Homework 3 (due on 9/20)

- Read Sections 9 and 10 for the next week.
- We are going to have the first Quiz held on 9/16.

4.7 & 5.6 Let S and T be nonempty subsets of $\mathbb{R}$.

(a) Prove if $S \subseteq T$, then $\inf T \le \inf S \le \sup S \le \sup T$.

(b) Prove $\sup(S \cup T) = \max\{\sup S, \sup T\}$.

Note: S and T may or may not be bounded above or below. In part (b), do not assume $S \subseteq T$, and we understand $\max\{+\infty, x\} = \max\{+\infty, +\infty\} = +\infty$ for any $x \in \mathbb{R}$.

4.18 Let S and T be nonempty subsets of $\mathbb{R}$ with the following property: $s \le t$ for all $s \in S$ and $t \in T$.

(a) Observe S is bounded above and T is bounded below.

(b) Prove $\sup S \le \inf T$.

(c) Give an example of such sets S and T where $S \cap T$ is nonempty.

(d) Give an example of sets S and T where $\sup S = \inf T$ and $S \cap T$ is the empty set.

4.12 The elements of $\mathbb{R} \setminus \mathbb{Q}$ are called irrational numbers. Prove if $a < b$, then there exists $x \in \mathbb{R} \setminus \mathbb{Q}$ such that $a < x < b$. Hint: First show $\{r + \sqrt{2} : r \in \mathbb{Q}\} \subseteq \mathbb{R} \setminus \mathbb{Q}$.

4.15 Let $a, b \in \mathbb{R}$. Show if $a \le b + \frac{1}{n}$ for all $n \in \mathbb{N}$, then $a \le b$. Compare Exercise 3.8.

4.16 Show $\sup\{r \in \mathbb{Q} : r < a\} = a$ for each $a \in \mathbb{R}$.

8.1 (a) Prove $\lim \frac{(-1)^n}{n} = 0$.

8.2 (e) Determine the limit of the sequence $s_n = \frac{1}{n} \sin n$, and then prove your claim. You may use any fact about $\sin x$ that you learned before.

8.3 Let (s_n) be a sequence of nonnegative real numbers, and suppose $\lim s_n = 0$. Prove $\lim \sqrt{s_n} = 0$. This will complete the proof for Example 5.

9.1 (c) Using the limit Theorems 9.2 - 9.7, prove the following. Justify all steps.

$$\lim_{n\to\infty} \frac{17n^5 + 73n^4 - 18n^2 + 3}{23n^5 + 13n^3} = \frac{17}{23}.$$

9.5 Let $t_1 = 1$ and $t_{n+1} = \frac{t_n^2+2}{2t_n}$ for $n \ge 1$. Assume (t_n) converges and find the limit.
Hint: You are not required to prove the convergence of (t_n). You may use the simple fact that when (t_n) converges, the sequence (t_{n+1}) converges to the same number.