Homework 11 Solutions

25.3 Let $f_n(x) = \frac{n+\cos x}{2n+\sin^2 x}$ for all real numbers x.

(a) Show f_n converges uniformly on $\mathbb{R}$. Hint: First decide what the limit function is; then show (f_n) converges uniformly to it. Use $|\cos x|, |\sin x| \leq 1$.

Proof. We write $f_n(x) = \frac{1+\frac{1}{n}\cos x}{2+\frac{1}{n}\sin^2 x}$. Since $\frac{1}{n}\cos x \to 0$ and $\frac{1}{n}\sin^2 x \to 0$ as $n \to \infty$, we have $f_n(x) \to \frac{1}{2}$ pointwise on $\mathbb{R}$. We now show that $f_n(x) \to \frac{1}{2}$ uniformly on $\mathbb{R}$. We calculate for $x \in \mathbb{R}$,

$$|f_n(x) - \frac{1}{2}| = \Big|\frac{n+\cos x}{2n+\sin^2 x} - \frac{1}{2}\Big| = \Big|\frac{2\cos x - \sin^2 x}{2(2n+\sin^2 x)}\Big| = \frac{|2\cos x - \sin^2 x|}{2|2n+\sin^2 x|} \leq \frac{3}{2(2n-1)},$$

where in the last inequality we used the triangle inequality: $|2\cos x - \sin^2 x| \leq |2\cos x| + |\sin^2 x| \leq 2+1 = 3$ and $|2n+\sin^2 x| \geq 2n - |\sin^2 x| \geq 2n-1 > 0$. Thus,

$$\sup\{|f_n(x) - \frac{1}{2}| : x \in \mathbb{R}\} \leq \frac{3}{2(2n-1)}, \quad n \in \mathbb{N}.$$

Since $\frac{3}{2(2n-1)} \to 0$, we get $f_n \to \frac{1}{2}$ uniformly on $\mathbb{R}$. □

25.6 (a) Show that if $\sum |a_n| < \infty$, then $\sum a_k x^k$ converges uniformly on $[-1,1]$ to a continuous function.

Proof. Since $|a_k x^k| \leq |a_k|$ for $x \in [-1,1]$ and $\sum |a_n| < \infty$, by Weierstrass M-test, $\sum a_k x^k$ converges uniformly on $[-1,1]$. Since $a_k x^k$ is continuous for each k, by Theorem 24.3 the limit of $\sum a_k x^k$ is continuous on $[-1,1]$. □

25.7 Show $\sum_{n=1}^{\infty} \frac{1}{n^2}\cos(nx)$ converges uniformly on $\mathbb{R}$.

Proof. Since $|\frac{1}{n^2}\cos(nx)| \leq \frac{1}{n^2}$ and $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges, by Weierstrass M-test, $\sum_{n=1}^{\infty} \frac{1}{n^2}\cos(nx)$ converges uniformly on $\mathbb{R}$. □

25.10 (a) Show $\sum \frac{x^n}{1+x^n}$ converges for $x \in [0,1)$.

(b) Show that the series converges uniformly on $[0,a]$ for each a, $0 < a < 1$.

(c) Does the series converge uniformly on $[0,1)$? Explain. Hint: Compare it with $\sum \frac{x^n}{2}$.

Proof. (a) For each $x \in [0,1)$, since $|\frac{x^n}{1+x^n}| = \frac{x^n}{1+x^n} \leq x^n$ and $\sum x^n$ converges, by Comparison test, $\sum \frac{x^n}{1+x^n}$ converges.

(b) For each $a \in (0,1)$, we have $|\frac{x^n}{1+x^n}| \leq x^n \leq a^n$ for $x \in [0,a]$. Since $\sum a^n$ converges, by Weierstrass M-test, $\sum \frac{x^n}{1+x^n}$ converges uniformly on $[0,a]$.

(c) The series does not converge uniformly on $[0,1)$. Suppose $\sum \frac{x^n}{1+x^n}$ converges uniformly on $[0,1)$. Then the series satisfies the Cauchy criterion uniformly on $[0,1)$, i.e., for any $\varepsilon > 0$, there is $N \in \mathbb{N}$, such that for $n \geq m > N$,

$$\Big| \sum_{k=m}^{n} \frac{x^k}{1+x^k} \Big| < \varepsilon, \quad \forall x \in [0,1).$$

Since $\frac{x^k}{1+x^k} \geq \frac{x^k}{2}$ on $[0,1)$ for each k, we then get

$$\Big| \sum_{k=m}^{n} x^k \Big| < 2\varepsilon, \quad \forall x \in [0,1).$$

This means that $\sum x^n$ satisfies the Cauchy criterion uniformly on $[0,1)$, which then implies that $\sum x^n$ converges uniformly on $[0,1)$. However, we have proved in class that $\sum x^n$ does not converge uniformly on $[0,1)$. So we get a contradiction. □

28.4 Let $f(x) = x^2 \sin \frac{1}{x}$ for $x \neq 0$ and $f(0) = 0$.

(a) Use Theorems 28.3 and 28.4 to show f is differentiable at each $a \neq 0$ and calculate $f'(a)$. Use, without proof, the fact that $\sin x$ is differentiable and that $\cos x$ is its derivative.

(b) Use the definition to show f is differentiable at $x = 0$ and $f'(0) = 0$.

(c) Show f' is not continuous at $x = 0$.

Proof. (a) By Theorems 28.3, we see that $\frac{1}{x}$ is differentiable on $\mathbb{R} \setminus \{0\}$ and $\frac{d}{dx}\frac{1}{x} = -\frac{1}{x^2}$. By Theorem 28.4 and that $\sin' x = \cos x$, $\sin \frac{1}{x}$ is differentiable on $\mathbb{R} \setminus \{0\}$, and $\frac{d}{dx} \sin \frac{1}{x} = (\sin' \frac{1}{x}) \cdot \frac{1}{x}$. By Theorem 28.3 again, $f(x) = x^2 \sin \frac{1}{x}$ is differentiable on $\mathbb{R} \setminus \{0\}$, and

$$f'(x) = \frac{d}{dx}(x^2) \sin \frac{1}{x} + x^2 \frac{d}{dx}\frac{1}{x} = 2x \sin \frac{1}{x} - \cos \frac{1}{x}, \quad x \neq 0.$$

(b) We calculate $\frac{f(x)-f(0)}{x-0} = \frac{f(x)}{x} = x \sin \frac{1}{x}$ for $x \neq 0$. Since $|x \sin \frac{1}{x}| \leq x$ and $\lim_{x \to 0} x = 0$, by squeeze lemma we get $\lim_{x\to 0} x \sin \frac{1}{x} = 0$. So f is differentiable at 0 and

$$f'(0) = \lim_{x \to 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0} x \sin \frac{1}{x} = 0.$$

(c) Let $x_n = \frac{1}{2n\pi}$. Then $x_n \to 0$ and $f'(x_n) = \frac{2}{2n\pi} \sin(2n\pi) - \cos(2n\pi) = -1 \not\to 0 = f'(0)$. So f' is not continuous at 0. □

28.7 Let $f(x) = x^2$ for $x \geq 0$ and $f(x) = 0$ for $x < 0$.

(b) Show f is differentiable at $x = 0$ and calculate $f'(0)$. Hint: You will have to use the definition of derivative. You may first consider one-sided derivatives.

(c) Calculate $f'(x)$ for $x > 0$ and $x < 0$.

(d) Is f' continuous on $\mathbb{R}$? differentiable on $\mathbb{R}$? Explain.

Proof. (b) For $x > 0$, $\frac{f(x)-f(0)}{x-0} = \frac{x^2}{x} = x$. So $\lim_{x\to 0^+} \frac{f(x)-f(0)}{x-0} = \lim_{x\to 0^+} x = 0$. For $x < 0$, $\frac{f(x)-f(0)}{x-0} = \frac{0-0}{x} = 0$. So $\lim_{x\to 0^-} \frac{f(x)-f(0)}{x-0} = 0$. Since $\lim_{x\to 0^+} \frac{f(x)-f(0)}{x-0} = \lim_{x\to 0^-} \frac{f(x)-f(0)}{x-0} = 0$, we get that f is differentiable at $x = 0$ and $f'(0) = 0$.

(c) Since $f(x) = x^2$ for $x > 0$, we have $f'(x) = \frac{d}{dx}x^2 = 2x$ for $x > 0$. Since $f(x) = 0$ for $x < 0$, we have $f'(x) = \frac{d}{dx}0 = 0$ for $x < 0$.

(d) We have $f'(x) = 2x$ for $x \geq 0$ and $f'(x) = 0$ for $x \leq 0$. So f' is continuous on $\mathbb{R}$. It is not differentiable on $\mathbb{R}$. Specifically, f' is not differentiable at 0. Note that for $x > 0$, $\frac{f'(x)-f'(0)}{x-0} = \frac{2x}{x} = 2$. So $\lim_{x\to 0^+} \frac{f'(x)-f'(0)}{x-0} = 2$; for $x < 0$, $\frac{f'(x)-f'(0)}{x-0} = \frac{0-0}{x} = 0$. So $\lim_{x\to 0^-} \frac{f(x)-f(0)}{x-0} = 0 \neq \lim_{x\to 0^+} \frac{f'(x)-f'(0)}{x-0}$. Thus, $\lim_{x\to 0} \frac{f(x)-f(0)}{x-0}$ does not exist. □

28.9 Let $h(x) = (x^4 + 13x)^7$. (a) Calculate $h'(x)$.

Solution. We write $h = g\circ f$, where $f(x) = x^4+13x$ and $g(x) = x^7$. Then $f'(x) = 4x^3+13$ and $g'(x) = 7x^6$. By the chain rule,

$$h'(x) = g'(f(x))f'(x) = 7(x^4 + 13x)^6(4x^3 + 13).$$

□

28.10 Let $h(x) = (\cos x + e^x)^{12}$. (a) Calculate $h'(x)$. You may use the fact that $\cos x$ and e^x are differentiable and that $-\sin x$ and e^x are their derivatives.

Solution. We write $h = g \circ f$, where $f(x) = \cos x + e^x$ and $g(x) = x^{12}$. Then $f'(x) = e^x - \sin x$ and $g'(x) = 12x^{11}$. By the chain rule,

$$h'(x) = g'(f(x))f'(x) = 12(\cos x + e^x)^{11}(e^x - \sin x).$$

□

28.11 Suppose f is differentiable at a, g is differentiable at $f(a)$, and h is differentiable at $g\circ f(a)$. State and prove the chain rule for $(h \circ g \circ f)'(a)$. Hint: Apply Theorem 28.4 twice.

Solution. We have that $h \circ g \circ f$ is differentiable at a, and

$$(h \circ g \circ f)'(a) = h'(g \circ f(a))g'(f(a)f'(a).$$

To prove this statement, we write $h \circ g \circ f = h \circ (g \circ f)$. By Theorem 28.4, $g \circ f$ is differentiable at a, and $(g \circ f)'(a) = g'(f(a))f'(a)$. Since h is differentiable at $(g \circ f)(a)$, by Theorem 28.4, $h \circ g \circ f = h \circ (g \circ f)$ is differentiable at a, and

$$(h \circ g \circ f)'(a) = h'(g \circ f(a))(g \circ f)'(a) = h'(g \circ f(a))g'(f(a)f'(a).$$

$\square$