Homework 2 Solutions

3.3 Prove (iv) and (v) of Theorem 3.1.

(iv) $(-a)(-b) = ab$ for all a, b;

(v) $ac = bc$ and $c \neq 0$ imply $a = b$.

Solutions. (iv) By A2, DL, and (ii), $(-a)(-b) + (-a)b = (-a)b + (-a)(-b) = (-a)(b + (-b)) = (-a)\cdot 0 = 0$. By M2, DL, and (ii), $ab+(-a)b = ba+b(-a) = b(a+(-a)) = b\cdot 0 = 0$. So $(-a)(-b) + (-a)b = ab + (-a)b$. By (i) we conclude $(-a)(-b) = ab$.

(v) $ac = bc$ implies $(ac)c^{-1} = (bc)c^{-1}$. By M1 and M3, $(ac)c^{-1} = a(cc^{-1}) = a \cdot 1 = a$ and $(bc)c^{-1} = b(cc^{-1}) = b \cdot 1 = b$. So $a = b$. □

3.4 Prove (v) and (vii) of Theorem 3.2.

(v) $0 < 1$;

(vii) If $0 < a < b$, then $0 < b^{-1} < a^{-1}$;

Solution. (v) By M3 and Theorem 3.2 (iv), $1 = 1 \cdot 1 = 1^2 \geq 0$. It now suffices to show that $1 \neq 0$. We prove this statement by contradiction, and use the assumption that a field contains more than one element. Suppose $1 = 0$. Then for any x in the field, by M3 and Theorem 3.1 (ii), $x = x \cdot 1 = x \cdot 0 = 0$. Then the field has only one element: 0, a contradiction.

(vii) By Theorem 3.2 (vi), $a^{-1}, b^{-1} > 0$. We now show $b^{-1} < a^{-1}$ by contradiction. Suppose it fails. Then $a^{-1} \leq b^{-1}$. By O5, $a^{-1}(ab) \leq b^{-1}(ab)$. By M1, M2, M3 and M4, $b = a^{-1}(ab)$ and $a = b^{-1}(ab)$. So we get $b \leq a$, which contradicts that $a < b$. □

3.7 (a) Show $|b| < a$ if and only if $-a < b < a$.

(b) Show $|a - b| < c$ if and only if $b - c < a < b + c$.

(c) Show $|a - b| \leq c$ if and only if $b - c \leq a \leq b + c$.

Solution. (a) Suppose $|b| < a$. Since $|b| \geq 0$ by Theorem 3.5 (i), we get $a > 0$ by O2 and O3. By Theorem 3.2 (i), $-a < 0$. If $b \geq 0$, then $b = |b| < a$ and $-a < 0 \leq b$. So $-a < b < a$. If $b \leq 0$, then$-b = |b| < a$. By Theorem 3.2 (i), $-a < b$. Since $b \leq 0 < a$, again we get $-a < b < a$. Now suppose $-a < b < a$. By Theorem 3.2 (i), $-a < -b < a$. Since $|b|$ equals either b or $-b$, we get $|b| < a$.

(b) By (a), $|a - b| < c$ iff $-c < a - b < c$, which is further equivalent to $b - c < a < b + c$ by O4.

(c) We first show that $|b| \leq a$ iff $-a \leq b \leq a$. We may simply repeat the proof of (a) with all "<" and ">" replaced by "≤" and "≥", respectively. Then we repeat the proof of (b) with the same modification. □

4.5 Let S be a nonempty subset of $\mathbb{R}$ that is bounded above. Prove if $\sup S$ belongs to S, then $\sup S = \max S$. Hint: Your proof should be very short.

Solution. Recall that $\sup S$ is the least upper bound of S. So $\sup S$ is an upper bound of S. This means that for any $s \in S$, $s \leq \sup S$. Since $\sup S \in S$, it satisfies the property of $\max S$, and so must be $\max S$. $\square$

4.6 Let S be a nonempty bounded subset of $\mathbb{R}$.

(a) Prove $\inf S \sup S$. Hint: This is almost obvious; your proof should be short.

(b) What can you say about S if $\inf S = \sup S$?

Solution. (a) Since $\inf S$ is a lower bound of S, and $\sup S$ is an upper bound of S, for any $s_0 \in S$, we have $\inf S \leq s_0$ and $s_0 \leq \sup S$. By O3, $\inf S \leq \sup S$.

(b) From (a), we know that for any $s \in S$, $\inf S \leq s \leq \sup S$. If $\inf S = \sup S$, then s has to be equal to $\inf S$. This means that S contains only one element. $\square$

E1 Show that for any nonempty finite subset S of $\mathbb{R}$, $\max S$ exists. Hint: Prove by induction on $|S|$, i.e., the number of elements of S.

Solution. Since S is nonempty and finite, we have $|S| \in \mathbb{N}$. We rewrite the statement as: for any $n \in \mathbb{N}$, if $|S| = n$, then $\max S$ exists. If $n = 1$, S contains only one element, say s. Then $s = \max S$. Suppose the statement holds true for n. We now show that it is also true for $n + 1$. Let S satisfy $|S| = n + 1$. Take any $s_0 \in S$. Let $S' = S \setminus \{s_0\}$. Then $|S'| = n$. By induction hypothesis, $\max S'$ exists. This means that there is $s' \in S'$ such that for any $s \in S'$, $s \leq s'$. Now either $s_0 \leq s'$ or $s' \leq s_0$. If $s_0 \leq s'$, then $s \leq s'$ for any $s \in S$, so $\max S = s'$. If $s' \leq s_0$, then for any $s \in S'$, $s \leq s_0$. Since $S = S' \cup \{s_0\}$, we get $\max S = s_0$. This finishes the induction step. So the statement holds for all $n \in \mathbb{N}$. $\square$