Homework 7 Solutions

14.2 (b) Determine the convergence of $\sum(-1)^n$ and justify your answer.

Solution. Since $|(-1)^n - 0| = 1$ for all $n \in \mathbb{N}$, we do not have $(-1)^n \to 0$. By limit test, $\sum(-1)^n$ diverges. □

14.3 (f) Determine the convergence of $\sum \frac{100^n}{n!}$ and justify your answer.

Solution. We use ratio test. Note that for $a_n := \frac{100^n}{n!}$,

$$|\frac{a_{n+1}}{a_n}| = \frac{100^{n+1}/(n+1)!}{100^n/n!} = \frac{100}{n+1} \to 0 < 1.$$

So the series converges by ratio test. □

14.6 (a) Prove that if $\sum |a_n|$ converges and (b_n) is a bounded sequence, then $\sum a_n b_n$ converges. Hint: Use Theorem 14.4.

Proof. Since $\sum |a_n|$ converges, by Theorem 14.4 it satisfies Cauchy criterion. Since (b_n) is bounded, there is $M \in \mathbb{R}$ with $M > 0$ such that $|b_n| \le M$ for any n. We are going to show that $\sum a_n b_n$ satisfies Cauchy criterion. Let $\varepsilon > 0$. Then $\frac{\varepsilon}{M} > 0$. Since $\sum |a_n|$ satisfies Cauchy criterion, there is $N \in \mathbb{N}$ such that for any $m \ge n > N$, $\sum_{k=n}^m |a_k| = |\sum_{k=n}^m |a_k|| < \frac{\varepsilon}{M}$, which together with triangle inequality and the upper bound of (b_n) implies that

$$\Big|\sum_{k=n}^m a_k b_k\Big| \le \sum_{k=n}^m |a_k b_k| = \sum_{k=n}^m |a_k||b_k| \le M \sum_{k=n}^m |a_k| < M \cdot \frac{\varepsilon}{M} = \varepsilon.$$

So $\sum a_n b_n$ also satisfies Cauchy criterion, and so by Theorem 14.4 it converges. □

14.12 Let (a_n) be a sequence such that $\liminf |a_n| = 0$. Prove that there is a subsequence (a_{n_k}) such that $\sum a_{n_k}$ converges.

Proof. Let $u_N = \inf\{|a_n| : n > N\}$, $N \in \mathbb{N}$. Since $|a_n| \ge 0$ for every n, we have $u_N \ge 0$ for every N. Since (u_N) is increasing and converges to $\liminf |a_n| = 0$, we get $u_N \le 0$ for every N. So $u_N = 0$ for all $N \in \mathbb{N}$. For any $N \in \mathbb{N}$ and $r > 0$, from $\inf\{|a_n| : n > N\} = 0 < r$ we see that there is $n > N$ such that $|a_n| < r$. So for any $r > 0$, there are infinitely many n such that $|a_n| < r$. Taking $r = \frac{1}{2}$, this implies that there is $n_1 \in \mathbb{N}$ such that $|a_{n_1}| < \frac{1}{2}$. We now construct a sequence (n_k) inductively. When $n_1 < \cdots < n_k$ are constructed, we use the fact that there are infinitely many n such that $|a_n| < \frac{1}{2^{k+1}}$ to find $n_{k+1} > n_k$ such that $|a_{n_{k+1}}| < \frac{1}{2^{k+1}}$. Such sequence (n_k) then satisfies (i) $n_1 < n_2 < \cdots$; (ii) $|a_{n_k}| \le \frac{1}{2^k}$ for every $k \in \mathbb{N}$. So (a_{n_k}) is a subsequence of (a_n). From comparison test and the convergence of $\sum \frac{1}{2^k}$ we conclude that $\sum a_{n_k}$ converges. □

E1 Prove that for any $r > 0$, $\frac{r^n}{n!} \to 0$ and $\frac{n!}{r^n} \to +\infty$. Hint: Use Ratio Test.

Proof. We first show that $\sum \frac{r^n}{n!}$ converges using ratio test. This is similar to 14.3 (f). Note that for $a_n := \frac{r^n}{n!}$,

$$|\frac{a_{n+1}}{a_n}| = \frac{r^{n+1}/(n+1)!}{r^n/n!} = \frac{r}{n+1} \to 0 < 1.$$

So the series $\sum \frac{r^n}{n!}$ converges, which in turn implies that $\frac{r^n}{n!} \to 0$. Since $\frac{r^n}{n!} > 0$, we then get $\frac{n!}{r^n} \to +\infty$. □

15.1 Determine which of the following series converge. Justify your answers.

$$(a) \sum \frac{(-1)^n}{n}; \qquad (b) \sum \frac{(-1)^n n!}{2^n}.$$

Solution. Both series are have alternative signs. Since $|\frac{(-1)^n}{n}| = \frac{1}{n}$ is decreasing and tends to 0, by alternative series test, $\sum \frac{(-1)^n}{n}$ converges. By the previous problem, we have $|\frac{(-1)^n n!}{2^n}| = \frac{n!}{2^n} \to +\infty$. So we do not have $\frac{(-1)^n n!}{2^n} \to 0$. By limit test, $\sum \frac{(-1)^n n!}{2^n}$ diverges. □