Homework 1 Solutions

1.3 Prove $1^3 + 2^3 + \cdots + n^3 = (1 + 2 + \cdots + n)^2$ for all positive integers n.

Solution. We prove by induction. The base case $1^3 = 1^2$ is true. Suppose the statement holds for n. So

$$1^3 + 2^3 + \cdots + n^3 = (1 + 2 + \cdots + n)^2 = (\frac{n(n+1)}{2})^2,$$

where the second "=" follows from $1 + 2 + \cdots + n = \frac{n(n+1)}{2}$. We now prove the statement for $n + 1$. By the displayed formula,

$$1^3 + 2^3 + \cdots + n^3 + (n+1)^3 = (\frac{n(n+1)}{2})^2 + (n+1)^3 = \frac{1}{4}(n+1)^2(n^2 + 4(n+1))$$

$$= \frac{1}{4}(n+1)^2(n+2)^2 = (\frac{(n+1)(n+2)}{2})^2 = (1 + 2 + \cdots + n + (n+1))^2.$$

Thus the induction step is also proven, and the claim is true. □

1.8 The principle of mathematical induction can be extended as follows. A list $P_m, P_{m+1}, \ldots$ of propositions is true provided (i) P_m is true, (ii) P_{n+1} is true whenever P_n is true and $n \geq m$.

(a) Prove $n^2 > n + 1$ for all integers $n \geq 2$.

(b) Prove $n! > n^2$ for all integers $n \geq 4$. [Recall $n! = n(n-1) \cdots 2 \cdot 1$; for example, $5! = 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 120$.]

Solution. (a) The base case $2^2 > 2 + 1$ is true. Suppose the statement holds for some $n \geq 2$. We now prove the statement for $n + 1$. We have

$$(n+1)^2 = n^2 + 2n + 1 > n^2 + 1 > (n+1) + 1.$$

So the induction step is proven, and the claim is true.

(b) The base case $4! > 4^2$ is true because $4! = 24$ and $4^2 = 16$. Suppose the statement holds for some $n \geq 4$. We now prove the statement for $n + 1$. We have

$$(n+1)! = (n+1)n! > (n+1)n^2 > (n+1)^2,$$

where the last step follows from (a). So the induction step is proven, and the claim is true. □

1.9 (a) Decide for which integers the inequality $2^n > n^2$ is true.

(b) Prove your claim in (a) by mathematical induction.

Proof. (a) We observe that $2^0 = 1$, $0^2 = 0$; $2^1 = 2$, $1^2 = 1$; $2^2 = 4$, $2^2 = 4$; $2^3 = 8$, $3^2 = 9$; $2^4 = 16$, $4^2 = 16$; $2^5 = 32$, $5^2 = 25$; $2^6 = 64$, $6^2 = 36$; $2^7 = 128$, $7^2 = 49$. For negative integers n, $2^n < 1$ and $n^2 \geq 1$. So we conjecture that $2^n > n^2$ holds if and only if $n \in \{0, 1\}$ or $n \geq 5$.

(b) We have excluded the case $n < 0$ and checked the case $n = 0, 1, 2, 3, 4$ one by one. We now show that $2^n > n^2$ for $n \geq 5$ by induction. The base case $2^5 > 5^2$ is also checked above. Suppose the statement holds for some $n \geq 5$. We now prove the statement for $n+1$. Note $n^2 - 2n + 1 = (n-1)^2 > 2$ implies $n^2 > 2n + 1$. So

$$2^{n+1} = 2 \cdot 2^n > 2n^2 = n^2 + n^2 > n^2 + 2n + 1 = (n+1)^2.$$

So the induction step is proven, and the claim is true. □

2.3 Show $\sqrt{2 + \sqrt{2}}$ is not a rational number.

Solution. We see that if $a = \sqrt{2 + \sqrt{2}}$, then we have $a^2 = 2 + \sqrt{2}$, $a^2 - 2 = \sqrt{2}$, and $(a^2 - 2)^2 = 2$. Expanding the formula, we see that a is a solution of the equation

$$x^4 - 4x^4 + 2 = 0.$$

If $a \in \mathbb{Q}$, then by a corollary of Rational Zeroes Theorem, a is an integer that divides 2. So a must be one of $1, -1, 2, -2$. Plugging these numbers into the equation, we see that none of them are roots. So a can not be a rational number. □

2.4 Show $\sqrt[3]{5 - \sqrt{3}}$ is not a rational number.

Solution. We now use a slightly different approach. First we show that $b = \sqrt{3}$ is not a rational number. Note that b solves the equation $x^2 - 3 = 0$. If $b \in \mathbb{Q}$, then by a corollary of Rational Zeroes Theorem, b is an integer that divides -3. So b must be one of $1, -1, 3, -3$. Plugging these numbers into the equation, we see that none of them are roots. So b can not be a rational number. Suppose now $a = \sqrt[3]{5 - \sqrt{3}}$ is a rational number, say $\frac{c}{d}$. Then $5 - \sqrt{3} = a^3 = \frac{c^3}{d^3}$ is also a rational number, and so $\sqrt{3} = 5 - \frac{c^3}{d^3}$ is a rational number, which contradicts the first part of the proof. So a is also not a rational number. □

2.7 Show the following irrational-looking expressions are actually rational numbers:

$$(a)\ \sqrt{4 + 2\sqrt{3}} - \sqrt{3}, \quad (b)\ \sqrt{6 + 4\sqrt{2}} - \sqrt{2}.$$

Solution. (a) We observe that $(1+\sqrt{3})^2 = 1^2 + 2 \cdot 1 \cdot \sqrt{3} + (\sqrt{3})^2 = 1 + 2\sqrt{3} + 3 = 4 + 2\sqrt{3}$. So $\sqrt{4 + 2\sqrt{3}} = 1 + \sqrt{3}$, and $\sqrt{4 + 2\sqrt{3}} - \sqrt{3} = 1$.

(b) We observe that $(2 + \sqrt{2})^2 = 2^2 + 2 \cdot 2 \cdot \sqrt{2} + (\sqrt{2})^2 = 4 + 4\sqrt{2} + 2 = 6 + 4\sqrt{2}$. So $\sqrt{6 + 4\sqrt{2}} = 2 + \sqrt{2}$, and $\sqrt{6 + 4\sqrt{2}} - \sqrt{2} = 2$. □