you need more space use the back of the page. Do not forget to write your name in the space below.

You may use any result from the book, lecture notes, or homework without proof, unless the exam problem is to prove a theorem or redo a homework problem.

You may use the following facts without proof: $\sin x$, $\cos x$, e^x are all differentiable on $\mathbb{R}$, $\log_e x$ and x^p, $p \in \mathbb{R}$, are differentiable on $(0, \infty)$, and the derivatives are $\cos x$, $-\sin x$, e^x, $\frac{1}{x}$, and px^{p-1}, respectively.

This is a sample exam. The actual exam problems may not be exactly similar to the sample exam problems, and may be harder.

NAME: ____

1. (a) [4pts] State Mean Value Theorem.

   (b) [6pts] Suppose that f is differentiable on $\mathbb{R}$ and $f'(x) > 0$ on $\mathbb{R}$. Prove that f is strictly increasing.

2. Let $f : \mathbb{R} \to \mathbb{R}$.

   (a) (4 pts) What does it mean for f to be differentiable at a?

   (b) (6 pts) Let $f(x) = x\sin(\frac{1}{x})$ when $x \neq 0$ and $f(0) = 0$. Is f differentiable at $x = 0$? Justify your answer.

3. (a) [5 pts] Prove that the exact interval of convergence of $\sum_{n=1}^{\infty} \frac{x^n}{n}$ in $[-1, 1)$.

   (b) [5 pts] What function does the power series above represent on $(-1, 1)$? Justify your answer.

4. (a) [4pts] Define $\liminf s_n$

   (b) [6pts] Prove that if $\limsup s_n = \liminf s_n = s \in \mathbb{R}$, then (s_n) converges to s.

5. (a) [4 pts.] State Weierstrass M-test.

(b) [6 pts.] Prove that the series of functions $\sum_{n=1}^{\infty} \frac{\sin(nx)}{n^3}$ converges to a continuous function on all of $\mathbb{R}$, being careful to justify all of your steps.

6. Suppose (s_n) is an increasing sequence of real numbers. Prove that $\lim_{n\to\infty} s_n = \sup\{s_n : n \in \mathbb{N}\}$. You need to consider two cases: (i) $\sup\{s_n : n \in \mathbb{N}\} \in \mathbb{R}$; and (ii) $\sup\{s_n : n \in \mathbb{N}\} = +\infty$.

7. (a) [4 pts] State L'Hospital's rule. Be sure to include all conditions.

   (b) [6 pts] Find the following limits

   - $\lim_{y\to\infty}(1+\frac{2}{y})^y$
   - $\lim_{x\to 0}\frac{\cos x-1}{e^x-1-x}$

8. (a) [4 pts] For $a, L \in \mathbb{R}$, define the expressions $\lim_{x\to a^-} f(x) = L$, $\lim_{x\to a^+} f(x) = L$, and $\lim_{x\to a} f(x) = L$.

   (b) [6 pts] Prove that if $\lim_{x\to a^-} f(x) = \lim_{x\to a^+} f(x) = L$, then $\lim_{x\to a} f(x) = L$.

This page is for scratch work. Feel free to tear it off. Do not write anything you want graded on this page unless you indicate very clearly that this is the case on the page of the corresponding problem.